Supplementary Material for Detection of non-Gaussianity

Teresa Ledwina and Grzegorz Wyłupek

1. Proofs

1.1. Proof of Proposition 1

Since C_j is location and scale invariant, therefore, we restrict attention to $\mu = 0$ and $\sigma = 1$. Consider L -statistics $\mathcal{L}_j = \sqrt{n} \int_0^1 Q_n(t) h_j(t) dt$, $j \geq 1$. The proof relies on showing asymptotic normality of properly standardized $\mathcal{L}_j$'s.

By (6),

$$\mathcal{L}_j = \sqrt{n} \left\{ w_{j1} X_{1:n} + w_{jn} X_{n:n} + \sum_{i=2}^{n-1} \left[\int_{(i-1)/n}^{i/n} h_j(t) \right] X_{i:n} \right\}. \quad (S1)$$

For normal order statistics it holds that $\sqrt{2 \log n} (X_{1:n} + b_n)$ and $\sqrt{2 \log n} (X_{n:n} - b_n)$ have nondegenerate limit laws, where b_n solves $\Phi(b_n) = 1 - 1/n$ and $b_n = \sqrt{2 \log n} + o(1)$; cf. David and Nagaraja (2003), Section 10.5.

The forms of w_{j1} and w_{jn} and the above imply that $\sqrt{n} w_{j1} X_{1:n}$ and $\sqrt{n} w_{jn} X_{n:n}$ tend to 0 in probability. Therefore, the problem reduces to asymptotic normality of the third term in (S1). This, however, follows from Example 1 in Shorack (1972), as all moments of the underlying distribution exist under **H**. The centering constant has the form $m_j(n) = \sqrt{n} \int_0^1 \Phi^{-1}(t) h_j(t) dt$ and equals $\sqrt{n}$ if $j = 1$ and 0 otherwise. The asymptotic variance of $\mathcal{L}_j - m_j(n)$ is given by

$$\sigma_j^2 = \int_0^1 \int_0^1 [\min\{s, t\} - st] h_j(s) h_j(t) d\Phi^{-1}(s) d\Phi^{-1}(t)$$

and $\sigma_j^2 = 1/(j+1)$. The last easily follows from the fact that $H_j(\Phi^{-1}(t))$ is the eigenfunction corresponding to the eigenvalue $1/(j+1)$ of the operator $\mathbf{L} : L_2(0, 1) \rightarrow L_2(0, 1)$, given by

$$\mathbf{L}f(t) = \int_0^1 \frac{\min\{s, t\} - st}{\varphi(\Phi^{-1}(s))\varphi(\Phi^{-1}(t))} f(s) ds,$$

cf. del Barrio et al. (2000), p. 38; note that the indices of the Hermite system are shifted by 1 in that paper.

The above and the Cramér-Wold device imply that $([\mathcal{L}_1 - m_1(n)]/\sigma_1, \dots, [\mathcal{L}_D - m_D(n)]/\sigma_D)$ has asymptotic D -dimensional standard normal distribution. Since $N_A \leq N_D$ the conclusion follows.

1.2. Proof of Proposition 2

It is enough to concentrate on A , as the proof for A^o is an analogous one.

Let $\xrightarrow{P}$ denote the convergence in probability. Note that L -statistics $\mathcal{L}_j^\circ = \int_0^1 Q_n(t) h_j(t) dt$, $j \geq 1$, satisfy (i) of Theorem 2.1 of van Zwet (1980) with $J_N = h_j$, $g = F^{-1}$, and $p = q = 2$. Hence $\mathcal{L}_j^\circ \xrightarrow{P} c_j = \int_0^1 h_j(t) F^{-1}(t) dt < +\infty$. In particular, $\hat{\sigma}_n = \mathcal{L}_1^\circ \xrightarrow{P} \int_0^1 \Phi^{-1}(t) F^{-1}(t) dt$.

The next fact we need is that $P(\mathcal{R}_n > r_\alpha) \rightarrow 1$, as $n \rightarrow \infty$, under any alternative F with finite second moment. Csörgő (2003), p. 757, briefly sketched some conditions for consistency of a large class of correlation tests. In our special case the above assumption and simple argument are sufficient. Namely, recall that $\mathcal{R}_n = n(1 - \hat{\sigma}_n^2/S_n^2)$. By Theorem 3 of del Barrio et al. (1999), under **H**, $\mathcal{R}_n - c_n$ converges in distribution while $c_n = \log \log n + \log 2 + \gamma + o(1)$, where γ is Euler's constant, cf. the formula (3.20) in del Barrio et al. (2000). Hence, the critical value r_α appearing in (8) tends to infinity and satisfies $r_\alpha = O(\log \log n)$ while $P(\mathcal{R}_n > r_\alpha) = P(\hat{\sigma}_n^2/S_n^2 < r_\alpha^*)$ with $r_\alpha^* = 1 + o(1)$. On the other hand, $\hat{\sigma}_n^2$ and S_n^2 are location

free. Therefore, without loss of generality, we can assume that $EX = 0$. Applying then the aforementioned law of large numbers and Schwarz's inequality we infer that $\hat{\sigma}_n^2/S_n^2 \xrightarrow{\mathcal{P}} g(F)$, where $g(F) < 1$ for all F outside $\mathbf{H}$.

To prove (i), observe that, by the above mentioned law of large numbers, $C_\ell^2 \xrightarrow{\mathcal{P}} +\infty$ and $N_\ell \xrightarrow{\mathcal{P}} +\infty$. Therefore, by $P(A \geq \ell) \geq P(A_2 \geq \ell, \mathcal{R}_n > r_\alpha)$, to justify that $P(A \geq \ell) \rightarrow 1$, it suffices to prove that $P(A_2 \geq \ell) \rightarrow 1$ as $n \rightarrow +\infty$. In view of

$$P(A_2 < \ell) \leq \sum_{l=1}^{\ell-1} P(N_l - 2l \geq N_\ell - 2\ell) = \sum_{l=1}^{\ell-1} P\left(\sum_{j=l+1}^{\ell} C_j^2 \leq 2(\ell - l)\right)$$

and the convergence $C_\ell^2 \xrightarrow{\mathcal{P}} +\infty$, the last expression tends to 0 and the proof of (i), in the case of the rule A , is concluded.

Since N_A is bounded in probability under $\mathbf{H}$, to prove (ii), it suffices to show that, under the alternative Q , $N_A \xrightarrow{\mathcal{P}} +\infty$. This, however, follows by the above.

2. Plots of $h_1, \dots, h_6$

Since the functions h_j 's, defined on $(0, 1)$, are unbounded, we confine ourselves to drawing them on $[0.001, 0.999]$.

Fig. S1. The functions $h_1, \dots, h_6$ on $[0.001, 0.999]$.

3. Simulations related to testing $\mathbf{H}$

3.1. Finite sample corrections $e_{j,0}$, $v_{j,0}$ and the related empirical quantiles

Table S1. *Monte Carlo estimates of means and standard deviations of C_j 's, $j = 2, \dots, 21$, under $\mathbf{H}$, $n = 75$ ($\mathbf{nr} = 100\,000$) and $n = 244^2$ ($\mathbf{nr} = 10\,000$).*

n	j		2	3	4	5	6	7	8	9	10	11
			0.000	-0.743	-0.001	-0.581	0.002	0.325	-0.002	0.066	0.002	-0.258
			-0.010	-0.087	-0.023	-0.303	-0.001	-0.589	0.006	-0.423	-0.006	0.185
			0.948	0.798	0.622	0.588	0.613	0.534	0.521	0.537	0.476	0.493
n	j		1.004	0.994	0.989	0.971	0.892	0.808	0.718	0.700	0.715	0.685
			-0.001	0.250	0.000	-0.129	0.000	-0.016	-0.001	0.129	0.001	-0.184
			0.004	0.272	0.000	-0.224	-0.006	-0.082	0.009	0.235	-0.008	-0.135
			0.482	0.445	0.474	0.433	0.442	0.441	0.410	0.437	0.399	0.416
n	j		0.640	0.644	0.647	0.608	0.600	0.604	0.579	0.565	0.580	0.551
			-0.001	0.250	0.000	-0.129	0.000	-0.016	-0.001	0.129	0.001	-0.184
			0.004	0.272	0.000	-0.224	-0.006	-0.082	0.009	0.235	-0.008	-0.135
			0.482	0.445	0.474	0.433	0.442	0.441	0.410	0.437	0.399	0.416
n	j		0.640	0.644	0.647	0.608	0.600	0.604	0.579	0.565	0.580	0.551
			-0.001	0.250	0.000	-0.129	0.000	-0.016	-0.001	0.129	0.001	-0.184
			0.004	0.272	0.000	-0.224	-0.006	-0.082	0.009	0.235	-0.008	-0.135
			0.482	0.445	0.474	0.433	0.442	0.441	0.410	0.437	0.399	0.416

Table S1 shows that the estimates are distinct from the limiting values even for n as large as 244^2 . Therefore, we also investigated the successive empirical quantiles of C_j 's under $\mathbf{H}$.

Table S2. *Selected empirical q -quantiles of $C_2, \dots, C_{21}$, calculated under $\mathbf{H}$ and $n = 75$, $\mathbf{nr} = 100\,000$, and asymptotic q -quantiles.*

j	q												
	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
2	-2.299	-1.655	-1.283	-0.846	-0.529	-0.253	0.009	0.255	0.525	0.846	1.295	1.666	2.330
3	-2.141	-1.599	-1.281	-0.884	-0.583	-0.312	-0.056	0.197	0.489	0.831	1.320	1.736	2.518
4	-2.330	-1.630	-1.288	-0.835	-0.513	-0.259	-0.006	0.263	0.526	0.859	1.308	1.665	2.427
5	-2.327	-1.682	-1.302	-0.836	-0.507	-0.232	0.031	0.283	0.557	0.864	1.301	1.674	2.330
6	-2.455	-1.682	-1.296	-0.860	-0.536	-0.258	-0.003	0.245	0.516	0.834	1.267	1.656	2.364
7	-2.374	-1.658	-1.291	-0.851	-0.534	-0.256	-0.002	0.240	0.509	0.843	1.281	1.659	2.299
8	-2.367	-1.663	-1.284	-0.833	-0.524	-0.255	0.000	0.263	0.538	0.864	1.290	1.679	2.390
9	-2.227	-1.621	-1.262	-0.836	-0.528	-0.266	-0.019	0.231	0.510	0.831	1.269	1.678	2.453
10	-2.387	-1.660	-1.300	-0.858	-0.528	-0.247	0.007	0.256	0.516	0.840	1.293	1.669	2.369
11	-2.363	-1.660	-1.290	-0.837	-0.501	-0.223	0.025	0.274	0.534	0.844	1.249	1.610	2.273
12	-2.417	-1.682	-1.293	-0.847	-0.533	-0.254	-0.005	0.247	0.526	0.859	1.292	1.670	2.335
13	-2.290	-1.621	-1.247	-0.837	-0.536	-0.277	-0.034	0.225	0.510	0.836	1.300	1.671	2.341
14	-2.276	-1.645	-1.280	-0.836	-0.518	-0.251	-0.001	0.262	0.528	0.838	1.294	1.688	2.425
15	-2.406	-1.705	-1.308	-0.836	-0.515	-0.239	0.019	0.280	0.535	0.841	1.288	1.622	2.363
16	-2.385	-1.654	-1.285	-0.848	-0.518	-0.251	-0.010	0.247	0.522	0.844	1.273	1.611	2.298
17	-2.413	-1.667	-1.271	-0.853	-0.527	-0.256	-0.009	0.262	0.539	0.859	1.318	1.673	2.303
18	-2.286	-1.620	-1.290	-0.834	-0.508	-0.237	0.005	0.251	0.514	0.832	1.248	1.620	2.337
19	-2.315	-1.632	-1.307	-0.872	-0.563	-0.279	-0.018	0.241	0.516	0.845	1.289	1.677	2.422
20	-2.288	-1.600	-1.251	-0.823	-0.504	-0.249	0.000	0.233	0.503	0.817	1.284	1.638	2.334
21	-2.393	-1.685	-1.287	-0.833	-0.511	-0.228	0.018	0.293	0.563	0.875	1.297	1.647	2.288
	-2.326	-1.645	-1.282	-0.842	-0.524	-0.253	0.000	0.253	0.524	0.842	1.282	1.645	2.326

The evidence shows that empirical quantiles are not very far from the limiting ones given in the last row of Table S2. Moreover, note that the consecutive quantiles of C_2 are closer to the limiting ones than the respective quantiles of C_3 . Recall also that, under $\mathbf{H}$, the impacts of C_2 and C_3 to the overall empirical distributions of $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{A}^o}^o$ are greater than 93% in the case when $q = 1 - \alpha = 0.95$; cf. Table 1. Additionally, observe that most of the 0.95 empirical quantiles of odd components $C_3, C_5, \dots, C_{21}$ are larger than the limiting ones. The above results in much larger 0.05 critical values of $\mathcal{N}_{\mathcal{A}^o}^o$ than those of $\mathcal{N}_{\mathcal{A}}$. For very large sample sizes the situation greatly improves; see Table 1. Also, in the case of testing simple null hypothesis $\mathbf{H}^*$ such a drawback does not appear; cf. Table S6.

Though the marginal distributions of C_j 's are reasonably close to the limiting ones $N(0, 1)$, the joint distribution of $(C_2, \dots, C_{21})$ under $\mathbf{H}$ is far from the standard multivariate normal law. The reason is that some of the correlations between C_i and C_j are relatively large even for n as large as 244^2 , see Table S3, below.

Table S3. Estimated covariances of $\mathcal{C}_2, \dots, \mathcal{C}_{21}$ under $\mathbf{H}$ and $n = 75$ ($\mathbf{nr} = 100\,000$), $n = 244^2$ ($\mathbf{nr} = 10\,000$). Upper triangular matrix $\text{cov}(\mathcal{C}_i, \mathcal{C}_j)$, $i \leq j$, $n = 75$. Lower triangular matrix $\text{cov}(\mathcal{C}_i, \mathcal{C}_j)$, $i \geq j$, $n = 244^2$.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	1.02	0.00	-0.41	0.00	-0.07	0.00	0.20	0.00	-0.14	0.00	0.01	-0.00	0.08	0.00	-0.13	-0.00	0.12	0.00	-0.07	0.00
2	1.00	1.01	0.00	-0.82	0.00	0.24	0.01	0.16	-0.01	-0.30	0.01	0.27	-0.00	-0.12	0.00	-0.04	-0.00	0.15	-0.00	-0.21
3	0.02	1.02	1.01	0.00	-0.86	0.00	0.57	0.00	-0.17	-0.00	-0.14	0.00	0.28	0.00	-0.30	0.00	0.23	0.00	-0.10	0.00
4	-0.02	0.01	0.99	1.01	0.00	-0.76	0.00	0.41	0.00	-0.17	0.00	-0.01	0.00	0.10	0.00	-0.13	0.00	0.11	0.00	-0.08
5	0.01	-0.02	-0.00	1.00	1.01	0.00	-0.90	0.00	0.52	0.00	-0.13	0.00	-0.11	0.00	0.23	0.00	-0.24	0.00	0.16	0.00
6	-0.02	0.01	-0.07	-0.00	1.03	1.02	0.00	-0.91	0.00	0.72	0.00	-0.43	-0.00	0.11	0.00	0.13	0.00	-0.25	0.00	0.30
7	-0.01	-0.05	-0.02	-0.22	-0.01	1.02	1.01	0.00	-0.86	0.00	0.55	0.01	-0.30	0.00	0.10	0.00	0.04	0.00	-0.11	0.00
8	0.00	0.01	-0.12	0.01	-0.46	0.00	1.04	1.02	0.00	-0.95	0.00	0.72	0.00	-0.37	0.00	0.07	0.00	0.13	0.00	-0.24
9	-0.00	-0.02	0.01	-0.18	0.02	-0.65	-0.01	1.02	1.01	0.00	-0.92	0.00	0.74	0.00	-0.53	0.00	0.30	0.00	-0.07	0.00
10	-0.01	-0.03	-0.01	-0.00	-0.18	0.00	-0.68	-0.00	0.99	1.02	0.00	-0.92	0.00	0.66	0.00	-0.37	0.00	0.15	0.00	0.01
11	0.00	0.01	0.01	0.03	-0.02	-0.01	0.01	-0.63	-0.01	1.02	1.01	0.00	-0.96	0.00	0.82	0.00	-0.59	0.00	0.23	0.00
12	0.02	0.03	0.05	-0.01	0.17	0.00	0.09	-0.01	0.02	0.99	1.01	1.01	-0.01	-0.92	0.01	0.72	-0.01	-0.52	0.01	0.33
13	-0.00	0.02	-0.02	0.10	0.01	0.28	-0.01	0.13	0.01	-0.77	-0.02	0.98	1.01	0.00	-0.96	0.00	0.79	0.00	-0.53	0.00
14	-0.01	-0.01	0.00	0.01	0.08	0.00	0.25	0.02	-0.02	-0.74	0.01	0.98	1.01	1.01	-0.01	-0.95	0.01	0.82	-0.01	-0.66
15	-0.01	-0.02	0.01	-0.06	-0.00	-0.07	0.01	0.14	-0.01	0.20	0.00	0.01	0.99	1.01	1.01	0.01	-0.94	0.00	0.75	0.00
16	0.01	-0.00	-0.03	-0.01	-0.17	0.01	-0.21	-0.02	0.01	0.30	0.01	-0.79	-0.02	0.98	1.01	1.01	-0.01	-0.97	0.01	0.87
17	0.01	-0.00	-0.01	-0.04	-0.00	-0.16	0.00	-0.15	0.00	0.20	0.01	-0.79	-0.02	0.02	1.01	1.01	1.01	0.01	-0.94	0.00
18	-0.01	0.01	0.00	0.00	0.06	-0.01	-0.01	0.00	-0.16	0.01	0.10	-0.02	0.29	-0.77	-0.01	1.00	1.01	1.01	-0.01	-0.97
19	-0.01	0.02	0.01	0.07	0.01	0.02	0.00	0.04	0.00	-0.27	-0.01	0.08	0.02	-0.02	-0.84	0.01	1.01	1.01	1.01	0.01
20	0.02	-0.01	0.03	-0.00	0.08	0.01	0.15	0.01	0.00	-0.02	-0.24	0.02	0.10	0.37	-0.00	-0.85	0.01	1.00	1.00	1.01
21	0.01	-0.01	-0.01	-0.02	-0.01	-0.04	0.01	0.06	0.00	0.09	0.00	-0.18	-0.01	0.01	0.40	-0.01	-0.83	-0.00	1.01	1.01
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

3.2. Empirical powers under moderate sample sizes. Symmetrical and asymmetrical alternatives

We supplement our simulations of Section 4 by three cases of symmetrical alternatives from Krauczi (2009) which are displayed in Fig. S2. Moreover, we present six cases of asymmetrical alternatives in Figs. S3 and S4. Some of the notations of the alternatives have already been described in the paper. The remaining ones are defined as follows.

$Beta(a, b)$, $a > 0$, $b > 0$, denotes the beta distribution. $Logistic(a, b)$, $a \in \mathbb{R}$, $b > 0$, stands for the logistic distribution with the shift parameter a and the scale parameter b . $SB(g, d)$, $g \in \mathbb{R}$, $d > 0$, denotes the bounded Johnson's distribution of a random variable $[\exp\{(Z - g)/d\}]/[1 + \exp\{(Z - g)/d\}]$.

Fig. S2. Symmetrical alternatives. First row: medians of $\mathcal{C}_{2j+1}$'s, $j = 1, \dots, 10$, estimated under $n = 150$. Second row: empirical powers (in percentages) against n ; $\alpha = 0.05$. $nr = 10\,000$.

Fig. S3. Asymmetrical alternatives. First row: medians of $\mathcal{C}_j$'s, $j = 2, \dots, 21$, estimated under $n = 150$. Second row: empirical powers (in percentages) against n ; $\alpha = 0.05$. $\text{nr} = 10\,000$.

Fig. S4. Asymmetrical alternatives. First row: medians of $\mathcal{C}_j$'s, $j = 2, \dots, 21$, estimated under $n = 150$. Second row: empirical powers (in percentages) against n ; $\alpha = 0.05$. $\text{nr} = 10\,000$.

4. Simulations related to testing $\mathbf{H}^*$

4.1. Definitions of $\mathcal{A}^*$ and the related objects

Analogously as in Section 2.3, now we model the function $Q - \Phi^{-1}$ using nested family of related models

$$\mathbf{M}^* : \{\delta_0 h_0; \delta_0 \in \mathbb{R}\}, \{\delta_0 h_0 + \delta_1 h_1; \delta_0, \delta_1 \in \mathbb{R}\}, \dots, \{\delta_0 h_0 + \dots + \delta_{d^*(n)-1} h_{d^*(n)-1}; \delta_0, \dots, \delta_{d^*(n)-1} \in \mathbb{R}\}$$

of the consecutive dimensions $1, 2, \dots, d^*(n)$, where $\{d^*(n)\}$ is nondecreasing sequence of natural numbers.

Since, similarly as in the case of testing $\mathbf{H}$, under finite sample sizes, both means and standard deviations of C_j^* 's, $j = 0, 1, \dots, 19$, are far from the limiting ones (see Table S4 for evidence), we introduce the rule based on the corrected components $\mathcal{C}_j^* = (C_j^* - e_{j,0}^*)/v_{j,0}^*$ and the related $\mathcal{N}_k^* = \sum_{j=0}^{k-1} \{\mathcal{C}_j^*\}^2$. The auxiliary selection rule has the form

$$\mathcal{A}_{s^*}^* = \min\{k : \mathcal{N}_k^* - s^*k \geq \mathcal{N}_j^* - s^*j, \quad k, j = 1, \dots, d^*(n)\}$$

while the final choice of a dimension k in the statistic $\mathcal{N}_k^*$ is done by

$$\mathcal{A}^* = \begin{cases} \mathcal{A}_{s^*(n,\alpha)}^* & \text{if } \mathcal{R}_n^* \leq r_\alpha^*, \\ \mathcal{A}_2^* & \text{otherwise,} \end{cases}$$

where $s^*(n, \alpha)$ is chosen as the smallest value of s^* such that, under $\mathbf{H}^*$, it holds that $P(\mathcal{A}_{s^*}^* = 1) \geq 1 - \alpha$ while r_α^* is the $(1 - \alpha)$ -quantile of the statistic $\mathcal{R}_n^*$.

When focusing on detection symmetrical alternatives to $\mathbf{H}^*$ we model $Q - \Phi^{-1}$ with use of

$$\mathbf{M}^{o*} : \{\delta_1 h_1; \delta_1 \in \mathbb{R}\}, \{\delta_1 h_1 + \delta_3 h_3; \delta_1, \delta_3 \in \mathbb{R}\}, \dots, \{\delta_1 h_1 + \dots + \delta_{d^{o*}(n)-1} h_{d^{o*}(n)-1}; \delta_1, \dots, \delta_{d^{o*}(n)-1} \in \mathbb{R}\}.$$

In a similar manner, as above, we define the rule $\mathcal{A}^{o*}$ which decides on the number of summands in $\mathcal{N}_k^{o*} = \sum_{l=0}^{k-1} \{\mathcal{C}_{2l+1}^*\}^2$.

Just as in the case of testing $\mathbf{H}$, in simulations, for each n , we fixed $d^*(n) = 20$ and $d^{o*}(n) = 10$.

4.2. Finite sample corrections and critical values

Table S4. Monte Carlo estimates of means and standard deviations of C_j^* 's, $j = 0, \dots, 19$, under $\mathbf{H}^*$, $n = 75$ ($\mathbf{nr} = 100\,000$) and $n = 244^2$ ($\mathbf{nr} = 10\,000$).

n		j	0	1	2	3	4	5	6	7	8	9
	75	$e_{j,0}^*$	0.000	-0.238	-0.001	-0.727	-0.004	-0.570	0.004	0.318	-0.002	0.067
	244^2		0.000	-0.018	-0.008	-0.077	0.005	-0.298	-0.010	-0.605	-0.002	-0.424
	75	$v_{j,0}^*$	0.993	0.985	0.929	0.789	0.608	0.580	0.600	0.526	0.510	0.528
244^2	1.002		0.998	1.001	1.000	1.001	0.962	0.903	0.814	0.718	0.712	

n		j	10	11	12	13	14	15	16	17	18	19
	75	$e_{j,0}^*$	0.000	-0.256	0.002	0.248	-0.002	-0.130	0.002	-0.013	-0.001	0.124
	244^2		0.011	0.195	-0.012	0.270	0.007	-0.232	-0.002	-0.071	-0.002	0.230
	75	$v_{j,0}^*$	0.467	0.483	0.472	0.436	0.464	0.424	0.432	0.432	0.401	0.430
244^2	0.711		0.689	0.642	0.634	0.647	0.600	0.597	0.610	0.577	0.575	

Empirical quantiles of $\mathcal{C}_j^*$, $j = 0, 1, \dots, 19$, behave similarly to the related quantiles of $\mathcal{C}_j$'s. For illustration, in Table S5, we present selected quantiles of the first five components. Note that now empirical behaviour of $\mathcal{C}_0^*$ and $\mathcal{C}_1^*$ determines to a large extent empirical quantiles of $\mathcal{N}_{\mathcal{A}^*}^*$ and $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$, respectively. We also investigated empirical covariances of $\mathcal{C}_0^*, \dots, \mathcal{C}_{19}^*$ both under $n = 75$ and $n = 244^2$. The first two components are almost uncorrelated with the remaining ones while the covariances of $\mathcal{C}_2^*, \dots, \mathcal{C}_{19}^*$ are similar to those presented in Table S3.

Table S5. *Selected empirical q -quantiles of $\mathcal{C}_0^*, \dots, \mathcal{C}_4^*$, calculated under $\mathbf{H}^*$ and $n = 75$, $\mathbf{nr} = 100\,000$, and asymptotic q -quantiles.*

	q												
j	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99
0	-2.346	-1.664	-1.291	-0.851	-0.531	-0.259	-0.003	0.253	0.524	0.852	1.292	1.653	2.345
1	-2.283	-1.622	-1.271	-0.842	-0.539	-0.271	-0.019	0.239	0.513	0.839	1.292	1.671	2.405
2	-2.346	-1.656	-1.284	-0.837	-0.518	-0.250	0.003	0.262	0.532	0.845	1.282	1.652	2.347
3	-2.174	-1.582	-1.256	-0.850	-0.553	-0.294	-0.045	0.211	0.493	0.832	1.313	1.719	2.535
4	-2.357	-1.650	-1.280	-0.835	-0.517	-0.246	0.005	0.259	0.535	0.854	1.297	1.671	2.363
	-2.326	-1.645	-1.282	-0.842	-0.524	-0.253	0.000	0.253	0.524	0.842	1.282	1.645	2.326

Table S6. *Simulated $(1 - \alpha)$ -quantiles of $\mathcal{N}_{\mathcal{A}^*}^*$, $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$, $\mathcal{R}_n^*$, $\mathcal{HC}$, $\mathcal{K}^*$ and the related quantities under $\mathbf{H}^*$, $n = 75$ and $\alpha = 0.05$ ($\mathbf{nr} = 100\,000$) as well as $n = 244^2$ and $\alpha = 0.01$ ($\mathbf{nr} = 10\,000$). Frequencies multiplied by 100.*

Statistic	n	75	244^2
	α	0.05	0.01
$\mathcal{N}_{\mathcal{A}^*}^*$	Quantile	7.061	9.788
	$s^*(n, \alpha)$	4.5	6.9
	$\hat{P}(\mathcal{A}_{s^*(n, \alpha)}^* = 1)$	95.42	99.06
	$\hat{P}(\mathcal{A}^* = 1)$	93.63	98.58
$\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$	Quantile	7.667	9.338
	$s^{o*}(n, \alpha)$	4.8	7.0
	$\hat{P}(\mathcal{A}_{s^{o*}(n, \alpha)}^{o*} = 1)$	95.44	99.05
	$\hat{P}(\mathcal{A}^{o*} = 1)$	94.04	98.66
$\mathcal{R}_n^*$		6.284	9.976
$\mathcal{HC}$		4.796	10.471
$\mathcal{K}^*$		1.637	2.593

5. Comparison of $\mathcal{N}_{\mathcal{A}^o}^o$ and $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$

In this section we study in some more detail empirical behaviour of the tests based on $\mathcal{N}_{\mathcal{A}^o}^o$ and $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$ under six symmetrical alternatives considered in Section 6, $n = 244^2$ and $\alpha = 0.01$.

First recall that when testing $\mathbf{H}$ and $\mathbf{H}^*$ different sets of components are inspected by our procedures, as $\mathcal{N}_{\mathcal{A}^o}^o$ exploits $\mathcal{C}_3, \mathcal{C}_5, \dots, \mathcal{C}_{21}$ while $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$ is based on $\mathcal{C}_1^*, \mathcal{C}_3^*, \dots, \mathcal{C}_{19}^*$. Therefore, for convenience, we start with presenting in Fig. S5 the estimated medians of the components $\mathcal{C}_1^*, \mathcal{C}_3^*, \dots, \mathcal{C}_{19}^*$ for the aforementioned alternatives.

Fig. S5. Medians of $\mathcal{C}_{2j+1}^*$'s, $j = 0, \dots, 9$, estimated under $n = 244^2$, $\mathbf{nr} = 10\,000$.

Table S7 shows, among others, the impact of the selection rules $\mathcal{A}^o$ and $\mathcal{A}^{o*}$ into the related empirical powers under the above presented alternatives. To describe the impact, analogously as in Section 4, introduce three counters:

K_1^* - indicating the percentage of cases when the 'switch' in $\mathcal{A}^{o*}$ works, i.e. $\{\mathcal{R}_n^* > r_\alpha^*\}$ holds;

K_2^* - counting the number of cases when the 'switch' works and $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$ exceeds the critical value, i.e. the new test rejects $\mathbf{H}^*$;

K_3^* - summarizing the cases when the 'switch' indicates $\mathbf{H}^*$ while $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$ rejects the null hypothesis $\mathbf{H}^*$.

Table S7. *Impact of the selection rules $\mathcal{A}^o$ and $\mathcal{A}^{o*}$ into the powers of the tests of $\mathbf{H}$ and $\mathbf{H}^*$, based on $\mathcal{N}_{\mathcal{A}^o}^o$ and $\mathcal{N}_{\mathcal{A}^{o*}}^{o*}$, respectively; $n = 244^2$, $\alpha = 0.01$, $\mathbf{nr} = 10\,000$. All frequencies are given in percentages.*

Alternative		$\mathbf{A}_7$		$\mathbf{A}_8$		$\mathbf{A}_9$		$\mathbf{A}_{10}$		$\mathbf{A}_{11}$		$\mathbf{A}_{12}$	
Hypothesis		$\mathbf{H}$	$\mathbf{H}^*$	$\mathbf{H}$	$\mathbf{H}^*$	$\mathbf{H}$	$\mathbf{H}^*$	$\mathbf{H}$	$\mathbf{H}^*$	$\mathbf{H}$	$\mathbf{H}^*$	$\mathbf{H}$	$\mathbf{H}^*$
Counts													
K_1	K_1^*	53	26	45	15	52	22	30	97	1	42	3	38
K_2	K_2^*	52	26	44	15	50	22	28	97	1	41	2	37
K_3	K_3^*	7	35	7	25	4	19	8	2	0	24	0	23

Recall that $K_1(K_1^*)$ coincide with empirical powers of the tests based on $\mathcal{R}_n(\mathcal{R}_n^*)$, respectively. Therefore, the counts $K_2(K_2^*)$ show that our postulate to construct solutions which reject $\mathbf{H}(\mathbf{H}^*)$ in most cases when the tests based on $\mathcal{R}_n(\mathcal{R}_n^*)$ do it, has been fulfilled. The last line shows that in case of testing $\mathbf{H}$ the rule $\mathcal{A}^o$ has a rather limited ability to impact extra to the power of $\mathcal{R}_n$. In case of testing $\mathbf{H}^*$ there is more room for improvement. This is obviously related to the magnitude and location of the leading components on the respective lists.

Table S8. *Empirical behaviour of selection rules $\mathcal{A}^o$ and $\mathcal{A}^{o*}$ under $\mathbf{A}_7$ - $\mathbf{A}_{12}$; $n = 244^2$, $\alpha = 0.01$, $\mathbf{nr} = 10\,000$.*

		Dimensions of models from the lists $\mathbf{M}^o$ and $\mathbf{M}^{o*}$									
		1	2	3	4	5	6	7	8	9	10
$\mathbf{A}_7$	$\mathcal{A}^o$	45	21	3	8	4	3	4	3	3	6
	$\mathcal{A}^{o*}$	38	22	21	2	5	2	2	2	2	4
$\mathbf{A}_8$	$\mathcal{A}^o$	45	13	1	8	1	6	4	1	11	10
	$\mathcal{A}^{o*}$	58	5	14	3	5	1	3	2	1	8
$\mathbf{A}_9$	$\mathcal{A}^o$	42	7	2	3	0	6	2	3	21	14
	$\mathcal{A}^{o*}$	58	2	6	5	3	0	5	1	3	17
$\mathbf{A}_{10}$	$\mathcal{A}^o$	86	6	2	2	1	1	0	1	0	1
	$\mathcal{A}^{o*}$	8	60	14	4	4	2	2	2	1	3
$\mathbf{A}_{11}$	$\mathcal{A}^o$	98	1	1	0	0	0	0	0	0	0
	$\mathcal{A}^{o*}$	84	6	2	1	2	1	1	1	1	1
$\mathbf{A}_{12}$	$\mathcal{A}^o$	96	1	1	0	0	0	0	0	1	1
	$\mathcal{A}^{o*}$	84	5	2	1	1	1	1	1	2	2

6. R codes

6.1. Introductory remarks

The computer programs, which we introduce below, allow one to compute values of the data driven test statistics for $\mathbf{H}$ and $\mathbf{H}^*$ and the related quantities on the base of the sample $\mathbf{x} = (x_1, \dots, x_n)$ of size n . The notations in the programs are consistent with or similar to those used in the paper.

We exploit, implemented in the CRAN, packages `polynom` and `orthopolynom`.

6.2. Composite null hypothesis $\mathbf{H}$

First, we put `d.n` = 20 (just as $d(n)$ was set in our simulations) and declare the function `H.j.stan` which returns the value of the standardized j th Hermite polynomial at the point x . Auxiliary functions `delta.k` and `C.vec` permit one to calculate the values of $\hat{\delta}_{k+1,n}$ and the vector of the $d(n)$ first components $C_2, \dots, C_{d(n)+1}$.

Second, an exemplary simple procedure for computing vectors of finite sample corrections `e.0`, `v.0`, ($e_{2,0}, \dots, e_{d(n)+1,0}$ and $v_{2,0}, \dots, v_{d(n)+1,0}$ in the paper), which are used in the sequel, is given. There, the sample size was set to be `n` = 100 while the number of Monte Carlo runs was adjusted to `nr` = 10 000.

Next, the function `cal.R` yields value of the statistic $\mathcal{R}_n$.

Finally, given `s`, `so`, and `r`, corresponding to $s(n, \alpha)$, $s^o(n, \alpha)$, and r_α in the paper, and the determined earlier `e.0` and `v.0` as well as given fixed `x`, `n`, and `d.n`, the routine `cal.N.test` provides the vector of the values of the following objects:

$$\mathcal{N}_{\mathcal{A}_{s(n,\alpha)}}, \mathcal{N}_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}_2}, \mathcal{A}_{s(n,\alpha)}, \mathcal{A}, \mathcal{A}_2, \mathcal{N}_{\mathcal{A}_{s^o(n,\alpha)}}^o, \mathcal{N}_{\mathcal{A}^o}^o, \mathcal{N}_{\mathcal{A}_2^o}^o, \mathcal{A}_{s^o(n,\alpha)}^o, \mathcal{A}^o, \mathcal{A}_2^o, C_2, \dots, C_{d(n)+1}.$$

Note that in the code $d^o(n) = \lfloor d(n)/2 \rfloor + 1$.

Recall that the constants `s` and `so` should be computed separately. In the simulation study, which we conducted, we used the above code and selected these quantities with the precision 0.1.

```

library(polynom)
library(orthopolynom)

d.n = 20
pols = hermite.h.polynomials(d.n)

H.j.stan <- function(x,j) {
  fpol = as.function(pols[[j+1]])
  2^(-j/2)*fpol(x/sqrt(2))/sqrt(factorial(j))
}

delta.k = function(x,n,k){
  x.s = sort(x)
  ps = 0:n/n
  phi = dnorm(qnorm( ps ))
  h.j = H.j.stan( qnorm( ps ), k )
  h.j[1] = 0
  h.j[n+1] = 0
  w.k = matrix(0,1,n)
  for(j in 1:n){
    w.k[1,j] = phi[j]*h.j[j] - phi[j+1]*h.j[j+1]
  }
  score = (w.k[1,] %*% x.s )/sqrt(k+1)
  return(score)
}

C.vec = function(x,n,d.n){
  score = matrix(0,1,d.n)
  for(j in 1:d.n)
    score[1,j] = delta.k(x,n,j)*sqrt((j+2)*n)/delta.k(x,n,0)
  return(score[1,])
}

n = 100
nr = 10000

C.0 = matrix(0,d.n,nr)
e.0 = matrix(0,1,d.n)
v.0 = matrix(0,1,d.n)

for(i in 1:nr){
  x = rnorm(n)
  C.0[,i] = C.vec(x,n,d.n)
}

for(j in 1:d.n){
  e.0[1,j] = mean(C.0[j,])
  v.0[1,j] = sd(C.0[j,])*sqrt((n-1)/n)
}

cal.R = function(x,n){
  S.sq = (sum((x-mean(x))^2))/n
  score = n*( 1 - delta.k(x,n,0)^2/S.sq )
}

```

```

    return(score)
}

cal.N.test = function(x,n,d.n,r,e.0,v.0,s,so){
  cal.C.vec = (C.vec(x,n,d.n)-e.0)/v.0
  cal.C.vec.sq = cal.C.vec^2
  cal.N.d.n = cumsum(cal.C.vec.sq)

  d.n.o = ceiling(d.n/2)

  cal.Co.vec.sq = cal.C.vec.sq[2*1:d.n.o]
  cal.No.d.n = cumsum(cal.Co.vec.sq)

  dim = 1:d.n
  dim.o = 1:d.n.o

  cal.A.s = which.max( cal.N.d.n - dim*s )
  cal.Ao.so = which.max( cal.No.d.n - dim.o*so )

  cal.A.2 = which.max( cal.N.d.n - dim*2 )
  cal.Ao.2 = which.max( cal.No.d.n - dim.o*2 )

  r.n = cal.R(x,n)

  if( r.n <= r ){
    cal.A = cal.A.s
    cal.Ao = cal.Ao.so
  }else{
    cal.A = cal.A.2
    cal.Ao = cal.Ao.2
  }

  cal.N.cal.A.s = cal.N.d.n[cal.A.s]
  cal.N.cal.A = cal.N.d.n[cal.A]
  cal.N.cal.A.2 = cal.N.d.n[cal.A.2]

  cal.No.d.n.cal.Ao.so = cal.No.d.n[cal.Ao.so]
  cal.No.d.n.cal.Ao = cal.No.d.n[cal.Ao]
  cal.No.d.n.cal.Ao.2 = cal.No.d.n[cal.Ao.2]

  result = c(cal.N.cal.A.s, cal.N.cal.A, cal.N.cal.A.2, cal.A.s, cal.A, cal.A.2,
             cal.No.d.n.cal.Ao.so, cal.No.d.n.cal.Ao, cal.No.d.n.cal.Ao.2,
             cal.Ao.so, cal.Ao, cal.Ao.2, cal.C.vec)
  return(result)
}

```

6.3. Simple null hypothesis H^*

We set $d.n.star = 20$ (similarly like $d^*(n)$ was taken in the simulation study). We use the functions `H.j.stan` and `delta.k` described in the case of composite null hypothesis. Then the function `C.vec.star` provides values of the components $C_0^*, \dots, C_{d^*(n)-1}^*$.

Just like before, a simple procedure allows one to compute the vectors of finite sample corrections `e.0.star` and `v.0.star` denoted, in the paper, by $e_{0,0}^*, \dots, e_{d^*(n)-1,0}^*$ and $v_{0,0}^*, \dots, v_{d^*(n)-1,0}^*$, respectively.

Next, the function `cal.R.star` returns value of the statistic $\mathcal{R}_n^*$. Critical value of this statistic is denoted by `r.star` in the code.

At last the algorithm `cal.N.test.star` is written to enable one, given `x`, `n`, and `d.n.star` as well as `r.star`, `e.0.star`, `v.0.star` and `s.star`, `so.star`, to calculate the vector of quantities

$$\mathcal{N}_{\mathcal{A}_{s^*(n,\alpha)}^*}^*, \mathcal{N}_{\mathcal{A}^*}^*, \mathcal{N}_{\mathcal{A}_2^*}^*, \mathcal{A}_{s^*(n,\alpha)}^*, \mathcal{A}^*, \mathcal{A}_2^*, \mathcal{N}_{\mathcal{A}_{s^{o^*}(n,\alpha)}^{o^*}}^*, \mathcal{N}_{\mathcal{A}^{o^*}}^*, \mathcal{N}_{\mathcal{A}_2^{o^*}}^*, \mathcal{A}_{s^{o^*}(n,\alpha)}^{o^*}, \mathcal{A}^{o^*}, \mathcal{A}_2^{o^*}, \mathcal{C}_0^*, \dots, \mathcal{C}_{d^*(n)-1}^*.$$

In the code we put $d^{o^*}(n) = \lfloor d^*(n)/2 \rfloor + 1$.

```
library(polynom)
library(orthopolynom)

d.n.star = 20
pols = hermite.h.polynomials(d.n.star)

H.j.stan <- function(x,j) {
  fpol = as.function(pols[[j+1]])
  2^(-j/2)*fpol(x/sqrt(2))/sqrt(factorial(j))
}

delta.k = function(x,n,k){
  x.s = sort(x)
  ps = 0:n/n
  phi = dnorm(qnorm( ps ))
  h.j = H.j.stan( qnorm( ps ), k )
  h.j[1] = 0
  h.j[n+1] = 0
  w.k = matrix(0,1,n)
  for(j in 1:n){
    w.k[1,j] = phi[j]*h.j[j] - phi[j+1]*h.j[j+1]
  }
  score = (w.k[1,] %*% x.s )/sqrt(k+1)
  return(score)
}

C.vec.star = function(x,n,d.n.star){
  C.0.star = sqrt(n)*mean(x)
  C.1.star = sqrt(2*n)*(delta.k(x,n,0)-1)
  score = matrix(0,1,d.n.star-2)
  for(j in 1:(d.n.star-2))
    score[1,j] = delta.k(x,n,j)*sqrt((j+2)*n)
  result = matrix(0,1,d.n.star)
  result[1,] = c(C.0.star, C.1.star, score[1,])
  return(result[1,])
}

n = 100
nr = 10000
```

```

C.0.star = matrix(0,d.n.star,nr)
e.0.star = matrix(0,1,d.n.star)
v.0.star = matrix(0,1,d.n.star)

for(i in 1:nr){
  x = rnorm(n)
  C.0.star[,i] = C.vec.star(x,n,d.n.star)
}

for(j in 1:d.n.star){
  e.0.star[1,j] = mean(C.0.star[j,])
  v.0.star[1,j] = sd(C.0.star[j,])*sqrt((n-1)/n)
}

cal.R.star = function(x,n){
  x.s = sort(x)
  score = n*(mean(x.s^2) - 2*delta.k(x,n,0) + 1)
  return(score)
}

cal.N.test.star = function(x,n,d.n.star,r.star,e.0.star,v.0.star,s.star,s0.star){

  cal.C.vec.star = (C.vec.star(x,n,d.n.star)-e.0.star)/v.0.star
  cal.C.vec.sq.star = cal.C.vec.star^2
  cal.N.d.n.star = cumsum(cal.C.vec.sq.star)

  d.n.star.o = ceiling(d.n.star/2)

  cal.Co.vec.sq.star = cal.C.vec.sq.star[2*1:d.n.star.o]
  cal.No.d.n.star = cumsum(cal.Co.vec.sq.star)

  dim = 1:d.n.star
  dim.o = 1:d.n.star.o

  cal.A.s.star = which.max( cal.N.d.n.star - dim*s.star )
  cal.Ao.so.star = which.max( cal.No.d.n.star - dim.o*so.star )

  cal.A.2.star = which.max( cal.N.d.n.star - dim*2 )
  cal.Ao.2.star = which.max( cal.No.d.n.star - dim.o*2 )

  r.n.star = cal.R.star(x,n)

  if( r.n.star <= r.star ){
    cal.A.star = cal.A.s.star
    cal.Ao.star = cal.Ao.so.star
  }else{
    cal.A.star = cal.A.2.star
    cal.Ao.star = cal.Ao.2.star
  }

  cal.N.cal.A.s.star = cal.N.d.n.star[cal.A.s.star]
  cal.N.cal.A.star = cal.N.d.n.star[cal.A.star]

```

```

cal.N.cal.A.2.star = cal.N.d.n.star[cal.A.2.star]

cal.No.d.n.cal.Ao.so.star = cal.No.d.n.star[cal.Ao.so.star]
cal.No.d.n.cal.Ao.star = cal.No.d.n.star[cal.Ao.star]
cal.No.d.n.cal.Ao.2.star = cal.No.d.n.star[cal.Ao.2.star]

result = c(cal.N.cal.A.s.star, cal.N.cal.A.star, cal.N.cal.A.2.star,
           cal.A.s.star, cal.A.star, cal.A.2.star, cal.No.d.n.cal.Ao.so.star,
           cal.No.d.n.cal.Ao.star, cal.No.d.n.cal.Ao.2.star, cal.Ao.so.star,
           cal.Ao.star, cal.Ao.2.star, cal.C.vec.star)
return(result)
}

```

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