Critical Values for the Durbin-Watson Test in Large Samples

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Abstract:

This paper estimates functional relationships between the Durbin-Watson critical bounds and the sample size. The functions estimated provide a quick and easy way to calculate critical values for sample sizes not reported as standard in books of statistical tables. They also allow estimation of the sample sizes necessary for the convergence of the Durbin-Watson lower and upper bounds to those derived from the normal distribution and each other. Our results indicate that what constitutes a ‘large’ sample is highly sensitive to the number of independent variables in the regression equation.

Keywords: Durbin-Watson statistic, convergence of distribution.

JEL classification: C12, C15, C22

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I. Introduction

This paper presents functions which allow the estimation of critical values for the Durbin-Watson statistics for sample sizes larger than those routinely presented in published tables. The functions presented are analogous to the response surface functions often used to calculate critical values for unit root tests (e.g. those presented in MacKinnon (1991)). However, the functions presented here are not based on Monte Carlo analysis but rather on approximations to the theoretically determined critical values presented in standard tables. The aim of the paper is to generalise the published critical values already available in table form and, by so doing, making it easier for the applied econometrician to obtain an appropriate critical value for sample sizes which lie outside those normally included in the published tables.

Our results therefore make it easier to quote accurate critical values in applied work. For example, Dimitropoulos, Hunt and Judge (2005) report the Durbin-Watson statistic for their models of energy demand but do not quote critical values since the sample size of 132 is greater than that normally reported in the printed tables. This problem is common in models estimated using financial data since the number of observations frequently exceeds the number reported in published tables. An example here is Smales (2017) who estimates a model of financial returns in the Australian stock market with 368 observations. However, as we will demonstrate, in both these cases the number of observations fall short of the value needed to justify the assumption of convergence to the normal distribution when the regression includes more than one exogenous variables. Our results also make it easier to compute more accurate critical values for tests in which the degrees of freedom lie within the standard range reported in published tables but do not match the published figures exactly. For example, Gallet (1996) reports Durbin-Watson statistics for equations relating output and price in the US steel industry for 39 observations – a value not reported as standard in most published tables.

II. The Durbin-Watson Test in Large Samples

Let us first define the von-Neumann ratio. Let e_t be a sequence of random variables. The von-Neumann ratio is defined as the ratio of the ‘mean successive difference’ to the variance of the series. This can be written:

$$\frac{\delta^2}{s^2} = \frac{\sum_{t=2}^T (e_t - e_{t-1})^2 / (T-1)}{\sum_{t=1}^T (e_t - \bar{e})^2 / T} \quad (1)$$

where T is the sample size and $\bar{e}$ is the mean value of e . If the e ’s are independent then, in large samples, the von Neumann ratio is approximately normally distributed with mean $2T / (T-1)$ and variance $4T^2 (T-2) / \{(T+1)(T-1)^3\}$. There is an obvious relationship with the Durbin-Watson statistic which can be written:

$$dw = \frac{\sum_{t=2}^T (e_t - e_{t-1})^2}{\sum_{t=1}^T e_t^2} = \frac{\delta^2}{s^2} \frac{T-1}{T} \quad (2)$$

where $\bar{e} = 0$ by construction when the e ’s are regression residuals. This suggests that, in large samples, the Durbin-Watson statistic will also follow a normal distribution with mean 2 and variance $4(T-2) / \{(T+1)(T-1)\}$ which, in large samples, is close to $4/T$. Note that the distributions of both these statistics are degenerate since the variance tends to zero as the sample size tends to infinity. If we appeal to the central limit theorem then a reasonable approximation is therefore that the Durbin-Watson statistic is approximately normally distributed with mean 2 and variance $4/T$ in large samples.

Determination of the small sample distribution of the Durbin-Watson statistic is complicated by a number of factors. Firstly, the e ’s are not independent by virtue of the fact

that they are regression residuals and therefore $\sum_{t=1}^T e_t = 0$ by construction. Secondly, it can be shown that the sampling distribution of the Durbin-Watson statistic depends on the particular data set being used. It is not possible to derive a unique sampling distribution for this statistic. However, Durbin and Watson (1950 and 1951) show that it is possible to show that the distribution is bounded between certain limits. It is therefore possible to derive upper and lower bounds for the critical values for the statistic in small samples. These are the ‘critical bounds’ reported in most books of statistical tables and econometric textbooks.

III. Numerical Approximations for the Durbin-Watson Critical Values

Durbin and Watson’s original tables allow tests for serial correlation for regressions with up to five regressors and sample sizes ranging from fifteen to one hundred. Most printed versions of these tables have kept these limits. However, this creates a problem in that a sample size of 100 is well below the level at which the distribution will have converged to the normal. For example, in the case of a single regressor and 100 observations, the Durbin-Watson 5% lower and upper bounds are 1.65 and 1.69 respectively. If we take the normal approximation to the critical value, we have $2 - 1.645 \times 2 / \sqrt{100} = 1.671$ which is approximately half-way between these values. Ideally, we need to extend the sample size to a level at which both the upper and lower critical bounds will have converged to the normal critical value.

Attempts have been made to extend the Durbin-Watson critical values in several ways. Savin and White (1977) extend Durbin and Watson’s tables to allow for sample sizes from six to two hundred and up to twenty regressors. However, they note that ‘the computations required for large values of n were extremely heavy, thus making any further extension of the tables impractical’. Again, we are left with the problem that the sample size may not be large enough to ensure that the distribution of the statistic has converged sufficiently to allow use of the normal distribution.

The approach taken in this paper is to model the critical bounds for the Durbin-Watson statistic explicitly as a function of the sample size using the reported values given in Savin and White (1977). Based on these values, we fit a regression which links the critical bound dw^b where b is equal to either l or u indicating lower and upper bounds: respectively

to the sample size T . The values for the critical bounds are taken from the webpage web.stanford.edu/~clint/bench/dwcrit.htm where they are reported to five decimal places. The form of the regression equation fitted is a Taylor series approximation to a general non-linear function and takes the form:

$$dw_T^b = 2 + \frac{\beta_1}{\sqrt{T}} + \frac{\beta_2}{T} + \frac{\beta_3}{T^{3/2}} + \frac{\beta_4}{T^2}$$

We experimented with a number of possible functional forms and found that this struck a reasonable balance between obtaining a good fit to the published critical bounds and maintaining a small number of parameters. Note that the function adopted is specified in powers of $\sqrt{T}$ rather than T . This is important because the large sample critical values under the assumption of a normally distributed dw statistic take the form $2 - z^{\alpha\%} \times 2\sqrt{(T-2)/\{(T+1)(T-1)\}}$ and it is important to adopt a functional form which allows the estimated critical values to converge to these. Our approach is to fit this relationship to the critical bound reported on the Stanford webpage for values of T up to 100. We then calculate predicted values for values $T=101, \dots, 200$ to assess how well the estimated functions perform ‘out of sample’.

The estimated functions for the 5% and 1% critical bounds are presented in Tables 1 and 2 respectively. These produce a very close match to the critical bounds reported on the Stanford webpage. To test this, we compared the Stanford critical values to our predicted critical values to an accuracy of two decimal places. These disagreed in only three cases out of the eight hundred and sixty considered and even here the disagreement was very small e.g. for $k=5$ the Stanford upper bound for the 5% test is 1.79 while our calculated value is 1.80. When the estimated functions are used to predict the critical bounds ‘out of sample’ i.e. for values of T in the range 101 to 200, the agreement between our values and the Stanford values is even closer. In fact, there are no cases in which the difference between the predicted and reported values exceeds 0.01 in absolute value. The estimated functions therefore provide a good approximation to published critical bounds. They therefore also provide a quick and easy way in which accurate critical bounds can be calculated for sample sizes not normally reported in published tables for the Durbin-Watson statistic.

IV. Convergence of the Critical Bounds

Econometricians are very fond of the phrase ‘in large samples’. This is because the small sample distributions of various statistics commonly used in econometric analysis prove difficult, if not impossible, to derive. Of course, this begs the question of how large is a large sample? Our estimates of the relationship between the sample size and the Durbin-Watson critical bounds offer a rare opportunity to address this question directly by allowing us to calculate how many observations it takes for the lower and upper critical bounds to converge to either the normal critical values or to each other.

Let $d_L(T)$ be the lower bound for the Durbin-Watson test, $d_U(T)$ be the upper bound and $d_Z(T)$ be the critical value based on the normal distribution – all based on a one-tailed test for positive first-order autocorrelation. We have $d_U(T) > d_Z(T) > d_L(T)$. Using the estimated parameters in Tables 1 and 2 we can compute values of T such that these critical values approach each other to a given degree of accuracy. The results are shown in Tables 3 and 4 for the 5% and 1% critical bounds respectively. These show the sample sizes at which the functional relationships converge to the desired degree of accuracy. For example, the number 5,676 in Table 3 is the solution T to the equation $d_U(T) - d_L(T) = 0.01$ where the parameters of the d_U and d_L functions are the values given in Table 1 for $k = 5$. From these values, we see that the rate of convergence is highly sensitive to the number of independent variables included in the regression equation. For example, in the case of one explanatory variable the upper and lower bound values converge to the normal critical value with approximately two hundred observations. However, increasing the number of explanatory variables to five means that more than one thousand observations are required. Similar increases are observed for the one percent critical bounds.

V. Conclusions

This paper has shown that it is possible to generate simple functional relationships which allow the calculation of critical bounds for the Durbin-Watson statistic. This, in turn, allows

the calculation of critical values for sample sizes which are not reported as standard in books of statistical tables. It has also been shown that these functional relationships are robust for sample sizes outside those used to estimate the functions. Using these functions, we can therefore determine the sample sizes necessary for convergence of the Durbin-Watson lower and upper bounds to those derived from the normal distribution and to each other. Our calculations show that what constitutes a ‘large sample’, as measured by convergence of the critical values, is highly sensitive to the number of independent variables (k) in the regression equation and increases sharply with the value of k .

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Number of regressors		$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	Maximum Error (within sample)	Maximum Error (outside sample)
$k = 1$	Lower	-3.269156	-2.217893	2.931711	4.263415	0.006711	0.000452
	Upper	-3.274405	1.917299	1.830633	10.64619	0.003945	0.000445
$k = 2$	Lower	-3.312097	-3.332536	-3.632166	19.31135	0.005421	8.03E-05
	Upper	-3.273537	4.017121	-0.171342	28.01835	0.004533	0.000542
$k = 3$	Lower	-3.372844	-3.988774	-14.24567	44.18308	0.004386	0.000528
	Upper	-3.212242	5.005001	4.192820	39.35740	0.005457	0.001348
$k = 4$	Lower	-3.447993	-4.229294	-28.91627	80.00972	0.003616	0.001070
	Upper	-3.083286	4.629474	17.34105	37.58253	0.006135	0.002777
$k = 5$	Lower	-3.535331	-4.085190	-47.63654	127.7127	0.003060	0.001611
	Upper	-2.889066	2.831871	40.52079	17.67699	0.006567	0.004700

Table 1: Durbin-Watson 5% critical bound function estimates

The following URL is for a web app which uses these functions to generate estimates of the Durbin-Watson critical bounds

<http://runbasichosting.com:8021/seaside/go/runbasicpersonal?app=durbin-watson>

Number of regressors		$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	Maximum Error (within sample)	Maximum Error (outside sample)
$k = 1$	Lower	-4.658992	-1.917947	6.992628	4.799035	0.005240	0.000153
	Upper	-4.875742	5.814534	-13.54258	38.75486	0.011335	0.003133
$k = 2$	Lower	-4.642915	-4.052984	5.966592	14.91894	0.003426	0.000214
	Upper	-4.985175	10.07143	-29.27794	82.95705	0.009963	0.004145
$k = 3$	Lower	-4.643183	-5.825048	1.825873	33.13285	0.002287	0.000389
	Upper	-4.998318	12.71578	-37.02585	122.0812	0.008208	0.003854
$k = 4$	Lower	-4.655069	-7.296073	-5.300441	60.11130	0.001774	0.000497
	Upper	-4.914934	13.54419	-34.15174	147.8301	0.005938	0.002506
$k = 5$	Lower	-4.675041	-8.518908	-15.25711	96.32291	0.001967	0.000573
	Upper	-4.743511	12.55718	-19.39023	154.4401	0.003522	0.000405

Table 2: Durbin-Watson 1% critical bound function estimates

Number of regressors	Lower Bound and Normal Distribution	Upper Bound and Normal Distribution	Lower and Upper Bounds
	$d_Z(T) - d_L(T) < 0.01$	$d_U(T) - d_Z(T) < 0.01$	$d_U(T) - d_L(T) < 0.01$
$k = 1$	194	210	400
$k = 2$	408	426	861
$k = 3$	676	718	1,585
$k = 4$	1,022	1,232	2,951
$k = 5$	1,476	2,740	5,676

Table 3: Sample sizes for convergence of 5% critical bound values

Number of regressors	Lower Bound and Normal Distribution	Upper Bound and Normal Distribution	Lower and Upper Bounds
	$d_Z(T) - d_L(T) < 0.01$	$d_U(T) - d_Z(T) < 0.01$	$d_U(T) - d_L(T) < 0.01$
$k = 1$	182	172	294
$k = 2$	377	278	502
$k = 3$	568	401	751
$k = 4$	767	584	1,132
$k = 5$	978	917	1,810

Table 4: Sample sizes for convergence of 1% critical bound values