**Online Supplement for “Dynamic Multi-Technology
Production-Inventory Problem with Emissions Trading”**
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Proof of Lemma 1

For any $(q, r) \in \Theta$, it is easy to verify that the linear programming problem in equation (5) is feasible, and does not have the unbound optimal value. It is clear that the rank of the coefficient matrix for (5) is 2. In other words, each basic feasible solution has no more than 2 non-zero elements. Therefore, it is optimal to produce with no more than two technologies for any $(q, r) \in \Theta$.

Proof of Lemma 2

From the discussion in subsection 4.1 and the algorithm that determines the effective technology set S , we know that if $r/\mu_{i-1} \leq q \leq r/\mu_i$ ($i=2, \dots, m$), then it is optimal to produce with technologies $i-1$ and i in S . The production quantities of technologies $i-1$ and i are $q_{i-1} = (r - \mu_i q)/(\mu_{i-1} - \mu_i)$ and $q_i = (\mu_{i-1} q - r)/(\mu_{i-1} - \mu_i)$, respectively, and the minimum production cost is

$$F(q, r) = (\mu_{i-1} q - r)(c_i - c_{i-1})/(\mu_{i-1} - \mu_i) + c_{i-1} q = \omega_i q - v_i r, \quad i = 2, \dots, m.$$

When finding technology $i+1$ in S , Step 3 in the algorithm that determines S implies that

$$(c_{i+2} - c_i)/(\mu_i - \mu_{i+2}) > (c_{i+1} - c_i)/(\mu_i - \mu_{i+1}), \quad i = 1, \dots, m-2.$$

Consequently, for $i = 1, \dots, m-2$,

$$v_{i+2} - v_{i+1} = \frac{c_{i+2} - c_{i+1}}{\mu_{i+1} - \mu_{i+2}} - \frac{c_{i+1} - c_i}{\mu_i - \mu_{i+1}} = \frac{(c_{i+2} - c_i) - (c_{i+1} - c_i)}{(\mu_i - \mu_{i+2}) - (\mu_i - \mu_{i+1})} - \frac{c_{i+1} - c_i}{\mu_i - \mu_{i+1}} > 0,$$

It follows that $\omega_{i+2} - \omega_{i+1} = (c_{i+1} + \mu_{i+1} v_{i+2}) - (c_{i+1} + \mu_{i+1} v_{i+1}) = \mu_{i+1} v_{i+2} - \mu_{i+1} v_{i+1} > 0$. Since $\mu_1 > \dots > \mu_m \geq 0$ and $0 < c_1 < \dots < c_m$, we have $0 < v_2 < \dots < v_m$ and $0 < \omega_2 < \dots < \omega_m$.

Proof of Lemma 3

We first show the convexity of $F(q, r)$. For any r^1 and r^2 , let $q^1 \in [r^1/\mu_{i-1}, r^1/\mu_i]$ and $q^2 \in [r^2/\mu_{j-1}, r^2/\mu_j]$. Lemma 2 implies that

$$F(q^1, r^1) = c_{i-1}q_{i-1}^1 + c_i q_i^1, \text{ and } F(q^2, r^2) = c_{j-1}q_{j-1}^2 + c_j q_j^2,$$

where $q_{i-1}^1 + q_i^1 = q^1$, $\mu_{i-1}q_{i-1}^1 + \mu_i q_i^1 = r^1$, $q_{j-1}^2 + q_j^2 = q^2$, and $\mu_{j-1}q_{j-1}^2 + \mu_j q_j^2 = r^2$. Therefore, for any $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda F(q^1, r^1) + (1-\lambda)F(q^2, r^2) &= c_{i-1}(\lambda q_{i-1}^1) + c_i(\lambda q_i^1) + c_{j-1}[(1-\lambda)q_{j-1}^2] + c_j[(1-\lambda)q_j^2] \\ &\geq F[\lambda q^1 + (1-\lambda)q^2, \lambda r^1 + (1-\lambda)r^2]. \end{aligned}$$

The inequality holds because λq_{i-1}^1 , λq_i^1 , $(1-\lambda)q_{j-1}^2$, and $(1-\lambda)q_j^2$ are the feasible solutions that minimize the production cost under given $(\lambda q^1 + (1-\lambda)q^2, \lambda r^1 + (1-\lambda)r^2)$. Consequently, $F(q, r)$ is convex.

Next, we prove the submodularity of $F(q, r)$. We only need to show that for all $q^1 \leq q^2$ and $r^1 \leq r^2$,

$$F(q^2, r^1) - F(q^1, r^1) \geq F(q^2, r^2) - F(q^1, r^2). \quad (\text{A.1})$$

We verify whether equation (A.1) holds or not in two cases: (1) there exists i such that $r_1/\mu_{i-1} \leq q^1 \leq q^2 \leq r_1/\mu_i$; and (2) there exist i and j such that $r_1/\mu_{i-1} \leq q^1 \leq r_1/\mu_i < r_1/\mu_{i+j} \leq q^2 \leq r_1/\mu_{i+j+1}$.

Case 1 $r_1/\mu_{i-1} \leq q^1 \leq q^2 \leq r_1/\mu_i$

In this case, $F(q^2, r^1) - F(q^1, r^1) = \omega_i q^2 - v_i r^1 - \omega_i q^1 + v_i r^1 = \omega_i (q^2 - q^1)$. We discuss two subcases: (1) there exists i_0 such that $r_2/\mu_{i_0-1} \leq q^1 \leq q^2 \leq r_2/\mu_{i_0}$; and (2) there exist i_0 and j_0 such that $r_2/\mu_{i_0-1} \leq q^1 \leq r_2/\mu_{i_0} < r_2/\mu_{i_0+j_0} \leq q^2 \leq r_2/\mu_{i_0+j_0+1}$.

Subcase 1 $r_2/\mu_{i_0-1} \leq q^1 \leq q^2 \leq r_2/\mu_{i_0}$

In this subcase,

$$F(q^2, r^2) - F(q^1, r^2) = \omega_{i_0} q^2 - v_{i_0} r^2 - \omega_{i_0} q^1 + v_{i_0} r^2 = \omega_{i_0} (q^2 - q^1).$$

Since $r^1 \leq r^2$, it is clear that $\mu_{i_0} \geq \mu_i$, or equivalently, $i_0 \leq i$. From Lemma 2, this implies that $\omega_{i_0} \leq \omega_i$. Consequently,

$$F(q^2, r^1) - F(q^1, r^1) = \omega_i (q^2 - q^1) \geq \omega_{i_0} (q^2 - q^1) = F(q^2, r^2) - F(q^1, r^2).$$

Subcase 2 $r_2/\mu_{i_0-1} \leq q^1 \leq r_2/\mu_{i_0} < r_2/\mu_{i_0+j_0} \leq q^2 \leq r_2/\mu_{i_0+j_0+1}$

It is clear that $r_1/\mu_{i-1} \leq q^1 \leq r_2/\mu_{i_0} < \dots < r_2/\mu_{i_0+j_0} \leq q^2 \leq r_1/\mu_i$. From Subcase 1, we have

$$F(r_2/\mu_{i_0}, r^1) - F(q^1, r^1) \geq F(r_2/\mu_{i_0}, r^2) - F(q^1, r^2),$$

$$F(r_2/\mu_{i'_0+1}, r^1) - F(r_2/\mu_{i'_0}, r^1) \geq F(r_2/\mu_{i'_0+1}, r^2) - F(r_2/\mu_{i'_0}, r^2), \quad i'_0 = i_0, \dots, i_0 + j_0 - 1,$$

$$F(q^2, r^1) - F(r_2/\mu_{i_0+j_0}, r^1) \geq F(q^2, r^2) - F(r_2/\mu_{i_0+j_0}, r^2).$$

Combining the above equations, we have

$$F(q^2, r^1) - F(q^1, r^1) \geq F(q^2, r^2) - F(q^1, r^2).$$

Case 2 $r_1/\mu_{i-1} \leq q^1 \leq r_1/\mu_i < r_1/\mu_{i+j} \leq q^2 \leq r_1/\mu_{i+j+1}$

From Case 1, we know that

$$F(r_1/\mu_i, r^1) - F(q^1, r^1) \geq F(r_1/\mu_i, r^2) - F(q^1, r^2),$$

$$F(r_1/\mu_{i'+1}, r^1) - F(r_1/\mu_{i'}, r^1) \geq F(r_1/\mu_{i'+1}, r^2) - F(r_1/\mu_{i'}, r^2), \quad i' = i, \dots, i+j-1,$$

$$F(q^2, r^1) - F(r_1/\mu_{i+j}, r^1) \geq F(q^2, r^2) - F(r_1/\mu_{i+j}, r^2).$$

Combining the above equations, we have

$$F(q^2, r^1) - F(q^1, r^1) \geq F(q^2, r^2) - F(q^1, r^2).$$

Equation (A.1) holds in both Case 1 and Case 2. Therefore, $F(q, r)$ is submodular.

In summary, $F(q, r)$ is convex and submodular. ,

Proof of Lemma 5

Let $\tilde{W}(x, z) = W(-x, z)$ and $\tilde{H}(y, w) = H(-y, w)$. Then

$$\tilde{W}(x, z) = \min_{(q, r) \in \Theta} \{F(q, r) + \tilde{H}(x - q, z - r)\}$$

$$= \min_{q+y=x, r+w=z, (q, r, y, w) \in \Omega} \{F(q, r) + \tilde{H}(y, w)\},$$

where $\Omega = \{(q, r, y, w) : r \geq 0, r/\mu_1 \leq q \leq r/\mu_m, y, w \in \square\}$. It can be verified that Ω is a nonempty closed convex sublattice.

The supermodularity and convexity of H imply that $\tilde{H}$ is submodular and convex. From Theorem 1 in Chen et al. (2013), we have that $\tilde{W}(x, z)$ is submodular and convex in $\square^2$. It follows that $W(x, z)$ is supermodular and convex. ,

Proof of Proposition 1

The supermodularity and convexity of $H_t(y, w, \mathbf{K})$, $W_t(x, \bar{z}, \mathbf{K})$, and $V_t(x, z, \mathbf{K})$ can be proved by induction. It is clear that $V_{T+1}(x, z)$ is supermodular and convex.

Suppose that $V_{t+1}(x, z, \mathbf{K})$ is supermodular and convex. This implies that $H_t(y, w, \mathbf{K}) = G_t(y) + \gamma \mathbf{E}_t[V_{t+1}(y - D_t, w, \tilde{\mathbf{K}}_{t+1})]$ is also supermodular and convex because (i) $G_t(y)$ is convex, and (ii) both supermodularity and convexity are preserved under expectation and

linear operation. As shown in Lemma 2, $F(q, r)$ is submodular and convex. Lemma 5 implies that $W_t(x, \bar{z}, \mathbf{K}) = \min\{F(q, r) + H_t(x + q, \bar{z} - r, \mathbf{K}) : (q, r) \in \Theta\}$ is supermodular and convex, so $W_t(-x, \bar{z}, \mathbf{K})$ is submodular and convex in $(x, \bar{z})$. Since C_t is a convex function, $C_t(\bar{z} - z)$ is submodular and convex in $(z, \bar{z})$ (Topkis 1998). It is well known that submodularity and convexity are preserved under minimization. Accordingly, we have the submodularity and convexity of $V_t(-x, z, \mathbf{K}) = \min\{C_t(\bar{z} - z, \mathbf{K}) + W_t(-x, \bar{z}, \mathbf{K}) : \bar{z} \in \square\}$ in (x, z) . This implies that $V_t(x, z, \mathbf{K})$ is supermodular and convex.

In summary, $H_t(y, w, \mathbf{K})$, $W_t(x, \bar{z}, \mathbf{K})$ and $V_t(x, z, \mathbf{K})$ are both supermodular and convex under any given $\mathbf{K}$ for all periods.

Proof of Theorem 1

Before proving Theorem 1, we rewrite problem (3) based on the formula of $F(q, r)$. For $i = 2, \dots, m$, we define

$$W_{it}(x, \bar{z}, \mathbf{K}) \square \min_{w \leq \bar{z}, (\bar{z} - w)/\mu_{i-1} \leq y - x \leq (\bar{z} - w)/\mu_i} \{\omega_i y + v_i w + H_t(y, w, \mathbf{K})\} - \omega_i x - v_i \bar{z},$$

$$V_{it}(x, z, \mathbf{K}) \square \min\{\min_{\bar{z} \geq z} \{K_t(\bar{z} - z) + W_{it}(x, \bar{z}, \mathbf{K})\}, \min_{\bar{z} \leq z} \{k_t(\bar{z} - z) + W_{it}(x, \bar{z}, \mathbf{K})\}\}.$$

Then problem (3) can be rewritten as

$$V_t(x, z, \mathbf{K}) = \min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\}.$$

Here $V_{it}(x, z, \mathbf{K})$ is the optimal cost for the problem using just technologies $i-1$ and i . Based on Lemma 5, it is easy to verify the supermodularity and convexity of $W_{it}(x, \bar{z}, \mathbf{K})$ and $V_{it}(x, z, \mathbf{K})$.

Define the following thresholds for the allowance level:

$$L_{it}(x, \mathbf{K}) \square \arg \min_{z \in \square} \{K_t z + W_{it}(x, z, \mathbf{K})\},$$

$$U_{it}(x, \mathbf{K}) \square \arg \min_{z \in \square} \{k_t z + W_{it}(x, z, \mathbf{K})\}, \quad i = 1, \dots, m,$$

$$l_{it}(x, \mathbf{K}) \square \sup\{z \in \square : (W_{it}(x, z, \mathbf{K}))'_{z^+} < -v_i\},$$

$$u_{it}(x, \mathbf{K}) \square \inf\{z \in \square : (W_{it}(x, z, \mathbf{K}))'_{z^+} > -v_i\}, \quad i = 2, \dots, m.$$

It is easy to verify that (1) $L_{it}(x, \mathbf{K}) \leq U_{it}(x, \mathbf{K})$ ($k_t \leq K_t$) and (2) $L_{it}(x, \mathbf{K}) \leq l_{it}(x, \mathbf{K}) \leq u_{it}(x, \mathbf{K}) \leq U_{it}(x, \mathbf{K})$ if $k_t < v_i < K_t$, from the convexity of $W_{it}(x, \bar{z}, \mathbf{K})$.

Similar to the proof of Theorems 2-4 in Gong and Zhou (2013), it can be proved from its supermodularity and convexity that the optimal solutions to $V_{it}(x, z, \mathbf{K})$ have the following properties, letting y_{it}^* and $\bar{z}_{it}^*$ be the optimal solutions to $V_{it}(x, z, \mathbf{K})$:

LEMMA A.1. *The following results hold:*

(i) *If $v_i \geq K_t$, then produce exclusively with technology $i-1$: when $z \geq U_{i-1,t}(x, \mathbf{K})$, sell allowances down to $\bar{z}_{it}^* = U_{i-1,t}(x, \mathbf{K})$, and then produce up to $y_{it}^* = \max\{\hat{S}_{i-1,t}^U(\mathbf{K}), x\}$; when $L_{i-1,t}(x, \mathbf{K}) < z < U_{i-1,t}(x, \mathbf{K})$, do not trade allowances, and then produce up to $y_{it}^* = \max\{s_{i-1,t}(\mu_{i-1}x + z, \mathbf{K}), x\}$; and when $z \leq L_{i-1,t}(x, \mathbf{K})$, purchase allowances up to $\bar{z}_{it}^* = L_{i-1,t}(x, \mathbf{K})$, and then produce up to $y_{it}^* = \max\{\hat{S}_{i-1,t}^L(\mathbf{K}), x\}$.*

(ii) *If $v_i \leq k_t$, then produce exclusively with technology i : when $z \geq U_{it}(x, \mathbf{K})$, sell allowances down to $\bar{z}_{it}^* = U_{it}(x, \mathbf{K})$, and then produce up to $y_{it}^* = \max\{\hat{S}_{it}^U(\mathbf{K}), x\}$; when $L_{it}(x, \mathbf{K}) < z < U_{it}(x, \mathbf{K})$, do not trade allowances, and then produce up to $y_{it}^* = \max\{s_{it}(\mu_i x + z, \mathbf{K}), x\}$; and when $z \leq L_{it}(x, \mathbf{K})$, purchase allowances up to $\bar{z}_{it}^* = L_{it}(x, \mathbf{K})$, and then produce up to $y_{it}^* = \max\{\hat{S}_{it}^L(\mathbf{K}), x\}$.*

(iii) *If $k_t < v_i < K_t$, then when $z \geq U_{i-1,t}(x, \mathbf{K})$, sell allowances down to $\bar{z}_{it}^* = U_{i-1,t}(x, \mathbf{K})$, and then produce exclusively with technology $i-1$ up to $y_{it}^* = \max\{\hat{S}_{i-1,t}^U(\mathbf{K}), x\}$; when $u_{it}(x, \mathbf{K}) \leq z < U_{i-1,t}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology $i-1$ up to $y_{it}^* = \max\{s_{i-1,t}(\mu_{i-1}x + z, \mathbf{K}), x\}$; when $l_{it}(x, \mathbf{K}) < z < u_{it}(x, \mathbf{K})$, do not trade allowances, and then produce $[\mu_i x + z - \mu_i \bar{S}_{it}(\mathbf{K}) - \bar{w}_{it}(\mathbf{K})]/(\mu_{i-1} - \mu_i)$ units with technology $i-1$ and $[\mu_{i-1} \bar{S}_{it}(\mathbf{K}) + \bar{w}_{it}(\mathbf{K}) - \mu_{i-1}x - z]/(\mu_{i-1} - \mu_i)$ units with technology i up to $y_{it}^* = \bar{S}_{it}(\mathbf{K})$; when $L_{it}(x, \mathbf{K}) < z \leq l_{it}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology i up to $y_{it}^* = \max\{s_{it}(\mu_i x + z, \mathbf{K}), x\}$; and when $z \leq L_{it}(x, \mathbf{K})$, purchase allowances up to $\bar{z}_{it}^* = L_{it}(x, \mathbf{K})$, and then produce exclusively with technology i up to $y_{it}^* = \max\{\hat{S}_{it}^L(\mathbf{K}), x\}$.*

The definitions of $\hat{S}_{it}^U(\mathbf{K})$, $\hat{S}_{it}^L(\mathbf{K})$, $s_{it}(\mu_i x + z, \mathbf{K})$, $\bar{S}_{it}(\mathbf{K})$, and $\bar{w}_{it}(\mathbf{K})$ can be found in Section 5. In Lemma A.1, $L_{it}(x, \mathbf{K})$, $U_{it}(x, \mathbf{K})$, $l_{it}(x, \mathbf{K})$ and $u_{it}(x, \mathbf{K})$ have the following specific formulas:

$$\begin{aligned} L_{it}(x, \mathbf{K}) &= \mu_i(\hat{S}_{it}^L(\mathbf{K}) - x)^+ + w_t^L(\max\{\hat{S}_{it}^L(\mathbf{K}), x\}), \\ U_{it}(x, \mathbf{K}) &= \mu_i(\hat{S}_{it}^U(\mathbf{K}) - x)^+ + w_t^U(\max\{\hat{S}_{it}^U(\mathbf{K}), x\}), \quad i = 1, \dots, m, \\ l_{it}(x, \mathbf{K}) &= \mu_i(\bar{S}_{it}(\mathbf{K}) - x)^+ + w_{it}(\max\{\bar{S}_{it}(\mathbf{K}), x\}), \\ u_{it}(x, \mathbf{K}) &= \mu_{i-1}(\bar{S}_{it}(\mathbf{K}) - x)^+ + w_{it}(\max\{\bar{S}_{it}(\mathbf{K}), x\}), \quad i = 2, \dots, m. \end{aligned}$$

Note that $x < \bar{S}_{it}(\mathbf{K})$ when $l_{it}(x, \mathbf{K}) < z < u_{it}(x, \mathbf{K})$. From Lemma A.1, we know that the optimal solutions to $V_{it}(x, z, \mathbf{K})$ have the following properties: (1) If $v_i \geq K_t$, then produce

exclusively with technology $i-1$, (ii) If $v_i \leq k_t$, then produce exclusively with technology i , (iii) If $k_t < v_i < K_t$, then (a) produce exclusively with technology $i-1$ when $z \geq u_{it}(x, \mathbf{K})$, (b) produce with technologies $i-1$ and i simultaneously when $l_{it}(x, \mathbf{K}) < z < u_{it}(x, \mathbf{K})$, and (c) produce exclusively with technology i when $z \leq l_{it}(x, \mathbf{K})$.

Lemma A.1 shows the optimal solutions y_{it}^* , $\bar{z}_{it}^*$ and the corresponding technology selection for $V_{it}(x, z, \mathbf{K})$. Please note that producing with technology i ($i=2, \dots, m-1$) is feasible to both $V_{it}(x, z, \mathbf{K})$ and $V_{i+1,t}(x, z, \mathbf{K})$. Consequently, the optimal solutions to $V_t(x, z, \mathbf{K}) = \min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\}$ can be characterized by discussing the optimal solutions to $V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})$ as shown in Lemma A.1. For example, if the optimal solution to $V_{it}(x, z, \mathbf{K})$ is also feasible to $V_{i+1,t}(x, z, \mathbf{K})$, then $\min\{V_{it}(x, z, \mathbf{K}), V_{i+1,t}(x, z, \mathbf{K})\} = V_{i+1,t}(x, z, \mathbf{K})$.

We next characterize the structures of the optimal policies when $v_2 \geq K_t$, $v_m \leq k_t$, and $v_{i_0} \leq k_t \leq K_t \leq v_{i_0+1}$, respectively.

Case 1 $K_t \leq v_2 < \dots < v_m$

From Lemma A.1(i), we know that when $v_i \geq K_t$, the optimal solution to $V_{it}(x, z, \mathbf{K})$ is producing exclusively with technology $i-1$. Since producing exclusively with technology $i-1$ ($i=3, \dots, m$) is feasible to both $V_{i-1,t}(x, z, \mathbf{K})$ and $V_{it}(x, z, \mathbf{K})$, we have $V_t(x, z, \mathbf{K}) = \min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\} = V_{2t}(x, z, \mathbf{K})$. Consequently, the optimal policies have the following structure: when $z \geq U_{1t}(x, \mathbf{K})$, sell allowances down to $\bar{z}_t^* = U_{1t}(x, \mathbf{K})$, and then produce exclusively with technology 1 up to $y_t^* = \max\{\hat{S}_{1t}^U(\mathbf{K}), x\}$; when $L_{1t}(x, \mathbf{K}) < z < U_{1t}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology 1 up to $y_t^* = \max\{s_{1t}(\mu_1 x + z, \mathbf{K}), x\}$; and when $z \leq L_{1t}(x, \mathbf{K})$, purchase allowances up to $\bar{z}_t^* = L_{1t}(x, \mathbf{K})$, and then produce exclusively with technology 1 up to $y_t^* = \max\{\hat{S}_{1t}^L(\mathbf{K}), x\}$.

Case 2 $v_2 < \dots < v_m \leq k_t$

From Lemma A.1(ii), we know that when $v_i \leq k_t$, the optimal solution to $V_{it}(x, z, \mathbf{K})$ is producing exclusively with technology i . Since producing exclusively with technology i ($i=2, \dots, m-1$) is feasible to both $V_{it}(x, z, \mathbf{K})$ and $V_{i+1,t}(x, z, \mathbf{K})$, we have

$$V_t(x, z, \mathbf{K}) = \min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\} = V_{mt}(x, z, \mathbf{K}).$$

Consequently, the optimal policies have the following structure: when $z \geq U_{mt}(x, \mathbf{K})$, sell allowances down to $\bar{z}_t^* = U_{mt}(x, \mathbf{K})$, and then produce exclusively with technology m up to

$y_t^* = \max\{\hat{S}_{mt}^U(\mathbf{K}), x\}$; when $L_{mt}(x, \mathbf{K}) < z < U_{mt}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology m up to $y_t^* = \max\{s_{mt}(\mu_m x + z, \mathbf{K}), x\}$; and when $z \leq L_{mt}(x, \mathbf{K})$, purchase allowances up to $\bar{z}_t^* = L_{mt}(x, \mathbf{K})$, and then produce exclusively with technology m up to $y_t^* = \max\{\hat{S}_{mt}^L(\mathbf{K}), x\}$.

Case 3 $v_2 < \dots < v_{i_0} \leq k_t \leq K_t \leq v_{i_0+1} < \dots < v_m$

According to the analyses in Cases 1 and 2, we have

$$V_t(x, z, \mathbf{K}) = \min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\} = \min\{V_{i_0t}(x, z, \mathbf{K}), V_{i_0+1,t}(x, z, \mathbf{K})\}.$$

From Lemma A.1(i, ii), we know that the optimal solutions to $V_{i_0t}(x, z, \mathbf{K})$ and $V_{i_0+1,t}(x, z, \mathbf{K})$ are the same. Thus, the structure of the optimal policies is: when $z \geq U_{i_0t}(x, \mathbf{K})$, sell allowances down to $\bar{z}_t^* = U_{i_0t}(x, \mathbf{K})$, and then produce exclusively with technology i_0 up to $y_t^* = \max\{\hat{S}_{i_0t}^U(\mathbf{K}), x\}$; when $L_{i_0t}(x, \mathbf{K}) < z < U_{i_0t}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology i_0 up to $y_t^* = \max\{s_{i_0t}(\mu_{i_0} x + z, \mathbf{K}), x\}$; and when $z \leq L_{i_0t}(x, \mathbf{K})$, purchase allowances up to $\bar{z}_t^* = L_{i_0t}(x, \mathbf{K})$, and then produce exclusively with technology i_0 up to $y_t^* = \max\{\hat{S}_{i_0t}^L(\mathbf{K}), x\}$.

Proof of Theorem 2

Recall that $v_2 < \dots < v_{i_0} \leq k_t < v_{i_0+1} < \dots < v_{j_0} < K_t \leq v_{j_0+1} < \dots < v_m$. Similar to Cases 1 and 2 in the proof of Theorem 1, we have

$$\min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{i_0t}(x, z, \mathbf{K})\} = V_{i_0t}(x, z, \mathbf{K}), \quad (\text{A.9})$$

$$\min\{V_{j_0+1,t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\} = V_{j_0+1,t}(x, z, \mathbf{K}). \quad (\text{A.10})$$

We first consider $\min\{V_{i_0t}(x, z, \mathbf{K}), V_{i_0+1,t}(x, z, \mathbf{K})\}$ and $\min\{V_{j_0t}(x, z, \mathbf{K}), V_{j_0+1,t}(x, z, \mathbf{K})\}$. Since $v_{i_0} \leq k_t$, Lemma A.1(ii) indicates that the optimal solution to $V_{i_0t}(x, z, \mathbf{K})$ is producing exclusively with technology i_0 . It is clear that this solution is also feasible to $V_{i_0+1,t}(x, z, \mathbf{K})$. Consequently,

$$\min\{V_{i_0t}(x, z, \mathbf{K}), V_{i_0+1,t}(x, z, \mathbf{K})\} = V_{i_0+1,t}(x, z, \mathbf{K}). \quad (\text{A.11})$$

Similarly, from $v_{j_0} < K_t \leq v_{j_0+1}$ and Lemma A.1(i), we have

$$\min\{V_{j_0t}(x, z, \mathbf{K}), V_{j_0+1,t}(x, z, \mathbf{K})\} = V_{j_0t}(x, z, \mathbf{K}). \quad (\text{A.12})$$

Concluding equations (A.9-12), we have

$$V_t(x, z, \mathbf{K}) = \min\{V_{2t}(x, z, \mathbf{K}), \dots, V_{mt}(x, z, \mathbf{K})\} = \min\{V_{i_0+1,t}(x, z, \mathbf{K}), \dots, V_{j_0t}(x, z, \mathbf{K})\}. \quad (\text{A.13})$$

If $j_0 = i_0 + 1$, Lemma A.1(iii) indicates that the optimal policies have the same structure as

shown in Theorem 2. We next consider the case with $j_0 = i_0 + 2$.

Since $k_t < v_{i_0+1}$ ($v_{i_0+2} < K_t$), we have $c_{i_0+1} + k_t \mu_{i_0+1} > c_{i_0} + k_t \mu_{i_0}$ ($c_{i_0+2} + \mu_{i_0+2} K_t > c_{i_0+1} + \mu_{i_0+1} K_t$). From the definitions of $\hat{S}_{it}^U(\mathbf{K})$ and $\hat{S}_{it}^L(\mathbf{K})$, we have $\hat{S}_{i_0+1,t}^U(\mathbf{K}) \leq \hat{S}_{i_0,t}^U(\mathbf{K})$ and $\hat{S}_{i_0+2,t}^L(\mathbf{K}) \leq \hat{S}_{i_0+1,t}^L(\mathbf{K})$. Therefore,

$$U_{i_0+1,t}(x, \mathbf{K}) \leq U_{i_0,t}(x, \mathbf{K}) \text{ and } L_{i_0+2,t}(x, \mathbf{K}) \leq L_{i_0+1,t}(x, \mathbf{K}) \text{ for all } x. \quad (\text{A.14})$$

Define $\tilde{y} \square \max\{s_{i_0+1,t}(\mu_{i_0+1}x + z, \mathbf{K}), x\}$. Now we discuss the relationship between $u_{i_0+2,t}(x, \mathbf{K})$ and $l_{i_0+1,t}(x, \mathbf{K})$. From Lemma A.1, we have

$$W_{i_0+2,t}(x, z, \mathbf{K}) = c_{i_0+1}(\tilde{y} - x) + H_t(\tilde{y}, \mu_{i_0+1}x + z - \mu_{i_0+1}\tilde{y}, \mathbf{K}), \text{ when } u_{i_0+2,t}(x, \mathbf{K}) \leq z < U_{i_0+1,t}(x, \mathbf{K}),$$

$$W_{i_0+1,t}(x, z, \mathbf{K}) = c_{i_0+1}(\tilde{y} - x) + H_t(\tilde{y}, \mu_{i_0+1}x + z - \mu_{i_0+1}\tilde{y}, \mathbf{K}), \text{ when } L_{i_0+1,t}(x, \mathbf{K}) < z \leq l_{i_0+1,t}(x, \mathbf{K}).$$

It is clear that $u_{i_0+2,t}(x, \mathbf{K})$, $l_{i_0+1,t}(x, \mathbf{K})$, and $L_{i_0+1,t}(x, \mathbf{K})$ can be rewritten as:

$$u_{i_0+2,t}(x, \mathbf{K}) = \inf\{z \in \mathbb{R} : (W_{i_0+2,t}(x, z, \mathbf{K}))'_{z^+} > -v_{i_0+2}, u_{i_0+2,t}(x, \mathbf{K}) \leq z < U_{i_0+1,t}(x, \mathbf{K})\},$$

$$l_{i_0+1,t}(x, \mathbf{K}) = \sup\{z \in \mathbb{R} : (W_{i_0+1,t}(x, z, \mathbf{K}))'_{z^+} < -v_{i_0+1}, L_{i_0+1,t}(x, \mathbf{K}) < z \leq l_{i_0+1,t}(x, \mathbf{K})\},$$

$$L_{i_0+1,t}(x, \mathbf{K}) = \inf\{z \in \mathbb{R} : (W_{i_0+1,t}(x, z, \mathbf{K}))'_{z^+} > -K_t, L_{i_0+1,t}(x, \mathbf{K}) < z \leq l_{i_0+1,t}(x, \mathbf{K})\}.$$

Since $v_{i_0+1} < v_{i_0+2} < K_t$, the convexities of $W_{i_0+2,t}(x, z, \mathbf{K})$ and $W_{i_0+1,t}(x, z, \mathbf{K})$ imply that

$$L_{i_0+1,t}(x, \mathbf{K}) \leq u_{i_0+2,t}(x, \mathbf{K}) \leq l_{i_0+1,t}(x, \mathbf{K}) \text{ for all } x. \quad (\text{A.15})$$

Recall that $L_{nt}(x, \mathbf{K}) \leq l_{nt}(x, \mathbf{K}) \leq u_{nt}(x, \mathbf{K}) \leq U_{nt}(x, \mathbf{K})$, $n = i_0 + 1, i_0 + 2$.

We next characterize the structure of the optimal policies when $j_0 = i_0 + 2$ in five cases.

Case 1 $z \geq U_{i_0,t}(x, \mathbf{K})$

Lemma A.1(iii) indicates that when $z \geq U_{i_0,t}(x, \mathbf{K})$, it is optimal to produce exclusively with technology $i - 1$ ($i = i_0 + 1, i_0 + 2$) and sell allowances. From $U_{i_0+1,t}(x, \mathbf{K}) \leq U_{i_0,t}(x, \mathbf{K})$ (equation (A.14)), we have

$$\min\{V_{i_0+1,t}(x, z, \mathbf{K}), V_{i_0+2,t}(x, z, \mathbf{K})\} = V_{i_0+1,t}(x, z, \mathbf{K}) \text{ when } z \geq U_{i_0,t}(x, \mathbf{K}), \quad (\text{A.16})$$

Because producing exclusively with technology $i_0 + 1$ with selling allowances is feasible to $V_{i_0+1,t}(x, z, \mathbf{K})$ when $z \geq U_{i_0,t}(x, \mathbf{K})$. Equation (A.16) indicates that when $z \geq U_{i_0,t}(x, \mathbf{K})$, it is optimal to sell allowances down to level $\bar{z}_t^* = U_{i_0,t}(x, \mathbf{K})$, and then produce exclusively with technology i_0 up to $y_t^* = \max\{\hat{S}_{i_0,t}^U(\mathbf{K}), x\}$.

Case 2 $l_{i_0+1,t}(x, \mathbf{K}) < z < U_{i_0,t}(x, \mathbf{K})$

Since $z \geq l_{i_0+1,t}(x, \mathbf{K}) \geq u_{i_0+2,t}(x, \mathbf{K})$ (equation (A.15)), Lemma A.1(iii) indicates that the optimal solution to $V_{i_0+2,t}(x, z, \mathbf{K})$ is producing exclusively with technology $i_0 + 1$. This solution is clearly feasible to $V_{i_0+1,t}(x, z, \mathbf{K})$. Hence,

$$\min\{V_{i_0+1,t}(x, z, \mathbf{K}), V_{i_0+2,t}(x, z, \mathbf{K})\} = V_{i_0+1,t}(x, z, \mathbf{K}) \quad \text{when } l_{i_0+1,t}(x, \mathbf{K}) < z < U_{i_0,t}(x, \mathbf{K}).$$

This implies that: (1) when $u_{i_0+1}(x, \mathbf{K}) \leq z < U_{i_0,t}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology i_0 up to $y_t^* = \max\{s_{i_0,t}(\mu_{i_0}x + z, \mathbf{K}), x\}$; and (2) when $l_{i_0+1,t}(x, \mathbf{K}) < z < u_{i_0+1,t}(x, \mathbf{K})$, do not trade allowances, and then produce $(\mu_{i_0+1}x + z - \mu_{i_0+1}\bar{S}_{i_0+1,t}(\mathbf{K}) - \bar{w}_{i_0+1,t}(\mathbf{K})) / (\mu_{i_0} - \mu_{i_0+1})$ units with technology i_0 and $(\mu_{i_0}\bar{S}_{i_0+1,t}(\mathbf{K}) + \bar{w}_{i_0+1,t}(\mathbf{K}) - \mu_{i_0}x - z) / (\mu_{i_0} - \mu_{i_0+1})$ units with technology $i_0 + 1$.

Case 3 $u_{i_0+2,t}(x, \mathbf{K}) \leq z \leq l_{i_0+1,t}(x, \mathbf{K})$

Since $L_{i_0+1,t}(x, \mathbf{K}) \leq u_{i_0+2,t}(x, \mathbf{K})$ (equation (A.15)) and $l_{i_0+1,t}(x, \mathbf{K}) \leq U_{i_0+1,t}(x, \mathbf{K})$, Lemma A.1(iii) implies that $V_{i_0+1,t}(x, z, \mathbf{K})$ and $V_{i_0+2,t}(x, z, \mathbf{K})$ have the same optimal solution when $u_{i_0+2,t}(x, \mathbf{K}) \leq z \leq l_{i_0+1,t}(x, \mathbf{K})$. Therefore, when $u_{i_0+2,t}(x, \mathbf{K}) \leq z \leq l_{i_0+1,t}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology $i_0 + 1$ up to $y_t^* = \max\{s_{i_0+1,t}(\mu_{i_0+1}x + z, \mathbf{K}), x\}$.

Case 4 $L_{i_0+2,t}(x, \mathbf{K}) < z < u_{i_0+2,t}(x, \mathbf{K})$

Since $z \leq u_{i_0+2,t}(x, \mathbf{K}) \leq l_{i_0+1,t}(x, \mathbf{K})$ (equation (A.15)), Lemma A.1(iii) suggests that the optimal solution to $V_{i_0+1,t}(x, z, \mathbf{K})$ is producing exclusively with technology $i_0 + 1$. This solution is also feasible to $V_{i_0+2,t}(x, z, \mathbf{K})$. Consequently,

$$\min\{V_{i_0+1,t}(x, z, \mathbf{K}), V_{i_0+2,t}(x, z, \mathbf{K})\} = V_{i_0+2,t}(x, z, \mathbf{K}) \quad \text{when } L_{i_0+2,t}(x, \mathbf{K}) < z < u_{i_0+2,t}(x, \mathbf{K}).$$

This implies that: (1) when $L_{i_0+2,t}(x, \mathbf{K}) \leq z < l_{i_0+2,t}(x, \mathbf{K})$, do not trade allowances, and then produce exclusively with technology $i_0 + 2$ up to $y_t^* = \max\{s_{i_0+2,t}(\mu_{i_0+2}x + z, \mathbf{K}), x\}$; and (2) when $l_{i_0+2,t}(x, \mathbf{K}) < z < u_{i_0+2,t}(x, \mathbf{K})$, do not trade allowances, and then produce $(\mu_{i_0+2}x + z - \mu_{i_0+2}\bar{S}_{i_0+2,t}(\mathbf{K}) - \bar{w}_{i_0+2,t}(\mathbf{K})) / (\mu_{i_0+1} - \mu_{i_0+2})$ units with technology $i_0 + 1$ and $(\mu_{i_0+1}\bar{S}_{i_0+2,t}(\mathbf{K}) + \bar{w}_{i_0+2,t}(\mathbf{K}) - \mu_{i_0+1}x - z) / (\mu_{i_0+1} - \mu_{i_0+2})$ units with technology $i_0 + 2$.

Case 5 $z \leq L_{i_0+2,t}(x)$

Lemma A.1(iii) indicates that it is optimal to produce exclusively with technology i ($i = i_0 + 1, i_0 + 2$) and purchase allowances when $z \leq L_{i,t}(x, \mathbf{K})$. From $L_{i_0+2,t}(x, \mathbf{K}) \leq L_{i_0+1,t}(x, \mathbf{K})$ (equation (A.14)), we have

$$\min\{V_{i_0+1,t}(x, z, \mathbf{K}), V_{i_0+2,t}(x, z, \mathbf{K})\} = V_{i_0+2,t}(x, z, \mathbf{K}) \quad \text{when } z \leq L_{i_0+2,t}(x, \mathbf{K}), \quad (\text{A.17})$$

because producing exclusively with technology $i_0 + 1$ with purchasing allowances is feasible to $V_{i_0+2,t}(x, z, \mathbf{K})$ when $z \leq L_{i_0+2,t}(x, \mathbf{K})$. Equation (A.17) indicates that: when $z \leq L_{i_0+2,t}(x, \mathbf{K})$, it is optimal to purchase allowances up to $\bar{z}_t^* = L_{i_0+2,t}(x, \mathbf{K})$, and then produce exclusively with

technology $i_0 + 2$ up to $y_t^* = \max\{\hat{S}_{i_0+2,t}^L(\mathbf{K}), x\}$.

So far we have fully characterized the optimal policies for the case with $j_0 = i_0 + 2$, which has the same structure as shown in Theorem 2. Similarly, for $j_0 > i_0 + 2$, it can be proved that the optimal policy has the same structure as shown in Theorem 2. Details are omitted for brevity.

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