Discontinuous Galerkin method with arbitrary polygonal finite elements

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Abstract

In the paper the efficient application of discontinuous Galerkin (DG) method on polygonal meshes is presented. Three versions of DG method are under consideration in which the approximation is constructed using sets of arbitrary basis functions. It means that in the presented approach there is no need to define nodes or to construct shape functions. The shape of a polygonal finite element (FE) can be quite arbitrary. It can have arbitrary number of edges and can be non-convex. In particular a single FE can have a polygonal hole or can even consists of two or more completely separated parts. The efficiency, flexibility and versatility of the presented approach is illustrated with a set of benchmark examples. The paper is limited to two-dimensional case. However, direct extension of the algorithms to three-dimension is possible.

Key words: *Discontinuous Galerkin method, polygonal finite elements, finite difference, elliptic problem*

1 Introduction

In this paper the discontinuous Galerkin (DG) method for polygonal meshes is considered. The application of DG method on the polygonal meshes is currently quite new topic and there are few papers tackling this problem, e.g. [39, 25, 26, 44, 8, 16, 1]. In this work the DG method on the polygonal meshes is extended for the cases where the finite element cells have arbitrary shape and can be convex or non-convex. Here the cells can be non-simply-connected (NSC) (i.e. a cell with holes inside) or even non-connected (NC) (i.e. a cell consists of two or more parts which are completely separated from each other). The approximation in the finite element cells are based on a set of basis functions which can consist of quite arbitrary functions. It means that there are no special requirements for those functions on edges or vertices of the cell.

Although the combination of DG method with polygonal meshes is quite new, the idea of polygonal finite element method (PFEM) reaches 1975 when Wachspress in [38] introduced the barycentric coordinates and barycentric interpolation and a rational basis on convex polygons was constructed. The renaissance of the Wachspress work took place in the early of our century when the Wachspress basis functions have been applied to finite element method (FEM) [14, 13]. Afterwards a series of other papers have appeared concerning the FEM on polygonal or polyhedral meshes, e.g. [33, 31, 34, 32, 24, 40, 29, 21, 20, 5, 36, 35]. There are two other methods that deals with polygonal/polyhedral meshes i.e. mimetic finite difference method (MFD) (e.q. [30, 22, 4]) and virtual element method (VEM) (e.q. [3, 6, 10, 11, 2, 12]). A nice survey on all the methods based on polygonal/polyhedral meshes is presented in [23]. The computer applications of the mentioned methods, namely PFEM, MFD and VEM are not easy since they either require complicated shape functions or complex algorithm to be implemented. To apply such methods a great computational effort have to be paid. A competitive alternative to these methods are the approaches based on DG idea. They do not require shape functions construction or any other sophisticated problem formulation. The DG methods are based on the standard Galerkin weak formulation and give a great flexibility in choosing approximation order or finite element shapes, which can be quite arbitrary.

In the DG method, in two-dimensional case, the same algorithm is applied for standard triangular or quadrilateral meshes as well as for meshes with convex and non-convex, NSC and NC polygons. The integration have to be performed over all finite elements, outer boundary and over the mesh skeleton (the finite elements boundaries). The integration over a finite element is performed in such a way that its region is subdivided into a set of triangles and then the standard Gauss method is applied for each of them.

In this paper three versions of DG method are considered:

1. discontinuous Galerkin with finite difference (DGFD) method [17],
2. interface discontinuous Galerkin (IDG) method [18],
3. standard discontinuous Galerkin (SDG) method [27].

In all the versions of DG method the approximation fields are constructed by a set of local basis functions in the same way as in [17]. Example results obtained from all the three methods are presented and compared.

The reminder of this paper is organized as follows. In Section 2 the elliptic mathematical problem is defined for which the DG methods are derived. The discontinuous Galerkin formulations on polygonal meshes are presented in 3 where the IDG, DGFD and SDG methods are described in 3.1, 3.2 and 3.3, respectively. Section 4 deals with approximation techniques where arbitrary basis functions can be used. The DG methods are illustrated with a series of benchmark examples in Section 5. The paper ends with some final conclusions.

2 Mathematical model

The DG methods are presented for scalar elliptic problem that has a physical interpretation of the stationary heat transport. The problem starts with the well-known local form of energy balance equation, the Fourier's law as well as boundary conditions of Dirichlet and Neumann types. The problem is defined in the domain Ω with outer boundary Γ as follow: find the continuous temperature field Θ that satisfy the following relations

$$\begin{aligned} \operatorname{div} \mathbf{q} - r &= 0, & \text{in } \Omega \\ \mathbf{q} &= -k \nabla \Theta, & \text{in } \Omega \\ \Theta &= \hat{\Theta} & \text{on } \Gamma_{\Theta} \\ \mathbf{q} \cdot \mathbf{n} &= \hat{h} & \text{on } \Gamma_q \end{aligned} \tag{1}$$

where $\mathbf{q}$ is the heat flux vector, r the heat source density, k is the heat conductivity parameter for a thermally isotropic material, $\hat{\Theta}$ and $\hat{h}$ are prescribed values of temperature and heat flux, respectively, Γ_{Θ} is the part of Γ where the temperature $\hat{\Theta}$ is prescribed, Γ_q is the part of Γ where the heat flux $\hat{h}$ is prescribed, $\mathbf{n}$ is the unit vector normal to the outer boundary. The heat flux vector is related to the temperature field by the Fourier's law in (1)₂.

The regarded domain is structured by polygonal mesh in 2D. The mesh consists of a set of cells, outer boundary and the mesh skeleton, what is illustrated in Fig. 1. Individual cell is a polygon which may not necessarily be convex, simply-connected or connected. The DG method requires integration along the mesh skeleton. In this paper such integration is performed with the help of the skeleton local coordinates. The local coordinates are defined by a set of mutually perpendicular unit vectors $\mathbf{n}^s, \mathbf{s}^s$ where vector $\mathbf{n}^s$ is normal to the skeleton and called in this paper the skeleton normal and $\mathbf{s}^s$ is referred as the skeleton tangent. The orientation of these vectors is arbitrary providing that they meet the above requirements. It means that the skeleton normal can be directed to either of the aligned cells, the same refers to the skeleton tangent. Fig. 2 shows the set of the skeleton local coordinates for polygonal mesh.

Figure 1: Polygonal mesh consisting of the set of cells, outer boundary and mesh skeleton

Figure 2: Mesh skeleton with the set of skeleton local coordinates: normal and tangent

The DG method is based on the discontinuous approximation with the discontinuity on the skeleton. The discontinuity has to be regarded in the global formulation of the problem defined by eq. (1), where the integration by parts is performed. In consequence it leads to the situation when the jump and mean values of the discontinuity have to be regarded. Their definitions are based on the skeleton normal, namely

$$[[f]] = \lim_{\epsilon \rightarrow 0} [[f]]_\epsilon, \quad \langle f \rangle(\mathbf{x}) = \lim_{\epsilon \rightarrow 0} \langle f \rangle_\epsilon \quad (2)$$

where the jump and mean values at distance ϵ are defined as follows

$$[[f]]_\epsilon = f(\mathbf{x} + \epsilon \mathbf{n}^s) - f(\mathbf{x} - \epsilon \mathbf{n}^s), \quad \langle f \rangle_\epsilon = 0.5 \left(f(\mathbf{x} + \epsilon \mathbf{n}^s) + f(\mathbf{x} - \epsilon \mathbf{n}^s) \right) \quad (3)$$

It is a well known property that the jump of two functions superposition can be expressed as follow

$$[[fg]] = [[f]] \langle g \rangle + \langle f \rangle [[g]] \quad (4)$$

3 Discontinuous Galerkin on polygonal meshes

As mentioned in the introduction, three versions of the DG method, i.e. IDG, DGFD and SDG are applied for polygonal meshes and used to solve the problem described in Section 2. The results obtained for the three approaches are mutually compared and some conclusions are formulated. The SDG is well known and widely described method e.g. [9, 27, 43, 7]. The other two approaches have been proposed by the first author of this paper and are not discussed in the literature. Thus, the special attention is paid on the presentation of details of IDG and DGFD method used for polygonal meshes.

3.1 Interface discontinuous Galerkin method

In the interface discontinuous Galerkin method it is assumed that the mesh skeleton Γ_s has a non-zero width w that is much smaller in comparison to characteristic sizes of neighbouring finite elements. In consequence the polygonal cells are diminished adequately to the width of the mesh skeleton. This feature is illustrated in Fig. 3. The skeleton segments are treated as a special finite elements where their lengths are much greater than their width. A special kind of approximation is constructed in these elements, and described subsequently in this Section. In consequence, contrary to the standard approach, the test function and the trial function are continuous in the whole domain. The domain area Ω can be expressed as a sum of the area of all regular finite elements Ω_E and the skeleton area Ω_s

$$\Omega = \Omega_E \cup \Omega_s, \quad \Omega_E \cap \Omega_s = \emptyset \quad (5)$$

The formulation of the IDG method starts with the weak form of the analysed problem (1), with the test function v

$$\int_{\Omega} v \operatorname{div} \mathbf{q} \, d\Omega - \int_{\Omega} vr \, d\Omega = 0, \quad \forall v \quad (6)$$

When integration by parts is performed on the first component from equation (6) which results in

$$\int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma - \int_{\Omega} \nabla v \cdot \mathbf{q} \, d\Omega - \int_{\Omega} vr \, d\Omega = 0 \quad (7)$$

In eq. (7) no jumps appear since all fields are supposed to be continuous.

Taking the relation (5) into account the integrals over Ω volume from equation (7) can be rewritten as a sum over these two volumes

$$\int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma - \int_{\Omega_E} \nabla v \cdot \mathbf{q} \, d\Omega - \int_{\Omega_s} \nabla v \cdot \mathbf{q} \, d\Omega - \int_{\Omega_E} vr \, d\Omega - \int_{\Omega_s} vr \, d\Omega = 0 \quad (8)$$

Figure 3: The polygonal mesh for IDG method with non-zero skeleton width and diminished cells

It is assumed in this paper that the width of the skeleton is very small in comparison to the other dimensions. Thus any integral over Ω_s can be approximated by the integral over Γ_s of the integrand multiplied by w . The justification of such approximation is presented in [19]. Then eq. (8) can be expressed as follows

$$\int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} d\Gamma - \int_{\Omega_E} \nabla v \cdot \mathbf{q} d\Omega - \int_{\Gamma_s} w \nabla v \cdot \mathbf{q} d\Gamma - \int_{\Omega_E} v r d\Omega - \int_{\Gamma_s} w v r d\Gamma = 0 \quad (9)$$

The third integral in (9) is over the skeleton Γ_s and in the integral the scalar product of the test function gradient and the heat flux vector have to be calculated. It means that they have to be calculated at all points on the Γ_s where the skeleton local coordinates are defined. The scalar product can be calculated in the global coordinates as well as in the skeleton local ones

$$\nabla v \cdot \mathbf{q} = \frac{\partial v}{\partial x} q_x + \frac{\partial v}{\partial y} q_y = \frac{\partial v}{\partial n} q_n + \frac{\partial v}{\partial s} q_s \quad \text{on } \Gamma_s \quad (10)$$

where $\frac{\partial v}{\partial n}$ and $\frac{\partial v}{\partial s}$ mean the partial derivatives in $\mathbf{n}^s$ and $\mathbf{s}^s$ directions, respectively, and q_n and q_s are the components of the heat flux vector in the skeleton local coordinates.

The skeleton finite elements are not typical because there are no standard shape functions in these elements. In the skeleton normal direction $\mathbf{n}^s$ the approximation is of first degree on the other side the approximation order in the $\mathbf{s}^s$ direction is inherited from the aligned elements. In the result the skeleton approximation in the $\mathbf{n}^s$ direction linearly interpolates approximations of two neighbouring elements. Fig. 4 presents the skeleton interpolation for one-dimensional case.

As can be seen in eq. (9), the integrals concerning the skeleton finite elements are limited to the integrals along the skeleton segments. In order to compute this kind of integrals we need to have the approximation of required quantities just in the segments points, i.e. on the mid-surface of the skeleton elements. It is achieved by the values on both sides of the skeleton finite elements. This is why in the last integral in (9), the test function v can be expressed as a mean value:

$$v = \langle v \rangle_{\frac{1}{2}w}, \quad \text{on } \Gamma_s \quad (11)$$

On the other hand, the derivatives in the skeleton normal direction can be then approximated with the help of the finite difference expression regarding Fourier law, namely

$$q_n = -k \frac{[\Theta]_{\frac{1}{2}w}}{w}, \quad \frac{\partial v}{\partial n} = \frac{[v]_{\frac{1}{2}w}}{w}, \quad \text{on } \Gamma_s \quad (12)$$

Figure 4: Skeleton interpolation in one-dimensional case

The other components of the inner product from eq. (10) are approximated by mean-values. Keeping in mind Fourier law, the scalar product from eq. (10) can now be written in the following form:

$$\nabla v \cdot \mathbf{q} = -\frac{k}{w^2} \llbracket v \rrbracket_{\frac{1}{2}w} \llbracket \Theta \rrbracket_{\frac{1}{2}w} - k \left\langle \frac{\partial v}{\partial s} \right\rangle_{\frac{1}{2}w} \left\langle \frac{\partial \Theta}{\partial s} \right\rangle_{\frac{1}{2}w}, \quad \text{on } \Gamma_s \quad (13)$$

The derivatives in $\mathbf{s}^s$ direction from equation (13) can be written by means of gradients in the global coordinates as follows:

$$\left\langle \frac{\partial v}{\partial s} \right\rangle_{\frac{1}{2}w} = \langle \nabla v \rangle_{\frac{1}{2}w} \cdot \mathbf{s}^s, \quad \left\langle \frac{\partial \Theta}{\partial s} \right\rangle_{\frac{1}{2}w} = \langle \nabla \Theta \rangle_{\frac{1}{2}w} \cdot \mathbf{s}^s, \quad \text{on } \Gamma_s \quad (14)$$

Finally, the scalar product from eq. (13) is expressed by the values of temperature, test function and their gradients on the edges of neighbouring regular finite elements

$$\begin{aligned} \nabla v \cdot \mathbf{q} &= -\frac{k}{w^2} \llbracket v \rrbracket_{\frac{1}{2}w} \llbracket \Theta \rrbracket_{\frac{1}{2}w} - k \langle \nabla v \rangle_{\frac{1}{2}w} \cdot \mathbf{s}^s \otimes \mathbf{s}^s \cdot \langle \nabla \Theta \rangle_{\frac{1}{2}w} \\ &= -\frac{k}{w^2} \llbracket v \rrbracket_{\frac{1}{2}w} \llbracket \Theta \rrbracket_{\frac{1}{2}w} - k \langle \nabla v \rangle_{\frac{1}{2}w} \cdot \mathbf{Q} \cdot \langle \nabla \Theta \rangle_{\frac{1}{2}w} \end{aligned} \quad \text{on } \Gamma_s \quad (15)$$

where $\mathbf{Q}$ is defined as

$$\mathbf{Q} = \mathbf{s}^s \otimes \mathbf{s}^s = \mathbf{I} - \mathbf{n}^s \otimes \mathbf{n}^s \quad (16)$$

Substituting the relation from eq. (15) into the equation (9) in the weak form, we obtain the following result

$$\begin{aligned} & \int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma + \int_{\Omega_E} k \nabla v \cdot \nabla \Theta \, d\Omega + \int_{\Gamma_s} \frac{k}{w} \llbracket v \rrbracket_{\frac{1}{2}w} \llbracket \Theta \rrbracket_{\frac{1}{2}w} \, d\Gamma \\ & + \int_{\Gamma_s} w k \langle \nabla v \rangle_{\frac{1}{2}w} \cdot \mathbf{Q} \cdot \langle \nabla \Theta \rangle_{\frac{1}{2}w} \, d\Gamma - \int_{\Omega_E} v r \, d\Omega - \int_{\Gamma_s} w \langle v \rangle_{\frac{1}{2}w} r \, d\Gamma = 0 \end{aligned} \quad (17)$$

The boundary conditions of Neumann and Dirichlet types, which are defined in eq. (1), are written in eq. (17) by means of the integral over the whole outer boundary Γ , i.e. $\int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma$. The integral along the outer boundary can be written as a sum of two integrals: over Γ_q and Γ_{Θ} , and the Neumann boundary condition can be substituted directly to:

$$\int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_q} v \hat{h} \, d\Gamma + \int_{\Gamma_{\Theta}} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma \quad (18)$$

In order to apply the Dirichlet boundary conditions, it is assumed here that along Γ_Θ there is a thin layer of material between the boundary and the regular finite elements which is discretized by the skeleton finite element but with $\frac{1}{2}w$ width. Under these assumptions the normal part of the heat flux can be expressed with the help of Fourier's law:

$$\mathbf{q} \cdot \mathbf{n} = q_n = -k \frac{\hat{\Theta} - \Theta}{\frac{1}{2}w} = -2k \frac{\hat{\Theta}}{w} + 2k \frac{\Theta}{w} \quad \text{on } \Gamma_\Theta \quad (19)$$

Equations (18) and (19) are substituted to eq. (17) to obtain the final weak form equation where both kinds of boundary conditions are regarded:

$$\begin{aligned} & \int_{\Gamma_q} v \hat{h} \, d\Gamma - 2 \int_{\Gamma_\Theta} v \frac{k}{w} \hat{\Theta} \, d\Gamma + 2 \int_{\Gamma_\Theta} \frac{k}{w} v \Theta \, d\Gamma + \int_{\Omega_E} k \nabla v \cdot \nabla \Theta \, d\Omega + \int_{\Gamma_s} \frac{k}{w} \llbracket v \rrbracket_{\frac{1}{2}w} \llbracket \Theta \rrbracket_{\frac{1}{2}w} \, d\Gamma \\ & + \int_{\Gamma_s} w k \langle \nabla v \rangle_{\frac{1}{2}w} \cdot \mathbf{Q} \cdot \langle \nabla \Theta \rangle_{\frac{1}{2}w} \, d\Gamma - \int_{\Omega_E} v r \, d\Omega - \int_{\Gamma_s} w \langle v \rangle_{\frac{1}{2}w} r \, d\Gamma = 0 \end{aligned} \quad (20)$$

Equation (20) can be written in the form of equality of bilinear and linear forms

$$A(\Theta, v) = F(v), \quad \forall v \quad (21)$$

where

$$\begin{aligned} A(\Theta, v) &= \int_{\Omega_E} k \nabla v \cdot \nabla \Theta \, d\Omega + \int_{\Gamma_s} \frac{k}{w} \llbracket v \rrbracket_{\frac{1}{2}w} \llbracket \Theta \rrbracket_{\frac{1}{2}w} \, d\Gamma + \int_{\Gamma_s} w k \langle \nabla v \rangle_{\frac{1}{2}w} \cdot \mathbf{Q} \cdot \langle \nabla \Theta \rangle_{\frac{1}{2}w} \, d\Gamma + 2 \int_{\Gamma_\Theta} \frac{k}{w} v \Theta \, d\Gamma \\ F(v) &= \int_{\Omega_E} v r \, d\Omega + \int_{\Gamma_s} w \langle v \rangle_{\frac{1}{2}w} r \, d\Gamma - \int_{\Gamma_q} v \hat{h} \, d\Gamma + 2 \int_{\Gamma_\Theta} \frac{k}{w} v \hat{\Theta} \, d\Gamma \end{aligned} \quad (22)$$

3.2 Discontinuous Galerkin with finite difference method

Formulation of the discontinuous Galerkin with finite difference method starts with the global description of the regarded problem, i.e. eq. (6). In this case, in contrary to IDG method, the skeleton width is zero and the approximation is truly discontinuous there. The discontinuity on Γ_s have to be taken into account in the further considerations.

After integration by parts of the first integral in eq. (6), the equation changes to

$$\int_{\Omega} \operatorname{div}(v \mathbf{q}) \, d\Omega - \int_{\Omega} \nabla v \cdot \mathbf{q} \, d\Omega - \int_{\Omega} v r \, d\Omega = 0 \quad (23)$$

The first integral in eq. (23) is over the domain area where its integrand is divergence. When $(v \mathbf{q})$ was continuous in Ω the integral would be equal to the integral along the outer boundary. However, here the discontinuity of the function $(v \mathbf{q})$ on the mesh skeleton Γ_s has to be regarded. In such a case the first integral in eq. (23) is the same as the integral along outer boundary reduced by the integral along skeleton, i.e.

$$\int_{\Omega} \operatorname{div}(v \mathbf{q}) \, d\Omega = \int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma - \int_{\Gamma_s} \llbracket v \mathbf{q} \rrbracket \cdot \mathbf{n}^s \, d\Gamma \quad (24)$$

Figure 5: The distance d from the skeleton which is used for finite difference relations

On the basis of relation in eq. (4), the eq. (24) can be rewritten as

$$\int_{\Omega} \operatorname{div}(v \mathbf{q}) \, d\Omega = \int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma - \int_{\Gamma_s} \llbracket v \rrbracket \langle \mathbf{q} \rangle \cdot \mathbf{n}^s \, d\Gamma - \int_{\Gamma_s} \langle v \rangle \llbracket \mathbf{q} \rrbracket \cdot \mathbf{n}^s \, d\Gamma \quad (25)$$

The heat flux in the skeleton normal direction is set to zero since no heat source is on the skeleton. It means, on the other side, that the heat flux is continuous in the $\mathbf{n}^s$ direction

$$\llbracket \mathbf{q} \rrbracket \cdot \mathbf{n}^s = 0 \quad \Rightarrow \quad \langle \mathbf{q} \rangle \cdot \mathbf{n}^s = \mathbf{q} \cdot \mathbf{n}^s \quad \text{on } \Gamma_s \quad (26)$$

The integral over outer boundary from eq. (25) can be expressed as a sum over Γ_{Θ} and Γ_q . Subsequently on Γ_q the Neumann boundary condition, from eq. (1)₄, can be applied

$$\int_{\Gamma} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_q} v \hat{h} \, d\Gamma + \int_{\Gamma_{\Theta}} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma \quad (27)$$

After taking into account equations (25) to (27) equation (23) changes to the following form

$$\int_{\Gamma_q} v \hat{h} \, d\Gamma + \int_{\Gamma_{\Theta}} v \mathbf{q} \cdot \mathbf{n} \, d\Gamma - \int_{\Gamma_s} \llbracket v \rrbracket \mathbf{q} \cdot \mathbf{n}^s \, d\Gamma - \int_{\Omega} \nabla v \cdot \mathbf{q} \, d\Omega - \int_{\Omega} v r \, d\Omega = 0 \quad (28)$$

The heat flux vector is expressed by Fourier's law, i.e. $\mathbf{q} = -k \nabla \Theta$ and then the eq. (28) takes the form

$$\int_{\Gamma_q} v \hat{h} \, d\Gamma - \int_{\Gamma_{\Theta}} k v \nabla \Theta \cdot \mathbf{n} \, d\Gamma + \int_{\Gamma_s} k \llbracket v \rrbracket \nabla \Theta \cdot \mathbf{n}^s \, d\Gamma + \int_{\Omega} k \nabla v \cdot \nabla \Theta \, d\Omega - \int_{\Omega} v r \, d\Omega = 0 \quad (29)$$

The third integral in eq. (29) requires the temperature derivative in the skeleton normal direction on the mesh skeleton. The derivative have to be evaluated with the help of finite difference relations, see [17]. The finite difference relations are calculated using the values at distance d from the skeleton, what is shown in Fig. 5

Taking advantage of the paper [17] the fourth degree finite difference evaluation of the derivative is chosen, as below

$$\nabla \Theta \cdot \mathbf{n}^s = \frac{3}{2} \cdot \frac{\llbracket \Theta \rrbracket_d}{2d} - \frac{1}{2} \cdot \langle \nabla \Theta \rangle_d \cdot \mathbf{n}^s \quad \text{on } \Gamma_s \quad (30)$$

The same approach is applied to the derivative on the outer boundary (the second integral in eq. (29)). In this case also the fourth degree finite difference is used, resulting in

$$\nabla \Theta \cdot \mathbf{n} = 6 \cdot \frac{\hat{\Theta} - \Theta(\mathbf{x}_{2d})}{2d} - 4\nabla_n \Theta(\mathbf{x}_d) - \nabla_n \Theta(\mathbf{x}_{2d}) \quad \text{on } \Gamma_\Theta \quad (31)$$

where $\mathbf{x}_d = \mathbf{x} - d\mathbf{n}$, $\mathbf{x}_{2d} = \mathbf{x} - 2d\mathbf{n}$ and $\nabla_n \Theta = \nabla \Theta \cdot \mathbf{n}$.

Finally, the equation (29) takes the following form, where the finite difference equations (30) and (31) are used

$$\begin{aligned} & \int_{\Gamma_q} v \hat{h} d\Gamma - \int_{\Gamma_\Theta} 3 \frac{k}{d} v \hat{\Theta} d\Gamma + \int_{\Gamma_\Theta} 3 \frac{k}{d} v \Theta(\mathbf{x}_{2d}) d\Gamma + \int_{\Gamma_\Theta} 4kv \nabla_n \Theta(\mathbf{x}_d) d\Gamma + \int_{\Gamma_\Theta} kv \nabla_n \Theta(\mathbf{x}_{2d}) d\Gamma \\ & + \int_{\Gamma_s} \frac{3k}{4d} \llbracket v \rrbracket \llbracket \Theta \rrbracket_d d\Gamma - \int_{\Gamma_s} \frac{k}{2} \llbracket v \rrbracket \mathbf{n}^s \cdot \langle \nabla \Theta \rangle_d d\Gamma + \int_{\Omega} k \nabla v \cdot \nabla \Theta d\Omega - \int_{\Omega} v r d\Omega = 0 \end{aligned} \quad (32)$$

Equation (32) can be written in the form of equality of bilinear and linear forms as in eq. (21), but now the definitions of those forms are

$$\begin{aligned} A(\Theta, v) = & \int_{\Omega} k \nabla v \cdot \nabla \Theta d\Omega + \frac{3}{4} \int_{\Gamma_s} \frac{k}{d} \llbracket v \rrbracket \llbracket \Theta \rrbracket_d d\Gamma - \frac{1}{2} \int_{\Gamma_s} k \llbracket v \rrbracket \mathbf{n}^s \cdot \langle \nabla \Theta \rangle_d d\Gamma \\ & + 3 \int_{\Gamma_\Theta} \frac{k}{d} v \Theta(\mathbf{x}_{2d}) d\Gamma + 4 \int_{\Gamma_\Theta} kv \nabla_n \Theta(\mathbf{x}_d) d\Gamma + \int_{\Gamma_\Theta} kv \nabla_n \Theta(\mathbf{x}_{2d}) d\Gamma \end{aligned} \quad (33)$$

$$F(v) = \int_{\Omega} v r d\Omega - \int_{\Gamma_q} v \hat{h} d\Gamma + 3 \int_{\Gamma_\Theta} \frac{k}{d} v \hat{\Theta} d\Gamma \quad (34)$$

3.3 The standard discontinuous Galerkin method

For the sake of clarity and completeness of this paper the formulation of the problem using the standard DG method is presented in this section. Since it is now the well known method only the final equations are presented. For more information on the subject the Reader is referred to for example in [27]. The SDG method can be written in the form as in eq. (21) but the definitions of the bilinear and linear forms are now as follows

$$\begin{aligned} A(\Theta, v) = & \int_{\Omega} k \nabla v \cdot \nabla \Theta d\Omega - \int_{\Gamma_s} k \llbracket v \rrbracket \mathbf{n}^s \cdot \langle \nabla \Theta \rangle d\Gamma \\ & + \kappa \int_{\Gamma_s} k \langle \nabla v \rangle \cdot \mathbf{n}^s \cdot \llbracket \Theta \rrbracket d\Gamma + \int_{\Gamma_s} \sigma \llbracket v \rrbracket \cdot \llbracket \Theta \rrbracket d\Gamma + \int_{\Gamma_\Theta} \sigma v \Theta d\Gamma \\ F(v) = & \int_{\Omega} v r d\Omega - \int_{\Gamma_q} v \hat{h} d\Gamma + \int_{\Gamma_\Theta} \sigma v \hat{\Theta} d\Gamma \end{aligned} \quad (35)$$

where σ is so-called discontinuity penalisation parameter. The σ parameter is taken to be of order $O(h^{-1})$, where h is the characteristic element size in a mesh, e.g [37, 41]. This parameter is set in this paper as $\sigma = \frac{\sigma_0}{h}$ where σ_0 is constant. The value of the penalisation parameter is assessed to be big enough to enforce continuity of the solution on the one hand and on the other hand not too big to avoid numerical instabilities. The κ parameter has the values +1, -1 or 0. Depending on its value the following DG schemes arise:

- $\kappa = -1$ – Symmetric Interior Penalty Galerkin (SIPG) [42]
- $\kappa = 1$ – Nonsymmetric Interior Penalty Galerkin (NIPG) [28]
- $\kappa = 0$ – Incomplete Interior Penalty Galerkin (IIPG) [15]

The SDG method in SIPG version is used in the examples presented in this paper.

4 Approximation

For the IDG method two kinds of approximation are used. The first one is performed on the regular elements and the second on the skeleton elements. In the DGFD and SDG methods the approximation is constructed only on the regular finite elements. The approximation on the mesh cells in IDG, DGFD and SDG methods is constructed in the same way and all the details are described in this Section.

In IDG method the degrees of freedom are connected with the approximation on the element cells. On the other hand the skeleton finite element has no degrees of freedom. In this kind of elements the linear approximation is constructed in the skeleton normal direction. The values of approximations are taken just from the two aligned regular finite elements and first degree polynomial is constructed (Fig. 4). The approach is justified by the fact that the skeleton finite element width is much smaller in comparison to the size of neighbouring regular finite elements. The skeleton finite element is a very narrow rectangular and its length is the same as the length of the two aligned regular finite elements. The approximation order in the $\mathbf{s}^s$ direction is naturally inherited from those aligned finite elements. In eq. (9) the integration over the skeleton area is replaced by the integration along the mid-line that goes along its length. It means that to calculate such integral we need only the values at points from the mid-line. In such a situation there is no need to construct the skeleton finite element approximation since it is enough to take the values on both sides of the element to calculate the integrals over skeleton volume, what is shown in (17).

In all the DG methods regarded in this paper the approximation in the e -th single finite element is constructed with the help of local basis functions and the local degrees of freedom

$$\Theta^e(\mathbf{x}) = \sum_{i=1}^{n_e} b_i^e(\mathbf{x}) \check{\Theta}_i^e = \mathbf{b}^e \check{\Theta}^e, \quad \text{for } \mathbf{x} \in \Omega^e \quad (36)$$

where Ω^e is the region of the e -th element, $\mathbf{b}^e$ is the vector of local basis functions and $\check{\Theta}^e$ is the vector of local degrees of freedom. The 'local degrees of freedom' means that such degrees of freedom are connected with the single finite element and 'local basis functions' means that the basis functions are defined in the local coordinates connected with the element and their support is limited to the element.

The choice of the local basis functions for each cell is quite arbitrary and depends on the considered problem. However, the polynomial basis functions are widely used. In the research associated with this work various basis have been tested. In this paper the results with the use of polynomial basis functions are presented.

In eq. (36) the approximation for one element cell is shown. It can be rewritten as an approximation for the whole domain as:

$$\Theta = \Phi \check{\Theta}, \quad \mathbf{x} \in \Omega_E \quad (37)$$

where

$$\Omega_E = \begin{cases} \Omega \setminus \Omega_s & \text{in IDG} \\ \Omega & \text{in DGFD and SDG} \end{cases} \quad (38)$$

$$\Phi = [s^1 \mathbf{b}^1 \quad s^2 \mathbf{b}^2 \quad \dots \quad s^{N_e} \mathbf{b}^{N_e}] , \quad \check{\Theta} = \begin{bmatrix} \check{\Theta}^1 \\ \check{\Theta}^2 \\ \vdots \\ \check{\Theta}^{N_e} \end{bmatrix} \quad (39)$$

and Ne is the number of finite element cells and s^e is the e -th element support, i.e.

$$s^e = \begin{cases} 1 & \text{for } \mathbf{x} \in \Omega^e \\ 0 & \text{for } \mathbf{x} \notin \Omega^e \end{cases} \quad (40)$$

The jump and mean values are approximated using the same vector of degrees of freedom, i.e.

$$[\![\Theta]\!] = [\![\Phi]\!] \check{\Theta}, \quad \langle \Theta \rangle = \langle \Phi \rangle \check{\Theta}, \quad \mathbf{x} \in \Gamma_s \quad (41)$$

In the same way the approximation of the the jump and mean values at distance ϵ from Γ_s is done

$$[\![\Theta]\!]_\epsilon = [\![\Phi]\!]_\epsilon \check{\Theta}, \quad \langle \Theta \rangle_\epsilon = \langle \Phi \rangle_\epsilon \check{\Theta} \quad (42)$$

In this paper the Galerkin formulation is regarded thus the same approximation as in equations (37), (41) and (42) are applied to the test function v .

In most cases the local basis functions are polynomials taken from the Taylor expansion. Each polynomial is defined in e -th cell with the help of local coordinates. In the two-dimensional case the local coordinates (x^e, y^e) are defined in the following way

$$x^e = \frac{x - x_m^e}{h_x^e}, \quad y^e = \frac{y - y_m^e}{h_y^e} \quad (43)$$

In eq. (43) the point (x_m^e, y_m^e) is a centre of gravity of the e -th element cell and h_x^e, h_y^e are the characteristic length of the e -th cell in the x and y directions, respectively. The vectors of basis functions of various degrees p , used in this paper, are then as follow

$$\begin{aligned} \text{for } p = 1 : \quad \mathbf{b}^e &= \mathbf{b}_1^e = [1 \quad x^e \quad y^e] \\ \text{for } p = 2 : \quad \mathbf{b}^e &= \mathbf{b}_2^e = [\mathbf{b}_1^e \quad x^e y^e \quad x^{e2} \quad y^{e2}] \\ \text{for } p = 3 : \quad \mathbf{b}^e &= \mathbf{b}_3^e = [\mathbf{b}_2^e \quad x^{e3} \quad x^{e2} y^e \quad x^e y^{e2} \quad y^{e3}] \\ \text{for } p = 4 : \quad \mathbf{b}^e &= \mathbf{b}_4^e = [\mathbf{b}_3^e \quad x^{e4} \quad x^{e3} y^e \quad x^{e2} y^{e2} \quad x^e y^{e3} \quad y^{e4}] \\ &\text{and so on ...} \end{aligned} \quad (44)$$

Instead of polynomial basis functions other kinds of functions can be used, e.g. the radial basis functions are suitable to find approximate solution. For instance, the vector of local radial basis functions for hexagonal finite element may consists of seven functions, i.e.

$$\mathbf{b}^e = [\exp(-r_1^2) \quad \exp(-r_2^2) \quad \exp(-r_3^2) \quad \exp(-r_4^2) \quad \exp(-r_5^2) \quad \exp(-r_6^2) \quad \exp(-r_7^2)] \quad (45)$$

where $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ and $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)$ are the coordinates of the hexagon vertices, and (x_7, y_7) is the hexagon mass centre. For other shapes such a vector can be constructed in similar manner.

When the approximations from equations (37), (41) and (42) are substituted to the weak form equation (21), the linear system of equations is obtained

$$\mathbf{K} \check{\Theta} = \mathbf{F} \quad (46)$$

where for IDG

$$\begin{aligned} \mathbf{K} &= \int_{\Omega_E} k \mathbf{B}^T \mathbf{B} \, d\Omega + \int_{\Gamma_s} \frac{k}{w} [\![\Phi]\!]_{\frac{1}{2}w}^T [\![\Phi]\!]_{\frac{1}{2}w} \, d\Gamma + \int_{\Gamma_s} w k \langle \mathbf{B} \rangle_{\frac{1}{2}w}^T \mathbf{Q} \langle \mathbf{B} \rangle_{\frac{1}{2}w} \, d\Gamma + 2 \int_{\Gamma_\Theta} \frac{k}{w} \Phi^T \Phi \, d\Gamma \\ \mathbf{F} &= \int_{\Omega_E} \Phi^T r \, d\Omega + \int_{\Gamma_s} \langle \Phi \rangle_{\frac{1}{2}w}^T w r \, d\Gamma - \int_{\Gamma_q} \Phi^T \hat{h} \, d\Gamma + 2 \int_{\Gamma_\Theta} \Phi^T \frac{k}{w} \hat{\Theta} \, d\Gamma \end{aligned} \quad (47)$$

For DGFD

$$\begin{aligned}
\mathbf{K} &= \int_{\Omega} k \mathbf{B}^T \mathbf{B} d\Omega + \frac{3}{4} \int_{\Gamma_s} \frac{k}{d} [\![\Phi]\!]^T [\![\Phi]\!]_d d\Gamma - \frac{1}{2} \int_{\Gamma_s} k [\![\Phi]\!]^T \mathbf{n}^{sT} \langle \mathbf{B} \rangle_d d\Gamma \\
&\quad + 3 \int_{\Gamma_{\Theta}} \frac{k}{d} \Phi^T \Phi(\mathbf{x}_{2d}) d\Gamma + 4 \int_{\Gamma_{\Theta}} k \Phi^T \mathbf{B}_n(\mathbf{x}_d) d\Gamma + \int_{\Gamma_{\Theta}} k \Phi^T \mathbf{B}_n(\mathbf{x}_{2d}) d\Gamma \\
\mathbf{F} &= \int_{\Omega} \Phi^T r d\Omega - \int_{\Gamma_q} \Phi^T \hat{h} d\Gamma + 3 \int_{\Gamma_{\Theta}} \frac{k}{d} \Phi^T \hat{\Theta} d\Gamma
\end{aligned} \tag{48}$$

where $\mathbf{B} = \nabla \Phi$ and $\mathbf{B}_n = \nabla_n \Phi$.

And for SDG

$$\begin{aligned}
\mathbf{K} &= \int_{\Omega} k \mathbf{B}^T \mathbf{B} d\Omega - \int_{\Gamma_s} k [\![\Phi]\!]^T \mathbf{n}^{sT} \langle \mathbf{B} \rangle d\Gamma + \kappa \int_{\Gamma_s} k \langle \mathbf{B} \rangle^T \mathbf{n}^s [\![\Phi]\!] d\Gamma \\
&\quad \int_{\Gamma_s} \sigma [\![\Phi]\!]^T [\![\Phi]\!] d\Gamma + \int_{\Gamma_{\Theta}} \sigma \Phi^T \Phi d\Gamma \\
\mathbf{F} &= \int_{\Omega} \Phi^T r d\Omega - \int_{\Gamma_q} \Phi^T \hat{h} d\Gamma + \int_{\Gamma_{\Theta}} \sigma \Phi^T \hat{\Theta} d\Gamma
\end{aligned} \tag{49}$$

5 Examples

In this section examples are discussed that illustrate already presented approaches. In the first three examples the boundary value problem on a square domain $[-1, 1] \times [-1, 1]$ is considered

$$-\Delta \Theta = f(x, y) \tag{50}$$

where the boundary conditions and right hand side function $f(x, y)$ are derived from the exact solution $\Theta(x, y)$ that is:

$$\Theta(x, y) = x^2 y + \sin(2\pi x) \sin(2\pi y) \tag{51}$$

and the right-hand side function is:

$$f(x, y) = 2y - 8\pi^2 \sin(2\pi x) \sin(2\pi y) \tag{52}$$

The exact solution of the problem is depicted in Fig. 6. It can be noticed that the function strongly fluctuates and has quite big gradients.

In the last example the problem of heat transfer through a domain with a rectangular hole is considered. Various configurations of meshes are regarded including NSC and NC elements.

In the discussed examples the error of the approximate solutions is measured. The definition of the error at an arbitrary point in Ω is defined in the following way

$$e = |\Theta - \bar{\Theta}| \quad \text{in } \Omega \tag{53}$$

where e is the error of the approximate solution, Θ is the exact solution, and $\bar{\Theta}$ is the approximate solution. In order to measure the global error of the approximate solution the L_2 norm and H_1 semi-norm are used, which are respectively defined as follows

$$\|e\|_{L_2} = \sqrt{\int_{\Omega} (\Theta - \bar{\Theta})^2 d\Omega}, \quad |e|_{H_1} = \sqrt{\int_{\Omega} (\nabla \Theta - \nabla \bar{\Theta})^T (\nabla \Theta - \nabla \bar{\Theta}) d\Omega} \tag{54}$$

Figure 6: Exact solution for the considered example

(a) Polygonal mesh

(b) Triangular mesh

Figure 7: Coarse meshes used in the first example

5.1 Examples for fixed meshes

The considered problem has been solved using three methods: DGFD, IDG and SDG on fixed polygonal and triangular meshes. The analysis has been performed for 18th-element coarse meshes, which are presented in Fig. 7. The calculations have been done for various approximation degree, but in this paper the results for 4th degree basis functions from eq. (44) are presented. The results are shown in the form of the maps with the approximate solution and the solution error. Fig. 8 shows the results for DGFD method. In Fig. 9 the IDG results are presented. Finally, the results obtained for SDG method are depicted in Fig. 10.

All the methods have given quite accurate results for both of the meshes. However, the best results are for DGFD method on the polygonal mesh. The conclusion is affirmed in the next example.

5.2 Convergence analysis

In this section the convergence analysis of the problem defined in eqs. from (50) to (52) for DGFD, IDG and SDG methods are presented. In these analysis both the polynomial and triangular meshes are used

(a) Approximate solution for polygonal mesh

(b) Solution error for polygonal mesh

(c) Approximate solution for triangular mesh

(d) Solution error for triangular mesh

Figure 8: DGFD solution for coarse polygonal and triangular meshes in first example

(a) Approximate solution for polygonal mesh

(b) Solution error for polygonal mesh

(c) Approximate solution for triangular mesh

(d) Solution error for triangular mesh

Figure 9: IDG solution for coarse polygonal and triangular meshes for first example

(a) Approximate solution for polygonal mesh

(b) Solution error for polygonal mesh

(c) Approximate solution for triangular mesh

(d) Solution error for triangular mesh

Figure 10: SDG solution for coarse polygonal and triangular meshes in first example

(a) DGFD method, polygonal mesh

(b) DGFD method, triangular mesh

Figure 11: Convergence of DGFD method for polygonal and triangular meshes

in order to compare the convergence for these two kinds of meshes. The calculations have been done for the approximation degrees from 1 up to 5. The results are presented in Figs. 11 to 13 and have the form of charts of number of degrees of freedom (ndof) in relation to the error in L_2 norm. The charts are in the logarithmic scales.

As can be seen from these figures the best results have been obtained for DGFD method for both the polygonal and triangular meshes. The IDG and SDG methods give quite similar results for both kinds of meshes but for higher approximation degrees the both methods stop to converge for higher number of degrees of freedom.

It can be concluded here that from these three methods the DGFD approach gives the best results and is still stable for higher degrees of approximation and higher number of degrees of freedom. The DGFD method can be successfully applied to the polygonal meshes as well as to the standard triangular meshes. For these two kinds of meshes the obtained results are quite good and much better in comparison to the other two approaches. That is why, following examples in this paper are presented only for the DGFD method.

5.3 Examples for non-typical finite elements

In the VEM the finite element cells do not have to be convex and simply-connected. The same refers to the approach presented in this paper since not only typical polygonal or triangular meshes can be used to obtain the approximate solution. The VFEM requires shape functions which calculation is quite complicated. In the proposed approach no shape functions are needed, in contrary to the VEM. That is why in methods based on DG approach the finite element cells do not necessary have to be simply-connected or connected regions. In the following examples such meshes with finite element cells which are non-convex, non-simply-disconnected and non-connected are presented. For the sake of clarity the meshes in these examples consist of three or two elements. The DGFD method has been used to perform the calculations in these examples.

The meshes with non-convex finite elements are shown in Fig. 14. The mesh in Fig. 14a has three finite elements coloured in red, green and blue. Fig. 14b shows the mesh with only two finite elements (red and green), where the red finite element is NC. It means that the element consists of two disconnected parts, but the two parts have a common set of degrees of freedom. For the three-element mesh the 10-th degree basis functions are used that gives 66 degrees of freedom for each element and 198 dof for the whole

(a) IDG method, polygonal mesh

(b) IDG method, triangular mesh

Figure 12: Convergence of IDG method for polygonal and triangular meshes

(a) SDG method, polygonal mesh

(b) SDG method, triangular mesh

Figure 13: Convergence of SDG method for polygonal and triangular meshes

(a) Mesh that consists of three finite elements with non-convex cell (red and green)

(b) Mesh that consists of two finite elements with non-convex cells (red and green) and non-connected cell (red)

Figure 14: Meshes with non-convex and non-connected finite elements

Figure 15: Approximate solution and solution error for mesh from Fig. 14a and with 10-th approximation degree

mesh. On the other side, for the two-element mesh the 12-th degree basis vector has been chosen it leads to 91 dof for an element and 192 dof for the whole mesh. The results which show approximate solutions and their errors for these meshes are presented in Figs. 15 and 16. It can be noticed that for both of the meshes the results are quite accurate in spite of non-convexity and non-connected finite elements.

Similar calculations for two meshes for NSC elements and additionally with NC finite element have been performed. The meshes are presented in Fig. 17. The mesh in Fig. 17a consists of three elements from which two are NSC (red and green). The mesh in Fig. 17b is constructed from two elements where one element is NSC (green) and one NC (red). Like in previous example the 10-th degree basis vector has been used for the 3-element mesh and in a case of 2-element mesh the 12-th degree basis vector has been chosen.

The results for that meshes are presented in Figs. 18 and 19. Also in this case the obtained results are quite satisfactory which means that it is possible to use NSC finite elements combined with NC finite elements.

As it is shown in these examples the DGFD method is very flexible and can be used for finite elements. In the approach the finite element mesh can consists of elements which are, in practice, arbitrary polygons and can be NSC or even NC. It allows to construct the finite element meshes in which the finite element cells can be constructed in arbitrary way to match the considered problem best.

5.4 Domain with a hole

In this example the heat flow through the domain with a hole is considered. Fig. 20 shows the domain with boundary conditions and the reference solution of the problem. The reference solution has been obtained on very dense mesh with 3-rd degree finite elements using standard FEM. For the sake of clarity all the values in this example are dimensionless and the conductivity parameter k is set to one.

In the standard approach the finite element cells have to surround the hole. It enforces the size of the finite elements since they have to be adjusted to the dimension of the hole. In contrary to the standard approach the DG methods allow to use just a few NSC finite elements to reproduce the hole in the domain, what is just presented in this example.

The problem defined in Fig. 20a has been solved by DGFD method using five meshes shown in Fig. 21. The first mesh, Fig. 21a, consists of 16-th quadrilateral finite elements that surrounds the hole in the domain. The second mesh, Fig. 21b, is constructed of four quadrilateral cells surrounding the hole.

Figure 16: Approximate solution and solution error for mesh from Fig. 14b and 12-th approximation degree

Figure 17: Meshes with non-simply-connected and non-connected finite elements

Figure 18: The approximate solution and solution error for mesh from Fig. 17a and 10-th approximation degree

(a) Approximate solution

(b) Solution error

Figure 19: The approximate solution and solution error for mesh from Fig. 17b and 12-th approximation degree

(a) Domain with boundary conditions

(b) Exact solution

Figure 20: Domain with a hole for heat transfer example

In the Fig. 21c the third mesh is presented that consists of four NSC finite elements. The fourth mesh in Fig. 21d is similar to the previous one but now it consists of only two NC finite elements. The last mesh, Fig. 21e, has only one NSC cell. The orders of the finite elements in these meshes are set in such a way that the global ndof is similar for each of the meshes.

The results are presented in Figs. from 22 to 26. In all the cases the errors are concentrated near the hole and the levels of the error is comparable. However, the worst results have been obtained for the mesh in Fig.21a.

In the example it is shown that the NSC finite elements can be used for the domain with holes. The results obtained for that meshes with NSC elements are comparable with the results based on the standard meshes. Nevertheless, it is much easier to use NSC elements to discretize domains with holes in comparison to standard meshing procedures.

5.5 Mesh with fish-shape cells

In this example the problem defined in eq. (50) is solved on the mesh in which each cell has the shape of fish, Fig. 27. Each cell is non-convex in the vicinity of the fish tail. The mesh consists of 40 cells in the $[-1, 1] \times [-1, 1]$ domain. Each cell, except those on close to outer boundary, is a polygonal with 14 vertices. The cells by the outer boundary are the appropriate parts of the fish-cell.

In spite of the fact that the mesh is non-typical the approximate solutions obtained on the mesh are quite well. The results for the approximation basis of degree 4, 6 and 8 are presented in Figs. from 28 to 30.

6 Conclusions

In the paper the discontinuous Galerkin (DG) method on the polygonal meshes was considered. Three versions of DG method were regarded: interface DG (IDG) method, DG with finite difference (DGFD) method and standard DG (SDG) method. All the versions of DG method are constructed with the help of a set of arbitrary basis functions. In the examples the polynomial basis functions were used, though other kinds of basis functions can be applied.

It has been shown in the examples that quite arbitrary polygonal finite elements can be used in the calculations. Not only can they be non-convex but also can they have holes i.e. non-simply-connected (NSC) or can consist of few separated parts, i.e. non-connected (NC). In the examples all of the types of finite elements have been used and the obtained results were quite satisfactory.

The three versions of DG method, i.e. IDG, DGFD and SDG methods, were compared in the paper. It has been evidently shown that the DGFD method gives better results than the other two methods and is more stable for higher degrees of approximation. In the paper the DGFD method has been used with fourth degree finite difference (FD) schemes for compatibility and boundary conditions. A further research is necessary to check the influence of FD degree on the approximate solution.

Besides that, the results obtained on polygonal meshes have been compared with the ones obtained on standard triangular ones. It can be concluded that the usage of polygonal meshes gives better results than for standard triangular meshes.

In the future research we plan to apply the DGFD method to other kinds of problems in three-dimensional analysis. The presented approach can be, for example, directly applied to elasticity or thermo-elasticity linear or non-linear problems with non-stationary heat flow.

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(a) Mesh with sixteen quadrilateral elements

(b) Mesh with four quadrilateral elements

(c) Mesh with four non-simply-connected elements

(d) Mesh with two (in red and green) non-connected elements

(e) Mesh with one non-simply-connected element

Figure 21: Meshes for domain with hole

(a) 1-st degree, 64 ndof

(b) 2-nd degree, 96 ndof

Figure 22: Approximate solutions for the mesh in Fig. 21a

(a) 1-st degree, 64 ndof

(b) 2-nd degree, 96 ndof

Figure 23: Approximate solutions for the mesh in Fig. 21b

(a) 4-th degree, 68 ndof

(b) 5-th degree, 96 ndof

Figure 24: Approximate solutions for the mesh in Fig. 21c

(a) 6-th degree, 64 ndof

(b) 8-th degree, 94 ndof

Figure 25: Approximate solutions for the mesh in Fig. 21d

(a) 10-th degree, 66 ndof

(b) 12-th degree, 95 ndof

Figure 26: Approximate solutions for the mesh in Fig. 21e

Figure 27: Mesh with fish-shape finite elements

(a) Approximate solution

(b) Error of approximate solution

Figure 28: Approximate solution and its error on the fish-mesh for 4th approximation degree

(a) Approximate solution

(b) Error of approximate solution

Figure 29: Approximate solution and its error on the fish-mesh for 6th approximation degree

(a) Approximate solution

(b) Error of approximate solution

Figure 30: Approximate solution and its error on the fish-mesh for 8th approximation degree

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