Supplement to “Estimation and inference of FAVAR models”

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Summary: In this supplement, we provide detailed derivations for the asymptotic results under IRb in Appendix C, and results under IRc in Appendix D. We also provide the derivations for the asymptotic results of the impulse response function in Appendix E.

Appendix C: The asymptotic results and their proofs under IRb

This appendix provides the detailed proofs of the theorems and corollaries in Section 4 under IRb. The idea of the derivation is to find the rotation matrix and its estimate, which transform the parameters under IRa to those of IRb. For ease of exposition, we adopt the symbols in the previous section to denote the parameters under IRa, e.g., Λ, Γ and F . We use the symbols with diamond to denote the parameters under IRb, e.g., $\Lambda^\diamond, \Gamma^\diamond$ and $F^\diamond$. We also use $R^\diamond$ to denote the rotation matrix which transforms the parameters set $(\lambda'_i, \gamma'_i)'$ into $(\lambda_i^{\diamond'}, \gamma_i^{\diamond'})'$. We point out that $R^\diamond$, unlike R^* in appendix A, is nonrandom. Then

$$\begin{aligned} z_t &= [\Lambda^\diamond, \Gamma^\diamond] \begin{bmatrix} f_t^\diamond \\ g_t \end{bmatrix} + e_t \\ &= [\Lambda, \Gamma] \begin{bmatrix} R_{11}^{\diamond'} & R_{21}^{\diamond'} \\ 0 & I_{r_2} \end{bmatrix} \begin{bmatrix} R_{11}^{\diamond'-1} & -R_{11}^{\diamond'-1} R_{21}^{\diamond'} \\ 0 & I_{r_2} \end{bmatrix} \begin{bmatrix} f_t \\ g_t \end{bmatrix} + e_t. \end{aligned}$$

Using the method in deriving (B.7), we have

$$\begin{aligned} \varepsilon_t^\diamond &= R_{11}^{\diamond'-1} \varepsilon_t - R_{11}^{\diamond'-1} R_{21}^{\diamond'} \mathbf{v}_t, \\ \mathbf{v}_t^\diamond &= \mathbf{v}_t. \end{aligned} \tag{C.1}$$

By $E(\varepsilon_t \mathbf{v}_t') = E(\varepsilon_t^\diamond \mathbf{v}_t^{\diamond'}) = 0$ and $E(\varepsilon_t \varepsilon_t') = E(\varepsilon_t^\diamond \varepsilon_t^{\diamond'}) = I_{r_1}$, we have

$$R_{21}^\diamond = 0; \quad R_{11}^{\diamond'} R_{11}^\diamond = I_{r_1}. \tag{C.2}$$

In addition, the identification conditions require

$$\Lambda_1 R_{11}^{\diamond'} = \Lambda_1^\diamond, \tag{C.3}$$

where $\Lambda_1, \Lambda_1^\diamond$ are the upper $r_1 \times r_1$ submatrices of $\Lambda, \Lambda^\diamond$, respectively. Accordingly, we have

$$\hat{R}_{21}^\diamond = 0, \quad \hat{R}_{11}^{\diamond'} \hat{R}_{11}^\diamond = I_{r_1}, \tag{C.4}$$

$$\hat{\Lambda}_1 \hat{R}_{11}^{\diamond'} = \hat{\Lambda}_1^\diamond. \tag{C.5}$$

By (C.2) and (C.4), we have $\hat{R}_{11}^{\diamond'} \hat{R}_{11}^\diamond R_{11}^{\diamond'} R_{11}^\diamond = I_{r_1}$. This implies

$$(R_{11}^\diamond \hat{R}_{11}^{\diamond'}) (R_{11}^\diamond \hat{R}_{11}^{\diamond'})' = I_{r_1}.$$

Furthermore, notice that both $\hat{\Lambda}_1^\diamond$ and $\Lambda_1^\diamond$ are lower triangular matrices, then $\Lambda_1^{\diamond-1}\hat{\Lambda}_1^\diamond$ is a lower triangular matrix. By (C.3) and (C.5), we have $\Lambda_1^{\diamond-1}\hat{\Lambda}_1^\diamond = R_{11}^\diamond \Lambda_1^{-1} \hat{\Lambda}_1 \hat{R}_{11}^{\diamond'}$. However, we have already proved that $\hat{\Lambda}_1 - \Lambda_1 \xrightarrow{p} 0$. Given this result, we have $R_{11}^\diamond \hat{R}_{11}^{\diamond'} \xrightarrow{p} \Lambda_1^{\diamond-1} \hat{\Lambda}_1^\diamond$, a lower triangular matrix. By $(R_{11}^\diamond \hat{R}_{11}^{\diamond'})(R_{11}^\diamond \hat{R}_{11}^{\diamond'})' = I_r$, we have $R_{11}^\diamond \hat{R}_{11}^{\diamond'}$ converges in probability to a diagonal matrix whose diagonal elements are either 1 or -1 . However, our sign restrictions rule out the possibility of -1 . Then it follows $\hat{R}_{11}^\diamond \xrightarrow{p} R_{11}^\diamond$.

Let $\widehat{\Delta R}_{11}^\diamond = \hat{R}_{11}^\diamond - R_{11}^\diamond$. By $\hat{R}_{11}^{\diamond'} \hat{R}_{11}^\diamond = R_{11}^{\diamond'} R_{11}^\diamond = I_r$, we have

$$\widehat{\Delta R}_{11}^{\diamond'} R_{11}^\diamond + R_{11}^{\diamond'} \widehat{\Delta R}_{11}^\diamond + \widehat{\Delta R}_{11}^{\diamond'} \widehat{\Delta R}_{11}^\diamond = 0$$

Pre-multiplying $R_{11}^\diamond$ and post-multiplying $R_{11}^{\diamond'}$ on both sides gives

$$(\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) + (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'})' + (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'})' (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) = 0 \quad (\text{C.6})$$

Also notice that

$$\hat{\Lambda}_1^{\diamond'} - \Lambda_1^{\diamond'} = \hat{R}_{11}^\diamond \hat{\Lambda}_1' - R_{11}^\diamond \Lambda_1' = \widehat{\Delta R}_{11}^\diamond \Lambda_1' + R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' + \widehat{\Delta R}_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)'$$

However, term $\widehat{\Delta R}_{11}^\diamond \Lambda_1' = (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) (R_{11}^\diamond \Lambda_1') = (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) \Lambda_1^{\diamond'}$. Given this result, we have

$$\hat{\Lambda}_1^{\diamond'} - \Lambda_1^{\diamond'} = (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) \Lambda_1^{\diamond'} + R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' + \widehat{\Delta R}_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)'$$

Post-multiplying $\Lambda_1^{\diamond'-1}$ on both sides, we have

$$(\hat{\Lambda}_1^{\diamond'} - \Lambda_1^{\diamond'}) \Lambda_1^{\diamond'-1} = (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) + R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1} + \widehat{\Delta R}_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1}$$

Matrix $(\hat{\Lambda}_1^{\diamond'} - \Lambda_1^{\diamond'}) \Lambda_1^{\diamond'-1}$ is an upper triangular matrix since both $\hat{\Lambda}_1^\diamond$ and $\Lambda_1^\diamond$ are lower triangular matrices. So we have

$$\text{lower}\{(\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'}) + R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1} + \widehat{\Delta R}_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1}\} = 0 \quad (\text{C.7})$$

where $\text{lower}\{\cdot\}$ denotes the elements of the argument strictly below the diagonal.

Consider $R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1}$. By Proposition B.1, we have

$$R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1} = R_{11}^\diamond \left(\sum_{t=1}^T \phi_t \phi_t' \right)^{-1} \left(\sum_{t=1}^T \phi_t \xi_t' \right) \Lambda_1^{\diamond'-1} + R_{11}^\diamond V \Lambda_1' \Lambda_1^{\diamond'-1} + O_p(T^{-1}) + O_p(N^{-1}),$$

where $\xi_t = (e_{1t}, e_{2t}, \dots, e_{rt})'$. Notice $\phi_t = f_t - E(f_t g_t') [E(g_t g_t')]^{-1} g_t$. So we have $R_{11}^\diamond \phi_t = f_t^\diamond - E(f_t^\diamond g_t') [E(g_t g_t')]^{-1} g_t$, which we use $\phi_t^\diamond$ to denote. So the first term is equal to

$$\left(\sum_{t=1}^T \phi_t^\diamond \phi_t^{\diamond'} \right)^{-1} \left(\sum_{t=1}^T \phi_t^\diamond \xi_t' \right) \Lambda_1^{\diamond'-1}$$

The second term is $R_{11}^\diamond V R_{11}^{\diamond'}$ by $\Lambda_1' \Lambda_1^{\diamond'-1} = R_{11}^{\diamond-1} = R_{11}^{\diamond'}$. Then it follows

$$R_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1} = R_{11}^\diamond V R_{11}^{\diamond'} + \left(\sum_{t=1}^T \phi_t^\diamond \phi_t^{\diamond'} \right)^{-1} \left(\sum_{t=1}^T \phi_t^\diamond \xi_t' \right) \Lambda_1^{\diamond'-1} + O_p(T^{-1}) + O_p(N^{-1})$$

Given the above result, (C.7) can be written as

$$\begin{aligned} \text{lower} \left\{ \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} + R_{11}^{\diamond} V R_{11}^{\diamond'} \right) + \left(\sum_{t=1}^T \phi_t^{\diamond} \phi_t^{\diamond'} \right)^{-1} \left(\sum_{t=1}^T \phi_t^{\diamond} \xi_t^{\diamond'} \right) \Lambda_1^{\diamond'-1} \right. \\ \left. + \widehat{\Delta R}_{11}^{\diamond} (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1} + O_p(N^{-1}) + O_p(T^{-1}) \right\} = 0 \end{aligned} \quad (\text{C.8})$$

By (C.6), we have

$$\left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} + R_{11}^{\diamond} V R_{11}^{\diamond'} \right) + \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} + R_{11}^{\diamond} V R_{11}^{\diamond'} \right)' + \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} \right)' \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} \right) = R_{11}^{\diamond} (V + V') R_{11}^{\diamond'}.$$

Notice

$$V + V' = \frac{1}{\bar{T}} \sum_{t=\bar{K}}^T (\varepsilon_t \varepsilon_t' - I_{r_1})$$

Then, by $\varepsilon_t^{\diamond} = R_{11}^{\diamond} \varepsilon_t$, we have

$$\begin{aligned} \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} + R_{11}^{\diamond} V R_{11}^{\diamond'} \right) + \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} + R_{11}^{\diamond} V R_{11}^{\diamond'} \right)' \\ + \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} \right)' \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} \right) = \frac{1}{\bar{T}} \sum_{t=\bar{K}}^T (\varepsilon_t^{\diamond} \varepsilon_t^{\diamond'} - I_{r_1}) \end{aligned} \quad (\text{C.9})$$

For now, define

$$\mathcal{V} = \left\{ \widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} + R_{11}^{\diamond} V R_{11}^{\diamond'} + \left[\sum_{t=1}^T \phi_t^{\diamond} \phi_t^{\diamond'} \right]^{-1} \left[\sum_{t=1}^T \phi_t^{\diamond} \xi_t^{\diamond'} \right] \Lambda_1^{\diamond'-1} \right\}'.$$

By neglecting the smaller order terms $\left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} \right)' \left(\widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} \right)$ and $\widehat{\Delta R}_{11}^{\diamond} (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1^{\diamond'-1}$, we see that $\mathcal{V}$ is an asymptotically lower triangular matrix such that

$$\begin{aligned} \mathcal{V} + \mathcal{V}' &= \frac{1}{\bar{T}} \sum_{t=\bar{K}}^T (\varepsilon_t^{\diamond} \varepsilon_t^{\diamond'} - I_{r_1}) + \left[\sum_{t=1}^T \phi_t^{\diamond} \phi_t^{\diamond'} \right]^{-1} \left[\sum_{t=1}^T \phi_t^{\diamond} \xi_t^{\diamond'} \right] \Lambda_1^{\diamond'-1} \\ &\quad + \Lambda_1^{\diamond-1} \left[\sum_{t=1}^T \xi_t \phi_t^{\diamond'} \right] \left[\sum_{t=1}^T \phi_t^{\diamond} \phi_t^{\diamond'} \right]^{-1} + O_p(T^{-1}) + O_p(N^{-1}). \end{aligned}$$

Let D_r^* be the matrix such that $\text{vec}(M) = D_r^* \text{vech}(M)$ for any lower triangular $r \times r$ matrix M , where $\text{vech}(\cdot)$ is the operator which stacks all the elements on and below the diagonal into a vector. It is worth noting that D_r^* is different from the usual duplication matrix since M is lower triangular, not symmetric. Then taking the vectorization operation on both sides and noticing that $\text{vec}(\mathcal{V}) = D_r^* \text{vech}(\mathcal{V})$, we have

$$2S_{r_1} D_{r_1}^* \text{vech}(\mathcal{V}) = \frac{1}{\bar{T}} \sum_{t=\bar{K}}^T [\varepsilon_t^{\diamond} \otimes \varepsilon_t^{\diamond} - \text{vec}(I_{r_1})] + 2S_{r_1} (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{\bar{T}} \sum_{t=1}^T (\xi_t \otimes \phi_t^{\diamond}) + O_p(T^{-1}) + O_p(N^{-1}).$$

where S_r is the symmetrizer matrix such that $S_r = (I_{r^2} + K_r)/2$, with K_r being the r -dimensional commutation matrix such that $\text{vec}(A') = K_r \text{vec}(A)$ for any $r \times r$ matrix A .

Let $V^\diamond$ be the leading term of $\widehat{\Delta R}_{11}^\diamond R_{11}' + R_{11}' V R_{11}^\diamond$. It is easy to see from (C.8) and (C.9) that $V^\diamond = O_p(T^{-1/2})$. So we have $\widehat{\Delta R}_{11}^\diamond R_{11}' + R_{11}' V R_{11}^\diamond = V^\diamond + O_p(T^{-1}) + O_p(N^{-1})$. Then by the definition of $\mathcal{V}$,

$$V^\diamond = \mathcal{V}' - \Delta_{\phi\phi}^{\diamond-1} \frac{1}{T} \sum_{t=1}^T \phi_t^\diamond \xi_t' \Lambda_1^{\diamond-1} + O_p(T^{-1}).$$

Taking vectorization operation on both sides, we have

$$\begin{aligned} \text{vec}(V^\diamond) &= K_{r_1} \text{vec}(\mathcal{V}) - (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^\diamond) + O_p(T^{-1}) \\ &= K_{r_1} D_{r_1}^* \text{vech}(\mathcal{V}) - (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^\diamond) + O_p(T^{-1}) \\ &= \mathbb{D}_2 \left[\frac{1}{T} \sum_{t=\bar{K}}^T [\varepsilon_t^\diamond \otimes \varepsilon_t^\diamond - \text{vec}(I_{r_1})] + 2S_{r_1} (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^\diamond) \right] \quad (\text{C.10}) \\ &\quad - (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^\diamond) + O_p(T^{-1}) \end{aligned}$$

where $\mathbb{D}_2 = K_{r_1} D_{r_1}^* (D_{r_1}' S_{r_1}' S_{r_1} D_{r_1}^*)^{-1} D_{r_1}' S_{r_1}' / 2$.

Now consider the asymptotic property of $\hat{\lambda}_i^\diamond - \lambda_i^\diamond$. By $\hat{\lambda}_i^\diamond = \hat{R}_{11}^\diamond \hat{\lambda}_i$ and $\lambda_i^\diamond = R_{11}^\diamond \lambda_i$,

$$\begin{aligned} \hat{\lambda}_i^\diamond - \lambda_i^\diamond &= \hat{R}_{11}^\diamond \hat{\lambda}_i - R_{11}^\diamond \lambda_i = \widehat{\Delta R}_{11}^\diamond \lambda_i + R_{11}^\diamond (\hat{\lambda}_i - \lambda_i) + \widehat{\Delta R}_{11}^\diamond (\hat{\lambda}_i - \lambda_i) \\ &= \widehat{\Delta R}_{11}^\diamond R_{11}' \lambda_i^\diamond + R_{11}^\diamond (\hat{\lambda}_i - \lambda_i) + O_p(T^{-1}) + O_p(N^{-2}) \end{aligned}$$

Substituting the first expression of Proposition B.1 into the above equation and noticing $R_{11}^\diamond \phi_t = \phi_t^\diamond$, we have

$$\hat{\lambda}_i^\diamond - \lambda_i^\diamond = (\widehat{\Delta R}_{11}^\diamond R_{11}' + R_{11}^\diamond V R_{11}') \lambda_i^\diamond + \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond \phi_t'^\diamond \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond e_{it} \right) + O_p(T^{-1}).$$

By $\widehat{\Delta R}_{11}^\diamond R_{11}' + R_{11}^\diamond V R_{11}' = V^\diamond + O_p(T^{-1}) + O_p(N^{-1})$ and $\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond \phi_t'^\diamond = \Delta_{\phi\phi}^\diamond + O_p(T^{-1/2})$, we have

$$\begin{aligned} \hat{\lambda}_i^\diamond - \lambda_i^\diamond &= V^\diamond \lambda_i^\diamond + \Delta_{\phi\phi}^{\diamond-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond e_{it} \right) + O_p(T^{-1}) + O_p(N^{-1}) \quad (\text{C.11}) \\ &= (\lambda_i'^\diamond \otimes I_{r_1}) \text{vec}(V^\diamond) + \Delta_{\phi\phi}^{\diamond-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond e_{it} \right) + O_p(T^{-1}) + O_p(N^{-1}). \end{aligned}$$

Substituting (C.10) into the above expression,

$$\begin{aligned} \hat{\lambda}_i^\diamond - \lambda_i^\diamond &= (\lambda_i'^\diamond \otimes I_{r_1}) \mathbb{D}_2 \left[\frac{1}{T} \sum_{t=\bar{K}}^T [\varepsilon_t^\diamond \otimes \varepsilon_t^\diamond - \text{vec}(I_{r_1})] + 2S_{r_1} (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^\diamond) \right] \\ &\quad - [(\lambda_i'^\diamond \Lambda_1^{\diamond-1}) \otimes \Delta_{\phi\phi}^{\diamond-1}] \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^\diamond) + \Delta_{\phi\phi}^{\diamond-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond e_{it} \right) + O_p(T^{-1}) + O_p(N^{-1}). \end{aligned}$$

We discuss the asymptotic representations in two cases.

Case one: $i \leq r_1$. In this case, $\lambda_i^{\diamond'} \Lambda_1^{\diamond-1} = \iota_i' \Lambda_1^{\diamond} \Lambda_1^{\diamond-1} = \iota_i'$ where ι_i is the i th column of $r_1 \times r_1$ identity matrix. Given this result, it is easy to see that the last two expressions are cancelled. Thus

$$\hat{\lambda}_i^{\diamond} - \lambda_i^{\diamond} = (\lambda_i^{\diamond'} \otimes I_{r_1}) \mathbb{D}_2 \left[\frac{1}{T} \sum_{t=\bar{K}}^T [\varepsilon_t^{\diamond} \otimes \varepsilon_t^{\diamond} - \text{vec}(I_{r_1})] + 2S_{r_1} (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^{\diamond}) \right] + O_p(T^{-1}) + O_p(N^{-1}).$$

When $\sqrt{T}/N \rightarrow 0$, we have

$$\sqrt{T}(\hat{\lambda}_i^{\diamond} - \lambda_i^{\diamond}) \xrightarrow{d} \mathcal{N}\left(0, (\lambda_i^{\diamond'} \otimes I_{r_1}) \mathbb{D}_2 \left(2I_{r_1^2} + \mathcal{H}^{\diamond} + 4S_{r_1} [(\Lambda_1^{\diamond-1} \Sigma_{\xi\xi} \Lambda_1^{\diamond-1'}) \otimes \Delta_{\phi\phi}^{\diamond-1}] S_{r_1}' \right) \mathbb{D}_2' (\lambda_i^{\diamond} \otimes I_{r_1}) \right).$$

Case two: $i > r_1$. In this case, the asymptotic representation can be written as

$$\begin{aligned} \hat{\lambda}_i^{\diamond} - \lambda_i^{\diamond} &= (\lambda_i^{\diamond'} \otimes I_{r_1}) \mathbb{D}_2 \frac{1}{T} \sum_{t=\bar{K}}^T [\varepsilon_t^{\diamond} \otimes \varepsilon_t^{\diamond} - \text{vec}(I_{r_1})] + \Delta_{\phi\phi}^{\diamond-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^{\diamond} e_{it} \right) \\ &\quad + (\lambda_i^{\diamond'} \otimes I_{r_1}) (2\mathbb{D}_2 S_{r_1} - I_{r_1^2}) (\Lambda_1^{\diamond-1} \otimes \Delta_{\phi\phi}^{\diamond-1}) \frac{1}{T} \sum_{t=1}^T (\xi_t \otimes \phi_t^{\diamond}) + O_p(T^{-1}) + O_p(N^{-1}). \end{aligned}$$

When $\sqrt{T}/N \rightarrow 0$, we have

$$\sqrt{T}(\hat{\lambda}_i^{\diamond} - \lambda_i^{\diamond}) \xrightarrow{d} \mathcal{N}\left(0, (\lambda_i^{\diamond'} \otimes I_{r_1}) \left(\mathbb{D}_2 (2I_{r_1} + \mathcal{H}^{\diamond}) \mathbb{D}_2' + \mathbb{D}_3 [(\Lambda_1^{\diamond-1} \Sigma_{\xi\xi} \Lambda_1^{\diamond-1'}) \otimes \Delta_{\phi\phi}^{\diamond-1}] \mathbb{D}_3' \right) (\lambda_i^{\diamond} \otimes I_{r_1}) + \sigma_i^2 \Delta_{\phi\phi}^{\diamond-1} \right).$$

with $\mathbb{D}_3 = 2\mathbb{D}_2 S_{r_1} - I_{r_1^2}$.

To derive the asymptotic $\hat{\gamma}_i^{\diamond} - \gamma_i^{\diamond}$, we notice $\hat{\gamma}_i^{\diamond} = \hat{\gamma}_i$ and $\gamma_i^{\diamond} = \gamma_i$. Thus, by (B.31),

$$\begin{aligned} \hat{\gamma}_i^{\diamond} - \gamma_i^{\diamond} &= \left(\frac{1}{T} \sum_{t=1}^T g_t g_t' \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T (g_t - \Xi_t) e_{it} \right) + W \lambda_i + O_p(N^{-1}) + O_p(T^{-1}) \\ &= \left(\frac{1}{T} \sum_{t=1}^T g_t g_t' \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T (g_t - \Xi_t^{\diamond}) e_{it} \right) + W^{\diamond} \lambda_i^{\diamond} + O_p(N^{-1}) + O_p(T^{-1}). \end{aligned}$$

where $\Xi_t^{\diamond} = \Delta_{gf}^{\diamond} \Delta_{\phi\phi}^{\diamond-1} \phi_t^{\diamond}$ and $W^{\diamond} = \Omega_{vv}^{\diamond-1} (\frac{1}{T} \sum_{t=\bar{K}}^T v_t^{\diamond} \varepsilon_t^{\diamond'})$. By the similar arguments in the previous section, the above asymptotic results has an alternative expression:

$$\hat{\gamma}_i^{\diamond} - \gamma_i^{\diamond} = \left[\sum_{t=1}^T \eta_t^{\diamond} \eta_t^{\diamond'} \right]^{-1} \left[\sum_{t=1}^T \eta_t^{\diamond} e_{it} \right] + W^{\diamond} \lambda_i^{\diamond} + O_p(N^{-1}) + O_p(T^{-1}).$$

Now consider $\hat{f}_t^{\diamond} - f_t^{\diamond}$. By $\hat{f}_t^{\diamond} = \hat{R}_{11}^{\diamond} \hat{f}_t$ and $f_t^{\diamond} = R_{11}^{\diamond} f_t$, we have

$$\begin{aligned} \hat{f}_t^{\diamond} - f_t^{\diamond} &= \widehat{\Delta R}_{11}^{\diamond} f_t + R_{11}^{\diamond} (\hat{f}_t - f_t) + \widehat{\Delta R}_{11}^{\diamond} (\hat{f}_t - f_t) \\ &= \widehat{\Delta R}_{11}^{\diamond} R_{11}^{\diamond'} f_t^{\diamond} + R_{11}^{\diamond} (\hat{f}_t - f_t) + O_p(N^{-1/2} T^{-1/2}) + O_p(T^{-1}). \end{aligned}$$

Substituting the third expression of Proposition B.1 into the above equation, we have

$$\begin{aligned}\hat{f}_t^\diamond - f_t^\diamond &= (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} - R_{11}^\diamond V' R_{11}^{\diamond'}) f_t^\diamond - R_{11}^\diamond \left(\sum_{t=1}^T \varepsilon_t \mathbf{v}_t' \right) \left(\sum_{t=1}^T \mathbf{v}_t \mathbf{v}_t' \right)^{-1} g_t \\ &\quad + R_{11}^\diamond \left(\sum_{i=1}^N \frac{1}{\sigma_i^2} \lambda_i \lambda_i' \right)^{-1} \left(\sum_{i=1}^N \frac{1}{\sigma_i^2} \lambda_i e_{it} \right) + O_p(N^{-1}) + O_p(T^{-1}).\end{aligned}$$

First notice that

$$\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} - R_{11}^\diamond V' R_{11}^{\diamond'} = (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond \widehat{\Delta R}_{11}^{\diamond'}) - (\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond V R_{11}^{\diamond'})' f_t^\diamond.$$

By (C.6), term $\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond \widehat{\Delta R}_{11}^{\diamond'}$ is negligible. In addition, as defined before, $V^\diamond$ is the leading term of $\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond V R_{11}^{\diamond'}$. Given this results, together with $\lambda_i^\diamond = R_{11}^\diamond \lambda_i$, $\varepsilon_t^\diamond = R_{11}^\diamond \varepsilon_t$, $\mathbf{v}_t^\diamond = \mathbf{v}_t$ and $R_{11}^{\diamond'} R_{11}^\diamond = I_{r_1}$, we have

$$\hat{f}_t^\diamond - f_t^\diamond = -V^{\diamond'} f_t^\diamond - W^{\diamond'} g_t + \left(\sum_{i=1}^N \frac{1}{\sigma_i^2} \lambda_i^\diamond \lambda_i^{\diamond'} \right)^{-1} \left(\sum_{i=1}^N \frac{1}{\sigma_i^2} \lambda_i^\diamond e_{it} \right) + O_p(N^{-1}) + O_p(T^{-1}).$$

We finally consider the asymptotic result of $\hat{\Phi}_k^\diamond - \Phi_k^\diamond$. Notice the derivation of (B.37) does not involve any special value of parameters. So it still holds in the present context. So we have

$$\hat{\Phi}_k^\diamond - \Phi_k^\diamond = \Phi_k^\diamond R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} - R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} \Phi_k^\diamond + R^{\diamond'-1} (\hat{\Phi}_k - \Phi_k) R^{\diamond'} + O_p(N^{-2}) + O_p(T^{-1}). \quad (\text{C.12})$$

Notice

$$R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} = R^\diamond \widehat{\Delta R}^{\diamond'} = \begin{bmatrix} R_{11}^\diamond \widehat{\Delta R}_{11}^{\diamond'} & 0 \\ 0 & 0 \end{bmatrix}.$$

By Theorem 4.5 together with the above result, we have

$$\begin{aligned}\hat{\Phi}_k^\diamond - \Phi_k^\diamond &= \Phi_k^\diamond \begin{bmatrix} \widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond V^{\diamond'} R_{11}^{\diamond'} & W^{\diamond'} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond V^{\diamond'} R_{11}^{\diamond'} & W^{\diamond'} \\ 0 & 0 \end{bmatrix} \Phi_k^\diamond \\ &\quad + \left(\sum_{t=\bar{K}}^T u_t^\diamond \psi_t^{\diamond'} \right) \left(\sum_{t=\bar{K}}^T \psi_t^\diamond \psi_t^{\diamond'} \right)^{-1} (i_k \otimes I_r) + O_p(N^{-1}) + O_p(T^{-1}).\end{aligned}$$

Let $B^\diamond$ is defined by

$$B^\diamond = \begin{bmatrix} V^\diamond & 0 \\ W^\diamond & 0 \end{bmatrix}.$$

By $\widehat{\Delta R}_{11}^\diamond R_{11}^{\diamond'} + R_{11}^\diamond V R_{11}^{\diamond'} = V^\diamond + O_p(T^{-1})$, we have

$$\hat{\Phi}_k^\diamond - \Phi_k^\diamond = \left(\sum_{t=\bar{K}}^T u_t^\diamond \psi_t^{\diamond'} \right) \left(\sum_{t=\bar{K}}^T \psi_t^\diamond \psi_t^{\diamond'} \right)^{-1} (i_k \otimes I_r) + \Phi_k^\diamond B^{\diamond'} - B^{\diamond'} \Phi_k^\diamond + O_p(N^{-1}) + O_p(T^{-1}).$$

This completes the whole proof. $\square$

Appendix D: The asymptotic results and their proofs under IRc

In this appendix, we provide the derivations for the asymptotic results under IRc, i.e., the detailed proofs of the theorems and corollaries in Section 4 under IRc. Similar to the previous section, we use the symbols without superscript to denote the parameters of IRa and the symbols with diamond to denote the parameters of IRc. We also use $R^\diamond$ to denote the rotation matrix, which transforms the parameters set (Λ, Γ, F) into $(\Lambda^\diamond, \Gamma^\diamond, F^\diamond)$, i.e.,

$$\begin{aligned} z_t &= [\Lambda^\diamond, \Gamma^\diamond] \begin{bmatrix} f_t^\diamond \\ g_t^\diamond \end{bmatrix} + e_t \\ &= [\Lambda, \Gamma] \begin{bmatrix} R_{11}^{\diamond'} & R_{21}^{\diamond'} \\ 0 & I_{r_2} \end{bmatrix} \begin{bmatrix} R_{11}^{\diamond'-1} & -R_{11}^{\diamond'-1} R_{21}^{\diamond'} \\ 0 & I_{r_2} \end{bmatrix} \begin{bmatrix} f_t \\ g_t \end{bmatrix} + e_t. \end{aligned}$$

Using the way in deriving (C.1), we have

$$\begin{aligned} \varepsilon_t^\diamond &= R_{11}^{\diamond'-1} \varepsilon_t - R_{11}^{\diamond'-1} R_{21}^{\diamond'} \mathbf{v}_t, \\ \mathbf{v}_t^\diamond &= \mathbf{v}_t. \end{aligned} \tag{D.1}$$

By $E(\varepsilon_t \mathbf{v}_t') = E(\varepsilon_t^\diamond \mathbf{v}_t^{\diamond'}) = 0$, we have

$$R_{21}^\diamond = 0. \tag{D.2}$$

By $\Lambda_1^\diamond = I_{r_1}$, we have

$$R_{11}^\diamond = \Lambda_1'^{-1}. \tag{D.3}$$

Accordingly, we have $\hat{R}_{21}^\diamond = 0$ and $\hat{R}_{11}^\diamond = \hat{\Lambda}_1'^{-1}$. Since we have already proved $\hat{\Lambda}_1 - \Lambda_1 \xrightarrow{p} 0$, then it follows $\hat{R}_{11}^\diamond \xrightarrow{p} R_{11}^\diamond$. We use $\widehat{\Delta R}_{11}^\diamond$ to denote $\hat{R}_{11}^\diamond - R_{11}^\diamond$. Apparently $\widehat{\Delta R}_{11}^\diamond \xrightarrow{p} 0$. Furthermore, Since $\hat{\Lambda}_1' = \Lambda_1' + O_p(T^{-1/2})$ by Proposition B.1, we have $\widehat{\Delta R}_{11}^\diamond = \hat{R}_{11}^\diamond - R_{11}^\diamond = -\hat{\Lambda}_1'^{-1}(\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1} = O_p(T^{-1/2})$. Now consider $\hat{\lambda}_i^\diamond - \lambda_i^\diamond$. Notice $\hat{\lambda}_i^\diamond = \hat{R}_{11}^\diamond \hat{\lambda}_i$ and $\lambda_i^\diamond = R_{11}^\diamond \lambda_i$. Then it follows

$$\hat{\lambda}_i^\diamond - \lambda_i^\diamond = \widehat{\Delta R}_{11}^\diamond \lambda_i + R_{11}^\diamond (\hat{\lambda}_i - \lambda_i) + \widehat{\Delta R}_{11}^\diamond (\hat{\lambda}_i - \lambda_i).$$

Notice $\widehat{\Delta R}_{11}^\diamond = O_p(T^{-1/2}) + O_p(N^{-1})$ and $\hat{\lambda}_i - \lambda_i = O_p(T^{-1/2}) + O_p(N^{-1})$. So the above result can be simplified as

$$\hat{\lambda}_i^\diamond - \lambda_i^\diamond = \widehat{\Delta R}_{11}^\diamond \lambda_i + R_{11}^\diamond (\hat{\lambda}_i - \lambda_i) + O_p(T^{-1}) + O_p(N^{-2}). \tag{D.4}$$

However, notice

$$\begin{aligned} \widehat{\Delta R}_{11}^\diamond &= -\hat{\Lambda}_1'^{-1}(\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1} \\ &= -\Lambda_1'^{-1}(\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1} - (\hat{\Lambda}_1'^{-1} - \Lambda_1'^{-1})(\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1} \\ &= -\Lambda_1'^{-1}(\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1} - \widehat{\Delta R}_{11}^\diamond (\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1}. \end{aligned}$$

The last term of right hand side of the above equality is $O_p(T^{-1}) + O_p(N^{-1})$. Then we have

$$\widehat{\Delta R}_{11}^\diamond = -\Lambda_1'^{-1}(\hat{\Lambda}_1 - \Lambda_1)' \Lambda_1'^{-1} + O_p(T^{-1}) + O_p(N^{-2}). \tag{D.5}$$

However, by Proposition B.1, we have

$$(\hat{\Lambda}_1 - \Lambda_1)' = V\Lambda_1' + \left(\frac{1}{T} \sum_{t=1}^T \phi_t \phi_t'\right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t \xi_t'\right) + O_p(T^{-1}) + O_p(N^{-1}), \quad (\text{D.6})$$

where $\xi_t = (e_{1t}, e_{2t}, \dots, e_{r_1 t})'$. Substituting (D.5) and (D.6) into (D.4), together with Proposition B.1 and $R_{11}^\diamond = \Lambda_1'^{-1}$ and $\lambda_i^\diamond = R_{11}^\diamond \lambda_i$, we have

$$\hat{\lambda}_i^\diamond - \lambda_i^\diamond = V^\diamond \lambda_i^\diamond + \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond \phi_t^{\diamond'}\right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond e_{it}\right) + O_p(T^{-1}) + O_p(N^{-1}), \quad (\text{D.7})$$

where $V^\diamond = -\mathbf{\Delta}_{\phi\phi}^{\diamond-1} \left(\frac{1}{T} \sum_{t=1}^T \phi_t^\diamond \xi_t'\right)$ and $\phi_t^\diamond = f_t^\diamond - \mathbf{\Delta}_{fg}^\diamond \mathbf{\Delta}_{gg}^{-1} g_t$.

We proceed to consider $\hat{\gamma}_i^\diamond - \gamma_i^\diamond$. Notice $\hat{\gamma}_i^\diamond = \hat{\gamma}_i$ and $\gamma_i^\diamond = \gamma$. Thus, by (B.31),

$$\begin{aligned} \hat{\gamma}_i^\diamond - \gamma_i^\diamond &= \hat{\gamma}_i - \gamma_i = \left[\frac{1}{T} \sum_{t=1}^T g_t g_t'\right]^{-1} \left[\frac{1}{T} \sum_{t=1}^T (g_t - \Xi_t) e_{it}\right] + W \lambda_i + O_p(N^{-1}) + O_p(T^{-1}) \\ &= \left(\frac{1}{T} \sum_{t=1}^T g_t g_t'\right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T (g_t - \Xi_t^\diamond) e_{it}\right) + W^\diamond \lambda_i^\diamond + O_p(N^{-1}) + O_p(T^{-1}), \end{aligned} \quad (\text{D.8})$$

where $\Xi_t^\diamond = (\sum_{t=1}^T g_t f_t^{\diamond'}) (\sum_{t=1}^T \phi_t^\diamond \phi_t^{\diamond'})^{-1} \phi_t^\diamond$ and $W^\diamond = (\sum_{t=1}^T \mathbf{v}_t^\diamond \mathbf{v}_t^{\diamond'})^{-1} (\sum_{t=1}^T \mathbf{v}_t^\diamond \varepsilon_t^{\diamond'})$. By the similar arguments in Section B, we have the following alternative expression:

$$\hat{\gamma}_i^\diamond - \gamma_i^\diamond = \left[\sum_{t=1}^T \eta_t^\diamond \eta_t^{\diamond'}\right]^{-1} \left[\sum_{t=1}^T \eta_t^\diamond e_{it}\right] + W^\diamond \lambda_i^\diamond + O_p(N^{-1}) + O_p(T^{-1}).$$

We further consider $\hat{f}_t^\diamond - f_t^\diamond$. Notice $\hat{f}_t^\diamond = \hat{R}_{11}^{\diamond'-1} \hat{f}_t$ and $f_t^\diamond = R_{11}^{\diamond'-1} f_t$. Given these results, together with $\hat{R}_{11}^{\diamond'-1} = \hat{\Lambda}_1$ and $R_{11}^{\diamond'-1} = \Lambda_1$, we have

$$\hat{f}_t^\diamond - f_t^\diamond = (\hat{\Lambda}_1 - \Lambda_1) f_t + \Lambda_1 (\hat{f}_t - f_t) + (\hat{\Lambda}_1 - \Lambda_1) (\hat{f}_t - f_t).$$

The last term is $O_p(T^{-1}) + O_p(N^{-2})$. By (D.6) and Proposition B.1, together with $\Lambda_1^{\diamond'-1} \lambda_i = R_{11}^\diamond \lambda_i = \lambda_i^\diamond$, we have

$$\hat{f}_t^\diamond - f_t^\diamond = \left(\frac{1}{N} \sum_{i=1}^N \frac{1}{\sigma_i^2} \lambda_i^\diamond \lambda_i^{\diamond'}\right)^{-1} \left(\frac{1}{N} \sum_{i=1}^N \frac{1}{\sigma_i^2} \lambda_i^\diamond e_{it}\right) - V^{\diamond'} f_t^\diamond - W^{\diamond'} g_t + O_p(N^{-1}) + O_p(T^{-1}). \quad (\text{D.9})$$

We finally consider $\hat{\Phi}_k^\diamond - \Phi_k^\diamond$. Using the way to derive (B.37), we have

$$\hat{\Phi}_k^\diamond - \Phi_k^\diamond = \Phi_k^\diamond R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} - R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} \Phi_k^\diamond + R^{\diamond'-1} (\hat{\Phi}_k - \Phi_k) R^{\diamond'} + O_p(N^{-2}) + O_p(T^{-1}). \quad (\text{D.10})$$

By Theorem 4.5 we have

$$\begin{aligned}\hat{\Phi}_k^\diamond - \Phi_k^\diamond &= \Phi_k^\diamond (R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} + R^{\diamond'-1} B' R^{\diamond'}) - (R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} + R^{\diamond'-1} B' R^{\diamond'}) \Phi_k^\diamond \\ &\quad + \left(\sum_{t=\bar{K}}^T u_t^\diamond \psi_t^{\diamond'} \right) \left(\sum_{t=\bar{K}}^T \psi_t^\diamond \psi_t^{\diamond'} \right)^{-1} (i_k \otimes I_r) + O_p(N^{-1}) + O_p(T^{-1}).\end{aligned}$$

However,

$$R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} = \begin{bmatrix} -R_{11}^{\diamond'-1} V' R_{11}^{\diamond'} - \left(\frac{1}{T} \sum_{t=1}^T \xi_t \phi_t' \right) \left(\frac{1}{T} \sum_{t=1}^T \phi_t \phi_t' \right)^{-1} R_{11}^{\diamond'} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$R^{\diamond'-1} B' R^{\diamond'} = \begin{bmatrix} R_{11}^{\diamond'-1} V' R_{11}^{\diamond'} & R_{11}^{\diamond'-1} W' \\ 0 & 0 \end{bmatrix}.$$

Given the above two results, together with $\phi_t^\diamond = R_{11}^{\diamond'-1} \phi_t$, $\varepsilon_t^\diamond = R_{11}^{\diamond'-1} \varepsilon_t$ and the definitions of $V^\diamond$ and $W^\diamond$, we have

$$R^{\diamond'-1} \widehat{\Delta R}^{\diamond'} + R^{\diamond'-1} B' R^{\diamond'} = B^\diamond,$$

where

$$B^\diamond = \begin{bmatrix} V^\diamond & 0 \\ W^\diamond & 0 \end{bmatrix}.$$

Thus,

$$\hat{\Phi}_k^\diamond - \Phi_k^\diamond = \left(\sum_{t=\bar{K}}^T u_t^\diamond \psi_t^{\diamond'} \right) \left(\sum_{t=\bar{K}}^T \psi_t^\diamond \psi_t^{\diamond'} \right)^{-1} (i_k \otimes I_r) + \Phi_k^\diamond B^{\diamond'} - B^{\diamond'} \Phi_k^\diamond + O_p(N^{-1}) + O_p(T^{-1}).$$

This completes the whole proof. $\square$

Appendix E: Proofs of the theorems in Section 5

In this section, we give detailed proofs of Theorems 5.2, 5.3 and 5.4. The proof of Theorem 5.1 is almost the same as that of the equation (11.7.4) in Hamilton (1994). Hence this proof is omitted. We first give a preliminary result, which is closely related to Theorem 5.2.

Theorem E.1 *Under Assumptions A-D, when $N, T \rightarrow \infty$ and $\sqrt{T}/N \rightarrow 0$, under IRa, IRb or IRc, we have*

$$\sqrt{T}(\tilde{\Omega}_{vv} - \Omega_{vv}) = \frac{1}{\sqrt{T}} \sum_{t=\bar{K}}^T (\mathbf{v}_t \mathbf{v}_t' - \Omega_{vv}) + o_p(1).$$

In addition, under IRc, we also have

$$\sqrt{T}(\hat{\Omega}_{\varepsilon\varepsilon} - \Omega_{\varepsilon\varepsilon}) = \frac{1}{\sqrt{T}} \sum_{t=\bar{K}}^T (\varepsilon_t \varepsilon_t' - \Omega_{\varepsilon\varepsilon}) + \Delta_{\phi\phi}^{-1} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^T \phi_t \xi_t' \right] + \left[\frac{1}{\sqrt{T}} \sum_{t=1}^T \xi_t \phi_t' \right] \Delta_{\phi\phi}^{-1} + o_p(1).$$

PROOF OF THEOREM E.1. Notice that $\mathbf{v}_t^* = \mathbf{v}_t$ and $\Omega_{\mathbf{v}\mathbf{v}}^* = \Omega_{\mathbf{v}\mathbf{v}}$. Then the first asymptotic representation is a direct result from (B.15). We then consider the second one. In this proof, we use Λ, Γ and F to denote the parameters under IRb and $\Lambda^\diamond, \Gamma^\diamond$ and $F^\diamond$ to denote the parameters under IRc. Notice that the rotation matrix R which transform the parameters set under IRb to the ones under IRc is $[\Lambda_1^{-1'}, 0; 0, I_{r_2}]$, i.e., $\lambda_i^\diamond = \Lambda_1^{-1'} \lambda_i$, $f_t^\diamond = \Lambda_1 f_t$ and $\varepsilon_t^\diamond = \Lambda_1 \varepsilon_t$. Given this result, we have $\Omega_{\varepsilon\varepsilon}^\diamond = \Lambda_1 \Lambda_1'$. Accordingly, we have $\hat{\Omega}_{\varepsilon\varepsilon}^\diamond = \hat{\Lambda}_1 \hat{\Lambda}_1'$. Thus,

$$\hat{\Omega}_{\varepsilon\varepsilon}^\diamond - \Omega_{\varepsilon\varepsilon}^\diamond = (\hat{\Lambda}_1 - \Lambda_1) \Lambda_1' + \Lambda_1 (\hat{\Lambda}_1 - \Lambda_1)' + (\hat{\Lambda}_1 - \Lambda_1)(\hat{\Lambda}_1 - \Lambda_1)'. \quad (\text{E.1})$$

By (C.11), we have

$$(\hat{\Lambda}_1 - \Lambda_1)' = V \Lambda_1' + \Delta_{\phi\phi}^{-1} \frac{1}{T} \sum_{t=1}^T \phi_t \xi_t' + O_p(T^{-1}) + O_p(N^{-1}).$$

Substituting the preceding equation into (E.1),

$$\hat{\Omega}_{\varepsilon\varepsilon}^\diamond - \Omega_{\varepsilon\varepsilon}^\diamond = \Lambda_1 (V + V') \Lambda_1' + \frac{1}{T} \sum_{t=1}^T \xi_t \phi_t' \Delta_{\phi\phi}^{-1} \Lambda_1' + \Lambda_1 \Delta_{\phi\phi}^{-1} \frac{1}{T} \sum_{t=1}^T \phi_t \xi_t' + O_p(T^{-1}) + O_p(N^{-1}).$$

However, as shown in Section C,

$$V + V' = \frac{1}{T} \sum_{t=1}^T (\varepsilon_t \varepsilon_t' - I_{r_1}) + O_p(T^{-1}).$$

Given this result, together with $\phi_t^\diamond = \Lambda_1 \phi_t$ by the definition of ϕ_t , we have

$$\hat{\Omega}_{\varepsilon\varepsilon}^\diamond - \Omega_{\varepsilon\varepsilon}^\diamond = \frac{1}{T} \sum_{t=1}^T (\varepsilon_t^\diamond \varepsilon_t^{\diamond'} - \Omega_{\varepsilon\varepsilon}^\diamond) + \frac{1}{T} \sum_{t=1}^T \xi_t \phi_t^{\diamond'} \Delta_{\phi\phi}^{\diamond-1} + \Delta_{\phi\phi}^{\diamond-1} \frac{1}{T} \sum_{t=1}^T \phi_t^\diamond \xi_t' + O_p(T^{-1}) + O_p(N^{-1}).$$

This completes the proof of Theorem E.1. $\square$

Based on the results in Theorem E.1, we can directly derive the limiting distribution results in Theorem 5.2. Details are omitted.

PROOF OF THEOREM 5.3. We first consider the impulse response function under IRa and IRb. Let $\mathcal{P}$ be the solution of the Cholesky decomposition of Ω . Under IRa and IRb, by the special structure of Ω , it is easy to see

$$\mathcal{P} = \begin{bmatrix} I_{r_1} & 0 \\ 0 & \mathcal{P}^* \end{bmatrix}.$$

with $\mathcal{P}^* \mathcal{P}^{*'} = \Omega_{\mathbf{v}\mathbf{v}}$. The impulse response function of h_t subject to one unit orthogonal innovation is

$$h_t = \mathcal{P} \omega_t + \Psi_1 \mathcal{P} \omega_{t-1} + \Psi_2 \mathcal{P} \omega_{t-2} + \cdots = \mathbb{C}_0 \omega_t + \mathbb{C}_1 \omega_{t-1} + \mathbb{C}_2 \omega_{t-2} + \cdots$$

with $\mathbb{C}_s = \Psi_s \mathcal{P}$. Notice that $\hat{\mathbb{C}}_s = \hat{\Psi}_s \hat{\mathcal{P}}$. To deliver the limiting distribution of $\hat{\mathbb{C}} - \mathbb{C}$, we need to derive the limiting distribution of $\hat{\mathcal{P}} - \mathcal{P}$. By the structure of $\mathcal{P}$, it suffices to consider $\hat{\mathcal{P}}^* - \mathcal{P}^*$. By $\Omega_{vv} = \mathcal{P}^* \mathcal{P}^{*'} and $\tilde{\Omega}_{vv} = \hat{\mathcal{P}}^* \hat{\mathcal{P}}^{*'} , we have$$

$$\tilde{\Omega}_{vv} - \Omega_{vv} = \hat{\mathcal{P}}^* \hat{\mathcal{P}}^{*'} - \mathcal{P}^* \mathcal{P}^{*'} = \mathcal{P}^* (\hat{\mathcal{P}}^* - \mathcal{P}^*)' + (\hat{\mathcal{P}}^* - \mathcal{P}^*) \mathcal{P}^{*'} + (\hat{\mathcal{P}}^* - \mathcal{P}^*) (\hat{\mathcal{P}}^* - \mathcal{P}^*)'$$

The last term is of smaller order and hence negligible. Therefore, we have (neglecting the last term)

$$\begin{aligned} \text{vec}(\tilde{\Omega}_{vv} - \Omega_{vv}) &= (\mathcal{P}^* \otimes I_{r_2}) \text{vec}(\hat{\mathcal{P}}^* - \mathcal{P}^*) + (I_{r_2} \otimes \mathcal{P}^*) \text{vec}[(\hat{\mathcal{P}}^* - \mathcal{P}^*)'] \\ &= [(\mathcal{P}^* \otimes I_{r_2}) + (I_{r_2} \otimes \mathcal{P}^*) K_{r_2}] \text{vec}(\hat{\mathcal{P}}^* - \mathcal{P}^*) \end{aligned}$$

where K_{r_2} is the r_2 -dimensional commutation matrix. Let $D_{r_2}^*$ be the duplication matrix such that $D_{r_2}^* \text{vech}(M) = \text{vec}(M)$ for any lower triangular matrix M , where $\text{vech}(\cdot)$ is the half-vectorization operator that stacks the elements on and below the diagonal into a vector. Then the preceding equation gives

$$\text{vec}(\tilde{\Omega}_{vv} - \Omega_{vv}) = [(\mathcal{P}^* \otimes I_{r_2}) + (I_{r_2} \otimes \mathcal{P}^*) K_{r_2}] D_{r_2}^* \text{vech}(\hat{\mathcal{P}}^* - \mathcal{P}^*),$$

which leads to

$$\text{vech}(\hat{\mathcal{P}}^* - \mathcal{P}^*) = \mathbb{W}_2 \text{vec}(\tilde{\Omega}_{vv} - \Omega_{vv}) = \mathbb{W}_2 D_{r_2}^* \text{vech}(\tilde{\Omega}_{vv} - \Omega_{vv}).$$

with $\mathbb{W}_2 = [D_{r_2}^{*'} \mathbf{W}_2' \mathbf{W}_2 D_{r_2}^*]^{-1} D_{r_2}^{*'} \mathbf{W}_2'$ and $\mathbf{W}_2 = (\mathcal{P}^* \otimes I_{r_2}) + (I_{r_2} \otimes \mathcal{P}^*) K_{r_2}$. By Theorem E.1 on $\hat{\Omega}_{vv} - \Omega_{vv}$, we have $\sqrt{T} \text{vech}(\hat{\mathcal{P}}^* - \mathcal{P}^*) \xrightarrow{d} \mathcal{N}(0, \mathbb{J}_2)$, with $\mathbb{J}_2$ equal to

$$\mathbb{J}_2 = \mathbb{W}_2 S_{r_2} \left[2\Omega_{vv} \otimes \Omega_{vv} + (\Omega_{vv}^{1/2} \otimes \Omega_{vv}^{1/2}) (\mathcal{D}_{r_2} \mathcal{W}_v \mathcal{D}_{r_2}') (\Omega_{vv}^{1/2} \otimes \Omega_{vv}^{1/2}) \right] S_{r_2}' \mathbb{W}_2', \quad (\text{E.2})$$

where we use the fact that $D_{r_2} D_{r_2}^+ = S_{r_2}$.

Now consider the asymptotic representation of $\hat{\mathbb{C}}_s - \mathbb{C}_s$. By definition, we have

$$\hat{\mathbb{C}}_s - \mathbb{C}_s = \hat{\Psi}_s \hat{\mathcal{P}} - \Psi_s \mathcal{P} = (\hat{\Psi}_s - \Psi_s) \mathcal{P} + \Psi_s (\hat{\mathcal{P}} - \mathcal{P}) + (\hat{\Psi}_s - \Psi_s) (\hat{\mathcal{P}} - \mathcal{P}).$$

Again the last term is of smaller order and hence negligible. Taking vectorization operation on both sides, we have (neglecting the last term)

$$\text{vec}(\hat{\mathbb{C}}_s - \mathbb{C}_s) = (\mathcal{P}' \otimes I_r) \text{vec}(\hat{\Psi}_s - \Psi_s) + (I_r \otimes \Psi_s) \text{vec}(\hat{\mathcal{P}} - \mathcal{P})$$

Let $\mathbb{D}_7$ be the matrix such that $\text{vec}(M) = \mathbb{D}_7 \text{vech}(M_1)$ with $M_{r \times r} = [0, 0; 0, M_1]$ where M_1 is an arbitrary $r_2 \times r_2$ lower triangular matrix. Then the above result gives

$$\text{vec}(\hat{\mathbb{C}}_s - \mathbb{C}_s) = (\mathcal{P}' \otimes I_r) K_r \text{vec}(\hat{\Psi}_s' - \Psi_s') + (I_r \otimes \Psi_s) \mathbb{D}_7 \text{vech}(\hat{\mathcal{P}}^* - \mathcal{P}^*)$$

It is easy to check that the above two expressions on the right hand side are asymptotically independent. Given this result, we have

$$\sqrt{T} \text{vec}(\hat{\mathbb{C}}_s - \mathbb{C}_s) \xrightarrow{d} \mathcal{N}\left(0, (\mathcal{P}' \otimes I_r) K_r \Upsilon_s \mathbb{J}_1 \Upsilon_s' K_r' (\mathcal{P} \otimes I_r) + (I_r \otimes \Psi_s) \mathbb{D}_7 \mathbb{J}_2 \mathbb{D}_7' (I_r \otimes \Psi_s')\right)$$

with $\mathbb{J}_1 = \Omega \otimes [E(\psi_t \psi_t')]^{-1} + \mathbb{D}_{10} J \mathbb{D}'_{10}$ and $\mathbb{J}_2$ defined in (E.2).

Now we consider the asymptotic representation under IRc. Let $\mathcal{P}$ be the solution of the Cholesky decomposition of Ω . Under IRc, $\mathcal{P}$ is of the form

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1^* & 0 \\ 0 & \mathcal{P}_2^* \end{bmatrix}.$$

with $\mathcal{P}_1^* \mathcal{P}_1^{*'} = \Omega_{\varepsilon\varepsilon}$ and $\mathcal{P}_2^* \mathcal{P}_2^{*'} = \Omega_{vv}$. $\hat{\mathcal{P}}_2^* - \mathcal{P}_2^*$ is already given. We only focus on $\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*$. By $\hat{\Omega}_{\varepsilon\varepsilon} = \hat{\mathcal{P}}_1^* \hat{\mathcal{P}}_1^{*'}$ and $\mathcal{P}_1^* \mathcal{P}_1^{*'} = \Omega_{\varepsilon\varepsilon}$, we have

$$\hat{\Omega}_{\varepsilon\varepsilon} - \Omega_{\varepsilon\varepsilon} = (\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) \mathcal{P}_1^{*'} + \mathcal{P}_1^* (\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*)' + (\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) (\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*)'.$$

Neglecting the last term and taking the vectorization operation on both sides, we get

$$\begin{aligned} \text{vec}(\hat{\Omega}_{\varepsilon\varepsilon} - \Omega_{\varepsilon\varepsilon}) &= [(\mathcal{P}_1^* \otimes I_{r_1}) + (I_{r_1} \otimes \mathcal{P}_1^*) K_{r_1}] \text{vec}(\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) \\ &= [(\mathcal{P}_1^* \otimes I_{r_1}) + (I_{r_1} \otimes \mathcal{P}_1^*) K_{r_1}] D_{r_1}^* \text{vech}(\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*). \end{aligned}$$

So we have

$$\text{vech}(\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) = \mathbb{W}_1 \text{vec}(\hat{\Omega}_{\varepsilon\varepsilon} - \Omega_{\varepsilon\varepsilon}) = \mathbb{W}_1 D_{r_1} \text{vech}(\hat{\Omega}_{\varepsilon\varepsilon} - \Omega_{\varepsilon\varepsilon})$$

with $\mathbb{W}_1 = (D_{r_1}^{*'} \mathbf{W}_1' \mathbf{W}_1 D_{r_1}^*)^{-1} D_{r_1}^{*'} \mathbf{W}_1'$ with $\mathbf{W}_1 = (\mathcal{P}_1^* \otimes I_{r_1}) + (I_{r_1} \otimes \mathcal{P}_1^*) K_{r_1}$. Given this result, we have $\text{vech}(\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) \xrightarrow{d} \mathcal{N}(0, \mathbb{J}_4)$ with

$$\mathbb{J}_4 = \mathbb{W}_1 S_{r_1} \left[2\Omega_{\varepsilon\varepsilon} \otimes \Omega_{\varepsilon\varepsilon} + (\Omega_{\varepsilon\varepsilon}^{1/2} \otimes \Omega_{\varepsilon\varepsilon}^{1/2}) (\mathcal{D}_{r_1} \mathcal{W}_\varepsilon \mathcal{D}_{r_1}') (\Omega_{\varepsilon\varepsilon}^{1/2} \otimes \Omega_{\varepsilon\varepsilon}^{1/2}) + 4S_{r_1} (\Sigma_{\xi\xi} \otimes \Delta_{\phi\phi}^{-1}) S_{r_1}' \right] S_{r_1}' \mathbb{W}_1'.$$

Let $\mathbb{D}_8$ be defined as $\text{vec}(M) = \mathbb{D}_8 \text{vech}(M_1) + \mathbb{D}_7 \text{vech}(M_2)$ for any M such that $M_{r \times r} = [M_1, 0; 0, M_2]$, where M_1 is $r_1 \times r_1$ and M_2 is $r_2 \times r_2$ and both are lower-triangular matrices. By these two definitions, we have

$$\text{vec}(\hat{\mathcal{P}} - \mathcal{P}) = \mathbb{D}_8 \text{vech}(\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) + \mathbb{D}_7 \text{vech}(\hat{\mathcal{P}}_2^* - \mathcal{P}_2^*),$$

implying

$$\text{vec}(\hat{\mathcal{P}} - \mathcal{P}) \xrightarrow{d} \mathcal{N}(0, \mathbb{D}_8 \mathbb{J}_4 \mathbb{D}_8' + \mathbb{D}_7 \mathbb{J}_2 \mathbb{D}_7').$$

Notice

$$\text{vec}(\hat{\mathbb{C}}_s - \mathbb{C}_s) = (\mathcal{P}' \otimes I_r) K_r \text{vec}(\hat{\Psi}'_s - \Psi'_s) + (I_r \otimes \Psi_s) \text{vec}(\hat{\mathcal{P}} - \mathcal{P}).$$

It is easy to check that the above two expressions on the right hand side are asymptotically independent. Given this result, we have

$$\sqrt{T} \text{vec}(\hat{\mathbb{C}}_s - \mathbb{C}_s) \xrightarrow{d} \mathcal{N}\left(0, (\mathcal{P}' \otimes I_r) K_r \Upsilon_s \mathbb{J}_1 \Upsilon_s' K_r' (\mathcal{P} \otimes I_r) + (I_r \otimes \Psi_s) \mathbb{J}_3 (I_r \otimes \Psi_s')\right)$$

where

$$\mathbb{J}_3 = \mathbb{D}_8 \mathbb{J}_4 \mathbb{D}_8' + \mathbb{D}_7 \mathbb{J}_2 \mathbb{D}_7'.$$

This completes the proof of Theorem 5.3. $\square$

PROOF OF THEOREM 5.4. We consider

$$\widehat{\frac{\partial z_{i,t+k}}{\partial \omega_t}} - \frac{\partial z_{i,t+k}}{\partial \omega_t} = (\hat{\mathbb{C}}_k - \mathbb{C}_k)' \begin{bmatrix} \lambda_i \\ \gamma_i \end{bmatrix} + \mathbb{C}'_k \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}$$

which has two components, arising from estimating the loadings (λ_i, γ_i) and the MA(∞) coefficients $\mathbb{C}_k$.

We can rewrite the above equation as

$$\begin{aligned} \sqrt{T} \left(\widehat{\frac{\partial z_{i,t+k}}{\partial \omega_t}} - \frac{\partial z_{i,t+k}}{\partial \omega_t} \right) &= (\lambda'_i, \gamma'_i) \otimes I_r \cdot K_r \cdot \sqrt{T} \text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k) + \mathbb{C}'_k \cdot \sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix} \\ &= m_1 + m_2, \text{ say} \end{aligned} \tag{E.3}$$

implying

$$\sqrt{T} \left(\widehat{\frac{\partial z_{i,t+k}}{\partial \omega_t}} - \frac{\partial z_{i,t+k}}{\partial \omega_t} \right) \xrightarrow{d} \mathcal{N} \left(0, \text{Avar} \left(\widehat{\frac{\partial z_{i,t+k}}{\partial \omega_t}} \right) \right)$$

where $\text{Avar} \left(\widehat{\frac{\partial z_{i,t+k}}{\partial \omega_t}} \right) = \text{Var}(m_1) + \text{Var}(m_2) + \text{Cov}(m_1, m_2) + \text{Cov}(m_2, m_1)$ with $\text{Var}(m_j)$ being the variance of m_j , for $j = 1, 2$; $\text{Cov}(m_j, m_{j'})$ being the covariance between m_j and $m_{j'}$, for $j, j' = 1, 2$ and $j \neq j'$. $\text{Var}(m_1)$ and $\text{Var}(m_2)$ are easy to obtain using the results in Corollaries 4.1 and 4.2, so we will focus on $\text{Cov}(m_j, m_{j'})$ in the following.

Notice that for m_1 ,

$$\sqrt{T} \text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k) = (\mathcal{P}' \otimes I_r) K_r \cdot \sqrt{T} \text{vec}(\hat{\Psi}'_s - \Psi'_s) + (I_r \otimes \Psi_s) \cdot \sqrt{T} \text{vec}(\hat{\mathcal{P}} - \mathcal{P})$$

where the part $\sqrt{T} \text{vec}(\hat{\mathcal{P}} - \mathcal{P})$ is different under different IRs. From the proof of Theorem 5.3, we know that under IRa and IRb,

$$\begin{aligned} \sqrt{T} \text{vec}(\hat{\mathcal{P}} - \mathcal{P}) &= \mathbb{D}_7 \sqrt{T} \text{vech}(\hat{\mathcal{P}}_2^* - \mathcal{P}_2^*) \\ &= \mathbb{D}_7 \mathbb{W}_2 \cdot \sqrt{T} \text{vec}(\tilde{\Omega}_{vv} - \Omega_{vv}) \\ &= \mathbb{D}_7 \mathbb{W}_2 \cdot \frac{1}{\sqrt{T}} \sum_{t=\bar{K}}^T (v_t v'_t - \Omega_{vv}) + o_p(1) \end{aligned} \tag{E.4}$$

and under IRc,

$$\begin{aligned} \sqrt{T} \text{vec}(\hat{\mathcal{P}} - \mathcal{P}) &= \mathbb{D}_8 \sqrt{T} \text{vech}(\hat{\mathcal{P}}_1^* - \mathcal{P}_1^*) + \mathbb{D}_7 \sqrt{T} \text{vech}(\hat{\mathcal{P}}_2^* - \mathcal{P}_2^*) \\ &= \mathbb{D}_8 \mathbb{W}_1 \cdot \sqrt{T} \text{vec}(\hat{\Omega}_{\varepsilon\varepsilon} - \Omega_{\varepsilon\varepsilon}) + \mathbb{D}_7 \mathbb{W}_2 \cdot \sqrt{T} \text{vec}(\tilde{\Omega}_{vv} - \Omega_{vv}) \\ &= \mathbb{D}_8 \mathbb{W}_1 \cdot \frac{1}{\sqrt{T}} \sum_{t=\bar{K}}^T (\varepsilon_t \varepsilon'_t - \Omega_{\varepsilon\varepsilon}) + \mathbb{D}_7 \mathbb{W}_2 \cdot \frac{1}{\sqrt{T}} \sum_{t=\bar{K}}^T (v_t v'_t - \Omega_{vv}) + o_p(1) \end{aligned} \tag{E.5}$$

The part $\sqrt{T}\text{vec}(\hat{\Psi}'_s - \Psi'_s)$ has a unified expression under all IRs,

$$\begin{aligned}
\sqrt{T}\text{vec}(\hat{\Psi}'_s - \Psi'_s) &= \Upsilon_k \cdot \sqrt{T}\text{vec}(\hat{\Phi}' - \Phi') \\
&= \left[I_r \otimes \left(\frac{1}{T} \sum_{t=1}^T \psi_t \psi_t' \right) \right]^{-1} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^T (u_t \otimes \psi_t) \right] + \mathbb{D}_{10} \sqrt{T}\text{vec}(B) + o_p(1) \\
&= i_1 + i_2 + o_p(1), \text{ say}
\end{aligned} \tag{E.6}$$

where $B = [V, 0; W, 0]$.

Also notice that for m_2 ,

$$\begin{aligned}
\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix} &= \sqrt{T} B (\lambda'_i, \gamma'_i)' + \left[\Delta_{\phi\phi}^{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \phi_t e_{it} \right)', \Delta_{\eta\eta}^{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \eta_t e_{it} \right)' \right]' + o_p(1) \\
&= (\lambda'_i, \gamma'_i) \otimes I_r \cdot \sqrt{T}\text{vec}(B) + \left[\Delta_{\phi\phi}^{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \phi_t e_{it} \right)', \Delta_{\eta\eta}^{-1} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T \eta_t e_{it} \right)' \right]' + o_p(1) \\
&= j_1 + j_2 + o_p(1), \text{ say}
\end{aligned} \tag{E.7}$$

Now we consider the covariance between $\sqrt{T}\text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k))$ and $\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}$. Notice that under all IRs, the part $\sqrt{T}\text{vec}(\hat{\mathcal{P}} - \mathcal{P})$ is asymptotically independent with V , W and j_2 , and hence asymptotically independent with $\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}$ under all IRs. Meanwhile, for the part $\sqrt{T}\text{vec}(\hat{\Psi}'_s - \Psi'_s)$, the first component i_1 is asymptotically independent with V , W and j_2 , and hence asymptotically independent with $\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}$ under all IRs. But the second component i_2 shares the same B as in j_1 , and then non-zero covariance arises. So the total covariance between $\sqrt{T}\text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k))$ and $\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}$ comes from the covariance between i_2 and j_1 . By calculation, we have

$$\text{Cov} \left(\sqrt{T}\text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k)), \sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix} \right) = (\mathcal{P}' \otimes I_r) K_r \Upsilon_k \mathbb{D}_{10} J [(\lambda'_i, \gamma'_i)' \otimes I_r]$$

where J is the limiting variance of $\sqrt{T}\text{vec}(B)$, defined in Corollary 4.4. Similarly, we have

$$\text{Cov} \left(\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}, \sqrt{T}\text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k)) \right) = [(\lambda'_i, \gamma'_i) \otimes I_r] J \mathbb{D}'_{10} \Upsilon'_k K'_r (\mathcal{P} \otimes I_r)$$

Hence,

$$\widehat{\text{Avar}} \left(\frac{\partial z_{i,t+k}}{\partial \omega_t} \right) = [(\lambda'_i, \gamma'_i) \otimes I_r] K_r \cdot \text{Avar}(\text{vec}(\hat{\mathbb{C}}_k)) \cdot K'_r \cdot (\lambda'_i, \gamma'_i)' \otimes I_r + \mathbb{C}'_k \cdot \text{Avar}(\hat{\lambda}'_i, \hat{\gamma}'_i) \cdot \mathbb{C}_k$$

$$\begin{aligned}
& + [(\lambda'_i, \gamma'_i) \otimes I_r] K_r \cdot (\mathcal{P}' \otimes I_r) K_r \Upsilon_k \mathbb{D}_{10} J[(\lambda'_i, \gamma'_i)' \otimes I_r] \cdot \mathbb{C}_k \\
& + \mathbb{C}'_k \cdot [(\lambda'_i, \gamma'_i) \otimes I_r] J \mathbb{D}'_{10} \Upsilon'_k K'_r (\mathcal{P} \otimes I_r) \cdot K'_r [(\lambda'_i, \gamma'_i)' \otimes I_r] \\
& = [(\lambda'_i, \gamma'_i) \otimes K_r] \cdot \text{Avar}(\text{vec}(\hat{\mathbb{C}}_k)) \cdot [(\lambda'_i, \gamma'_i)' \otimes K'_r] + \mathbb{C}'_k \cdot \text{Avar}(\hat{\lambda}'_i, \hat{\gamma}'_i) \cdot \mathbb{C}_k \\
& + [(\lambda'_i, \gamma'_i) \otimes K_r] \cdot (\mathcal{P}' \otimes I_r) K_r \Upsilon_k \mathbb{D}_{10} J[(\lambda'_i, \gamma'_i)' \otimes \mathbb{C}_k] \\
& + [(\lambda'_i, \gamma'_i) \otimes \mathbb{C}'_k] J \mathbb{D}'_{10} \Upsilon'_k K'_r (\mathcal{P} \otimes I_r) \cdot [(\lambda'_i, \gamma'_i)' \otimes K'_r] \tag{E.8}
\end{aligned}$$

where the last two terms on the right hand side capture the covariance between $\sqrt{T} \text{vec}((\hat{\mathbb{C}}_k - \mathbb{C}_k))$ and $\sqrt{T} \begin{bmatrix} \hat{\lambda}_i - \lambda_i \\ \hat{\gamma}_i - \gamma_i \end{bmatrix}$; $\text{Avar}(\text{vec}(\hat{\mathbb{C}}_k))$ is the asymptotic variance of $\text{vec}(\hat{\mathbb{C}}_k)$, as defined in Theorem 5.3. In addition, $\text{Avar}(\hat{\lambda}'_i, \hat{\gamma}'_i) = \text{diag}(\text{Avar}(\hat{\lambda}_i), \text{Avar}(\hat{\gamma}_i))$, where $\text{Avar}(\hat{\lambda}_i)$ and $\text{Avar}(\hat{\gamma}_i)$ are the asymptotic variances of $\hat{\lambda}_i$ and $\hat{\gamma}_i$ respectively. Details of all these three asymptotic variances and the notations $K_r, \mathcal{P}, \Upsilon_k, \mathbb{J}, \mathbb{D}_{10}$ in (E.8) are given in Theorem 5.4.

This completes the proof of Theorem 5.4. $\square$