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Continuous Nonsingular Terminal Sliding Mode Control for Nonlinear Systems Subject to Mismatched Terms [†]

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Abstract This paper proposes a continuous nonsingular terminal sliding mode (NTSM) control approach for nonlinear systems subject to mismatched terms in order to achieve finite time exact tracking and disturbance rejection. The controller is constructed using a composite method that utilizes a power integrator and combines the output regulation theory, disturbance observation technique, feedback domination and sliding mode control technique. The performance analysis demonstrates that the proposed continuous NTSM controller can drive the system output to the desired reference signal in a finite time in the presence of mismatched time-varying disturbances and nonsmoothed nonlinearities in each channel. The finite time Lyapunov theory is utilized to ensure finite-time convergence of the closed-loop system. The simulation results validate the effectiveness of the proposed continuous NTSM controller.

Key words Continuous nonsingular terminal sliding mode control, finite-time exact tracking, finite-time disturbance observer, Lyapunov function.

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1 INTRODUCTION

Sliding mode control (SMC) is proved to be effective in dealing with internal parameter variations and external disturbances with low computational complexity [1, 2]. The conventional SMC adopts the linear sliding manifold and the system states reach their equilibrium points asymptotically. In recent years, the terminal sliding mode (TSM) [3] is proposed, which can drive the system states to their equilibrium points in a finite-time by designing a nonlinear sliding manifold. However, the TSM method has the singular problem [3]. In order to overcome this disadvantage, the nonsingular terminal sliding mode (NTSM) has been proposed in [4]. The NTSM approaches can provide finite-time stability, fast dynamic response, low steady-state errors, more degrees of freedom for parameter choice of the controller and avoid the singular problem. Due to these advantages, the NTSM controllers are widely used in various applications, such as electronic throttle systems [5], autonomous underwater vehicles [6], dual-motor driving servo systems [7], robot manipulators [8] and so on.

In practice, with the development of industrial systems, there are not only matched disturbances/uncertainties in the systems, but also mismatched disturbances and nonlinearities [9, 10]. For example, the load torque is the mismatched disturbance in permanent magnet asynchronous motor system [11]. In [12], a nonlinear sliding mode controller based on linear matrix inequality has been proposed to handle the mismatched disturbances. In [13], a novel sliding surface has been proposed to simultaneously achieve low overshoot and short settling time for the system with mis-

matched disturbances. However, the controllers proposed in [12] and [13] cannot drive the system states to the desired values in a finite-time. Even though the NTSM controllers have several advantages, they cannot effectively cope with the mismatched disturbances/uncertainties, which is the same problem as in the traditional SMC [1], therefore, designing a NTSM controller to counteract the mismatched disturbances/uncertainties and nonlinearities is challenging and significant.

To date, several methods have been proposed to solve the mismatched problem in NTSM by combining different robust approaches. In [14], a continuous nonsingular terminal sliding mode controller has been proposed for small-scale unmanned helicopter with mismatched disturbances. In [15], an anti-disturbance control method based on the non-singular terminal sliding mode theory has been proposed for leader-follower multi-agent system with mismatched disturbances. In [16], the NTSM control has been used to solve the load disturbances in DC-DC buck converters. Moreover, a non-singular terminal sliding mode controller for flexible air-breathing hypersonic vehicle system has been proposed in [17]. In general, the existing methods can be divided into two categories. The first one is based on the principle to remove all the mismatched disturbances in the control channel through coordinate transformation [18, 19], and the disturbances can be suppressed by conventional NTSM controller [20, 21]. In [22], an adaptive fuzzy TSM controller has been proposed, which requires the mismatched disturbances to be H_2 norm-bounded. A full-order TSM controller has been proposed in [23] for MIMO systems subject to mismatched uncertainties, but the system output cannot converge to zero in a finite-

time. In [24], an integral TSM controller has been proposed for hypersonic vehicles system with mismatched disturbances, but as is well known, the integral action will always bring adverse effects to control systems. It should be pointed out that the above-mentioned approaches will sacrifice the performance of the closed-loop system. The second category utilizes the disturbance observation technique [9] to online estimate the mismatched disturbances and compensate them in the controller. In [25], a continuous NTSM control method based on the finite-time disturbance observer (FTDO) [26] has been proposed to drive the system states to the set-point in a finite-time in the presence of mismatched disturbance. In [27], a disturbance observer based NTSM control method has been proposed for spacecraft system. Nevertheless, if nonlinearities exist in each channel of the system, the above-mentioned disturbance observer-based SMC methods will not be effective.

This paper aims to solve the finite-time exact tracking and disturbance rejection problem for nonlinear systems with mismatched disturbances. A novel continuous NTSM control strategy is proposed based on the basic idea of output regulation theory [28, 29, 30]. The proposed control approach uses a power integrator and combines disturbance observation technique, feedback domination and sliding mode technique. Firstly, the paper provides a continuous NTSM control algorithm for handling the nonlinear systems with matched and mismatched disturbances as well as matched and mismatched nonsmoothed nonlinearities. Secondly, it is shown by finite-time Lyapunov theory that the proposed sliding mode controller achieves the finite-time exact tracking for the closed-loop system under mismatched time-

varying disturbances. Finally, the proposed sliding mode controller is continuous, which largely reduces the adverse influence caused by the discontinuous switching of the control law.

2 Preliminaries

2.1 Useful lemmas

Firstly, some useful lemmas are presented that play important roles in the analysis. Define a $\mathcal{C}^0$ function as $\lfloor x \rfloor^\alpha = |x|^\alpha \text{sgn}(x)$.

Lemma 1 [31] *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, $p_1 > 0$ and $0 < p_2 \leq 1$, there is $|\lfloor x \rfloor^{p_1 p_2} - \lfloor y \rfloor^{p_1 p_2}| \leq 2^{1-p_2} |\lfloor x \rfloor^{p_1} - \lfloor y \rfloor^{p_1}|^{p_2}$.*

Lemma 2 [31] *For $x \in \mathbb{R}$, $y \in \mathbb{R}$, $c > 0$, $d > 0$ and any real-valued function $\gamma(x, y) > 0$, the following inequality holds: $|x|^c |y|^d \leq \frac{c}{c+d} \gamma(x, y) |x|^{c+d} + \frac{d}{c+d} \gamma^{-\frac{c}{d}}(x, y) |y|^{c+d}$.*

Lemma 3 [32] *For $x_i \in \mathbb{R}$, $i = 1, \dots, n$ and $0 < p \leq 1$, there is $(|x_1| + \dots + |x_n|)^p \leq |x_1|^p + \dots + |x_n|^p$.*

2.2 Problem formulation

Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_i(t) &= x_{i+1}(t) + f_i(\bar{x}_i(t)) + \omega_i(t), \\ \dot{x}_n(t) &= u(t) + f_n(\bar{x}_n(t)) + \omega_n(t), \\ y(t) &= x_1(t),\end{aligned}\tag{1}$$

for $\bar{x}_i := [x_1, \dots, x_i]^T$, $i \in \mathbb{N}_{1:n-1}$ and $\mathbb{N}_{j:k} = \{j, j+1, \dots, k\}$ is a set of nonnegative integers. $x_i \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ are the system state, the control input and the control output, respectively. $\omega_i \in \mathbb{D} \subset \mathbb{R}$ with $\mathbb{D}$ is a compact set that denotes the external disturbance. It is assumed that the

external disturbance $\omega_i(t)$, $i \in \mathbb{N}_{1:n}$ is unknown but its amplitude and derivatives are bounded by $\sup_{t>0} |\frac{d^j \omega_i(t)}{dt^j}| \leq \mathcal{F}_{\omega_i}^+$ for $j \in \mathbb{N}_{0:n+1-i}$, where $\mathcal{F}_{\omega_i}^+$ is a positive constant. The state of x_i and the nonlinear function $f_i(\bar{x}_i)$ are assumed to be available for control design, and $f_i(\bar{x}_i)$ is supposed to satisfy the following condition:

$$|f_i(\bar{x}_i) - f_i(\bar{y}_i)| \leq \rho_i(\bar{x}_i, \bar{y}_i) \sum_{j=1}^i |x_j - y_j|^{(r_i+\tau)/r_j}, \quad (2)$$

for $i \in \mathbb{N}_{1:n}$, $r_{i+1} = r_i + \tau$, $\tau \in (-\infty, 0)$, with $r_1 > 0$ is a constant, and $\rho_i(\bar{x}_i, \bar{y}_i) > 0$ is a $\mathcal{C}^0$ function.

Control Objective: This paper attempts to utilize the nonsingular terminal sliding mode control theory to address the finite-time exact tracking and disturbance rejection problem for a class of lower triangular nonlinear system (1), in which nonlinearities and time-varying disturbances exist in each channel. For system (1) satisfying (2), a recursive sliding manifold and continuous nonsingular terminal sliding mode controller are designed to ensure the finite-time exact tracking of the nonlinear system in the presence of matched and/or mismatched disturbances. Under the proposed controller, it is shown that there exists a finite-time T such that $y(t) = r(t)$ for $t \geq T$.

2.3 Motivations

To begin with, the basic principle of TSM control is briefly introduced [3]. Consider the following second-order uncertain nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(x_1, x_2) + g(x_1, x_2, d) + u, \end{aligned} \quad (3)$$

where $f(x_1, x_2)$ is a smooth nonlinear function, and $g(x_1, x_2, d)$ represents the uncertainties and

disturbances satisfying $|g(x_1, x_2, d)| < g^*$ with $g^* > 0$. u is the control input.

The traditional first-order terminal sliding manifold is designed as $s = x_2 + \beta x_1^{q/p}$, where $\beta > 0$ is a constant to be designed, p, q are positive integers satisfying $p > q$. The sufficient condition for the existence of TSM is $\frac{1}{2} \frac{d}{dt} s^2 \leq -k|s|$, where $k > 0$ is a constant. For system (3), the TSM controller is designed as

$$u = - \left[f(x_1, x_2) + \beta q / p x_1^{q/p-1} x_2 + (g^* + k) \text{sgn}(s) \right]. \quad (4)$$

It is observed that if $s(0) \neq 0$, the system states will reach the sliding manifold $s = 0$ in a finite-time t_r with $t_r \leq \frac{|s(0)|}{\eta}$. When the sliding manifold is reached, the system dynamics are given by $x_2 + \beta x_1^{q/p} = \dot{x}_1 + \beta x_1^{q/p} = 0$. Therefore, the system states x_1 and x_2 converge to zero in a finite-time.

Since the controller (4) contains the term $x_1^{q/p-1} x_2$, it may cause a singularity to occur if $x_2 \neq 0$ when $x_1 = 0$. The TSM controller (4) cannot ensure a bounded control signal for the case of $x_2 \neq 0$ when $x_1 = 0$ before the system states reach the sliding manifold. Moreover, the singularity problem may also occur after the sliding manifold is reached due to the uncertain factors in the closed-loop system. It can be ensured that the system states will always remain in the sliding mode especially near the equilibrium point and the case of $x_2 \neq 0$ while $x_1 = 0$ may occur from time to time [4].

3 Main results

3.1 Steady-state estimations

A set of FTDO [26] are designed as follows to estimate the disturbances and their derivatives in

system (1):

$$\begin{aligned}\dot{z}_0^i &= v_o^i + x_{i+1} + f_i(\bar{x}_i(t)), \\ \dot{z}_1^i &= v_1^i, \dots, \dot{z}_{n-i+1}^i = v_{n-i+1}^i,\end{aligned}\quad (5)$$

for $i = 1, \dots, n$, with definitions $x_{n+1} = u$, $v_o^i = -\lambda_0^i L_i^{\frac{1}{n-i+2}} [z_0^i - x_i]^{\frac{n-i+1}{n-i+2}} + z_1^i$, and $v_j^i = -\lambda_j^i L_i^{\frac{1}{n-i+2-j}} [z_j^i - v_{j-1}^i]^{\frac{n-i+1-j}{n-i+2-j}} + z_{j+1}^i$ for $i \in \mathbb{N}_{1:n}, j \in \mathbb{N}_{0:n-i+1}$. $\lambda_j^i > 0$ and $L_i > 0$ is the observer gains to be designed. The estimations of x_i and $\omega_i^{(k-1)}$ are given by $\hat{x}_i = z_0^i$ and $\hat{\omega}_i^{(k-1)} = z_k^i$, for $i \in \mathbb{N}_{1:n}, k \in \mathbb{N}_{1:n+1-i}$. According to [26], if the observer gain L_i is selected to satisfy $L_i > \mathcal{F}_{\omega_i}^+$, then the above estimations converge to their true values in a finite-time T_1 .

The steady-states of system (1) are determined by

$$\begin{aligned}x_{1s}(t) &= r(t), \\ x_{(i+1)s}(t) &= \frac{dx_{is}(t)}{dt} - f_i(\bar{x}_{is}(t)) - \omega_i(t), \\ u_s(t) &= \frac{dx_{ns}(t)}{dt} - f_n(\bar{x}_{ns}(t)) - \omega_n(t),\end{aligned}\quad (6)$$

where $i \in \mathbb{N}_{2:n}$, $x_{is}(t)$ is the steady-state of $x_i(t)$. By virtue of disturbance estimations, the steady-state estimations are obtained as follows:

$$\begin{aligned}\hat{x}_{1s}(t) &= r(t), \\ \hat{x}_{(i+1)s}(t) &= \frac{d\hat{x}_{is}(t)}{dt} - f_i(\hat{\bar{x}}_{is}(t)) - z_1^i(t), \\ \hat{u}_s(t) &= \frac{d\hat{x}_{ns}(t)}{dt} - f_n(\hat{\bar{x}}_{ns}(t)) - z_1^n(t),\end{aligned}\quad (7)$$

where $\hat{x}_{1s}, \hat{x}_{is}, \hat{u}_s$ are estimations of x_{1s}, x_{is}, u_s for $i \in \mathbb{N}_{2:n}$. Suppose that $\hat{x}_{is} \in \mathcal{C}^1$ for $i \in \mathbb{N}_{1:n}$ and $\hat{u}_s \in \mathcal{C}^0$ with respect to their respective arguments for all $t \geq 0$.

Lemma 4 *The disturbance observers (5) ensure that the steady-state estimations of system (1) converge to their real steady-states in a finite-time T_2 , that is, $\hat{x}_{is} = x_{is}, \hat{u}_s = u_s$ for all $t \geq T_2$.*

Proof: Since $\hat{x}_{is} \in \mathcal{C}^1$ and $\hat{u}_s \in \mathcal{C}^0$, it can be derived from the boundedness of disturbances, references and disturbance estimations that $x_i(t) \in \mathcal{L}_\infty$, $\dot{x}_i(t) \in \mathcal{L}_\infty$, $u_s(t) \in \mathcal{L}_\infty$, $\hat{x}_i(t) \in \mathcal{L}_\infty$, $\dot{\hat{x}}_i(t) \in \mathcal{L}_\infty$, and $\hat{u}_s(t) \in \mathcal{L}_\infty$. Since for all $i \in \mathbb{N}_{1:n}$ and $k \in \mathbb{N}_{1:n+1-i}$ the convergence of the disturbance observer (5) reveals that $z_k^i = \omega_i^{(k-1)}$ for all $t > T_1$, we can draw the conclusion that $\hat{x}_{is} = x_{is}, \hat{u}_s = u_s$ for all $t > T_2$.

Remark 1 *Combining system dynamics (1) with the finite-time disturbance observer (5), the estimation error dynamics are given by*

$$\begin{aligned}\dot{e}_0^i &= -\lambda_0^i L_i^{\frac{1}{n-i+2}} [e_0^i]^{\frac{n-i+1}{n-i+2}} + e_1^i, \\ \dot{e}_j^i &= -\lambda_j^i L_i^{\frac{1}{n-i+2-j}} [e_j^i - e_{j-1}^i]^{\frac{n-i+1-j}{n-i+2-j}} + e_{j+1}^i, \\ \dot{e}_{n-i+1}^i &\in -\lambda_{n-i+1}^i L_i \text{sgn}(e_{n-i+1}^i - e_{n-i}^i) + [-L_i, L_i],\end{aligned}\quad (8)$$

where the estimation errors are defined as $e_0^i = z_0^i - x_i$, $e_j^i = z_j^i - \omega_i^{(j-1)}$. According to [2], the differential inclusion is homogeneous with homogeneity degree -1 weights $n-i-j+2, n-i-j+1, \dots, 1$ of $e_0^i, e_1^i, \dots, e_{n-i+1}^i$, respectively, and the estimation errors will converge to zero in a finite-time T_1 . In [26], it has been proven that the finite-time T_1 is related to the initial conditions and the observer parameters. Furthermore, the value of T_1 can be arbitrary small if the observer gains are selected sufficiently large enough [26], and the value of T_2 depends on T_1 and system dynamics.

3.2 Continuous NTSM controller design

Denote $\eta_i = x_i - \hat{x}_{is}$. Let ρ satisfy $\rho \geq \max_{i \in \mathbb{N}_{1:n}} \{r_i\}$. A new NTSM manifold is designed as

$$s(t) = \eta_n(t) - \eta_n(0) + \int_0^t \lambda_n(\bar{\eta}_n) [\sigma(\tau)]^{\frac{r_{n+1}}{\rho}} d\tau, \quad (9)$$

where $\sigma(t) = \sum_{i=1}^n \lambda_i(\bar{\eta}_n) [\eta_i]^{\frac{\rho}{r_i}}$, $\lambda_i(\bar{\eta}_n) = \prod_{l=i}^n \kappa_i^{\frac{\rho}{r_{l+1}}}(\bar{\eta}_l)$, and $\kappa_i(\bar{\eta}_i), i \in \mathbb{N}_{1:n}$ are nonlinear functions with respect to $\bar{\eta}_i$.

With the sliding manifold (9), a recursive NTSM controller is designed as follows:

$$u = -[u_{eq}(t) + u_r(t)], \quad (10)$$

where the equivalent control law is constructed as

$$u_{eq} = \hat{u}_s - \lambda_n(\bar{\eta}_n) [\sigma(t)]^{\frac{r_{n+1}}{\rho}} - f_n(\bar{x}_n) + f_n(\hat{x}_{ns}), \quad (11)$$

and the reaching law is designed as

$$u_r = k_1 [s]^{\frac{1}{2}} + k_2 \int_0^t [s(\tau)]^0 d\tau, \quad (12)$$

with $k_1 > 0$ and $k_2 > 0$ are parameters to be determined.

4 Performance and stability analysis

The performance and stability analysis is based on Lyapunov theory, which is elaborated in this section.

Design the following dynamic compensators:

$$\begin{aligned} \eta_1^* &= 0, \xi_1 = [\eta_1]^{\frac{\rho}{r_1}}, \\ \eta_i^* &= -\lambda_{i-1}(\bar{\eta}_{i-1}) [\xi_{i-1}]^{\frac{r_i}{\rho}}, \\ \xi_i &= [\eta_i]^{\frac{\rho}{r_i}} - [\eta_i^*]^{\frac{\rho}{r_i}}, \end{aligned} \quad (13)$$

for $i \in \mathbb{N}_{2:n}$, $\mu \geq \max_{i \in \mathbb{N}_{1:n}} \{r_{i+1}, \rho\}$. Next, a positive definite Lyapunov function is designed as

$$V_i(\bar{\eta}_i) = \sum_{k=1}^i W_k(\bar{\eta}_k), \quad (14)$$

with $W_k(\bar{\eta}_k) = \int_{\eta_i^*}^{\eta_i} \left[[q]^{\frac{\rho}{r_i}} - [\eta_i^*]^{\frac{\rho}{r_i}} \right]^{\frac{2\mu-r_{i+1}}{\rho}} dq$.

Using the lemmas given in Section 2 and the Lyapunov function (14), the main result of this paper can be obtained, which is summarized by the following theorem.

Theorem 1 For the n -th order nonlinear system (1) with mismatched terms, the finite-time exact tracking for the system output can be guaranteed using the finite-time disturbance observer based NTSM controller (5), (9)-(12), i.e. $y(t) \rightarrow r(t)$ as $t \rightarrow T$ with T is a finite-time.

Proof: Taking the time-derivative of η_i , one obtains

$$\begin{aligned} \dot{\eta}_i &= \eta_{i+1} + f_i(\bar{x}_i) - f_i(\hat{x}_{is}) + \varepsilon_i(t), i \in \mathbb{N}_{1:n-1}, \\ \dot{\eta}_n &= u - \hat{u}_s + f_n(\bar{x}_n) - f_n(\hat{x}_{ns}) + \varepsilon_n(t), \end{aligned} \quad (15)$$

where

$$\begin{aligned} \varepsilon_1(t) &= f_1(\hat{x}_{1s}) - f_1(x_{1s}) + \hat{x}_{2s} - x_{2s}, \\ \varepsilon_i(t) &= f_i(\hat{x}_{is}) - f_i(\bar{x}_{is}) + \hat{x}_{(i+1)s} - x_{(i+1)s} + \hat{x}_{is}, \end{aligned} \quad (16)$$

for $i \in \mathbb{N}_{2:n}$. To proceed the proof, the following proposition is required.

Proposition 1. With the help of Lemmas 1 and 2, there is the following inequality:

$$\begin{aligned} \dot{V}_n(\bar{\eta}_n) &\leq \left(\tilde{c}_{n-1}(\bar{\eta}_{n-1}) + \hat{c}_n(\bar{\eta}_n) + c_n \right) [\xi_n]^{\frac{2\mu}{\rho}} \\ &\quad + \frac{1}{2} \sum_{j=1}^{n-1} [\varepsilon_j]^{\frac{2\mu}{r_{j+1}}} - \sum_{j=1}^{n-1} [\xi_j]^{\frac{2\mu}{\rho}} \\ &\quad + [\xi_n]^{\frac{2\mu-r_{n+1}}{\rho}} \left(u - \hat{u}_s + f_n(\bar{x}_n) - f_n(\hat{x}_{ns}) + \varepsilon_n \right), \end{aligned} \quad (17)$$

with $\tilde{c}_{n-1}(\bar{\eta}_{n-1})$ and $\hat{c}_n(\bar{\eta}_n)$ are nonlinear functions with respect to $\bar{\eta}_{n-1}$ and $\bar{\eta}_n$, respectively, and c_n is a constant.

The detailed proof for Proposition 1 can be referred to Appendix A.

Define $v(t) = \varepsilon_n(t) - k_2 \int_0^t [s(\tau)]^0 d\tau$. Taking the derivatives of s and $v(t)$ along system dynamics, one obtains

$$\begin{aligned} \dot{s}(t) &= -k_1 [s(t)]^{\frac{1}{2}} + v(t), \\ \dot{v}(t) &= -k_2 [s(t)]^0 + o(t), \end{aligned} \quad (18)$$

where $o(t) = \dot{\varepsilon}_n$. It is supposed that there exists a positive constant δ such that $o(t) \leq \delta$. Define a quadratic Lyapunov function $V_s(s, v) = \zeta^T P \zeta$ with $\zeta = \left[|s(t)|^{\frac{1}{2}}, v(t) \right]^T$ and $P = \begin{bmatrix} 4k_2 + k_1^2 & -k_1 \\ -k_1 & 2 \end{bmatrix}$. The derivative of $V_s(s, v)$ is given as follows

$$\begin{aligned} \dot{V}(s, v) &= \frac{1}{|s|^{\frac{1}{2}}} \zeta^T (A^T P + AP) \zeta \\ &\leq -\frac{1}{|s|^{\frac{1}{2}}} \zeta^T Q \zeta \leq -\gamma_s V_s^{\frac{1}{2}}, \end{aligned} \quad (19)$$

where $Q = \frac{k_1}{2} \begin{bmatrix} 4k_2 + 2k_1^2 + 4\delta & -2k_1 - \frac{4\delta}{k_1} \\ -2k_1 - \frac{4\delta}{k_1} & 2 \end{bmatrix}$, $\gamma_s = \frac{\lambda_{\min}^{\frac{1}{2}}\{P\} \lambda_{\min}\{Q\}}{\lambda_{\max}\{P\}}$, with $\lambda_{\min}\{Q\}$ is the minimum eigenvalue of matrix Q , $\lambda_{\max}\{P\}$ and $\lambda_{\min}\{P\}$ are maximum and minimum eigenvalues of P , respectively.

It follows from [33] and (19) that the system states will reach the sliding manifold in a finite time, i.e. there exists a finite time $T_r = \frac{2V_s^{\frac{1}{2}}(s(0), v(0))}{\gamma_s}$ such that $V_s(s(t), v(t)) = 0$, $s(t) = 0$ and $v(t) = 0$ for $t \geq T_r$. It can be further obtained from (18) that $\dot{s} = 0$ for $t \geq T_r$.

According to Bihari's inequality, it obtains

$$\begin{aligned} \|\zeta\| &\leq V_s^{\frac{1}{2}} / \lambda_{\min}\{P\} \leq V_s^{\frac{1}{2}}(0) / \lambda_{\min}\{P\} \\ &= \zeta_{\max}. \end{aligned} \quad (20)$$

It can be derived from (20) that

$$\begin{aligned} |\varepsilon_n - k_1 |s|^{\frac{1}{2}} - k_2 \int_0^t |s|^0 d\tau| &= |v - k_1 |s|^{\frac{1}{2}}| \\ &\leq \|\zeta\| \sqrt{2 \max\{k_1^2, 1\}} \leq \zeta_{\max} \sqrt{2 \max\{k_1^2, 1\}}. \end{aligned} \quad (21)$$

By Lemma 2 and inequality (21), one has

$$\begin{aligned} |\xi_n|^{\frac{2\mu - r_{n+1}}{\rho}} \left(\varepsilon_n - k_1 |s|^{\frac{1}{2}} - k_2 \int_0^t |s|^0 d\tau \right) \\ \leq \check{c} |\xi_n|^{\frac{2\mu}{\rho}} + m, \end{aligned} \quad (22)$$

where $\check{c}$ is a positive constant that can be arbitrarily small, and

$$\begin{aligned} m &= \frac{r_{n+1}}{2\mu} \left(\frac{2\mu - r_{n+1}}{2\mu \check{c}} \right)^{\frac{2\mu(2\mu - r_{n+1})}{\rho r_{n+1}}} \\ &\quad \times \left(\zeta_{\max} \sqrt{2 \max\{k_1^2, 1\}} \right)^{\frac{2\mu}{r_{n+1}}}. \end{aligned}$$

Substituting (9)-(12) and (22) into (17) obtains

$$\dot{V}_n(\bar{\eta}_n) \leq -\sum_{j=1}^n |\xi_j|^{\frac{2\mu}{\rho}} + \sum_{j=1}^{n-1} |\varepsilon_j|^{\frac{2\mu}{r_{j+1}}} + m, \quad (23)$$

where the control gain $\lambda_n(\bar{\eta}_n) \geq -1 + \tilde{c}_{n-1}(\bar{\eta}_{n-1}) + \hat{c}_n(\bar{\eta}_n) + \bar{c}_n + \check{c} + m$.

Since $f_i \in \mathcal{C}^0$, it directly follows from Lemma 4 and (16) that $\varepsilon_i \in \mathcal{L}_\infty$ and $\varepsilon_i = 0$ for $t > T_2$. Therefore, there exists a constant $\beta > 0$ such that $\beta = \sup_{t>0} \left(\sum_{j=1}^{n-1} |\varepsilon_j|^{\frac{2\mu}{r_{j+1}}}(t) \right)$. From the analysis above, we can conclude that the closed-loop system will not escape to infinity during the transient of FTDO. Since $\varepsilon_i \in \mathcal{L}_\infty$, $\eta_i \in \mathcal{L}_\infty$ and $\varepsilon_i(t) = 0$ for $t \geq T_2$, we have

$$\dot{V}_n(\bar{\eta}_n) \leq -\sum_{j=1}^n |\xi_j|^{\frac{2\mu}{\rho}} + m, \quad (24)$$

for all $t \geq T_2$. To proceed the stability analysis, we have the following proposition whose proof is included in Appendix.

Proposition 2. The closed-loop system states will not escape into infinity in a finite time and can be bounded by

$$|\eta_i| \leq \delta_i(\bar{\eta}_n(0), \tau, \bar{r}), \quad (25)$$

where $\delta_i(\bar{\eta}_n(0), \tau, \bar{r})$ is a constant in terms of initial states and controller parameters.

Since $\dot{s} = 0$ for all $t \geq T_r$, according to (18) we obtain $\varepsilon_n - k_1 |s|^{\frac{1}{2}} - k_2 \int_0^t |s|^0 d\tau = 0$ for $t \geq T_r$. Based on *Proposition 2*, we have $\dot{V}_n(\bar{\eta}_n) \leq -\sum_{j=1}^n |\xi_j|^{\frac{2\mu}{\rho}}$. According to LaSalle invariant principal, $\xi_i(t) \rightarrow 0$, $\eta_i(t) \rightarrow 0$ and hence

$x_1(t) \rightarrow r(t)$ as $t \rightarrow \infty$, which shows the asymptotically exact tracking performance of the closed-loop system.

By definition of V_n and Lemma 1, we have the following inequality

$$V_n(\bar{\eta}_n) \leq \vartheta \sum_{j=1}^n |\xi_j|^{\frac{2\mu-\tau}{\rho}}, \quad (26)$$

where ϑ is a positive constant. Let $\alpha = \frac{1}{2}\vartheta^{\frac{2\mu-\tau}{2\mu}}$, it can be induced from (24) and (26) that

$$\dot{V}_n(\bar{\eta}_n) + \alpha V_n^{\frac{2\mu}{2\mu-\tau}}(\bar{\eta}_n) \leq -\frac{1}{2} \sum_{j=1}^n |\xi_n|^{\frac{2\mu}{\rho}}, \quad (27)$$

for $t \geq \max\{T_r, T_2\}$. By Lemma 3, it follows from (27) that $\xi_i(t) \rightarrow 0, \eta_i(t) \rightarrow 0$ and subsequently $x_i(t) \rightarrow x_{is}(t)$ as $t \rightarrow T$, where $\epsilon = (r_1/\rho - 1)^{2\mu/(2\mu-\tau)}, T = \max\{T_r, T_2\} + 2^{n-1}(2\mu - \tau)V_n^{\frac{-\tau}{2\mu-\tau}}(\bar{\eta}(0))/(-\epsilon\tau)$.

The block diagram of the proposed continuous nonsingular terminal sliding mode control scheme is shown in Fig. 1.

Remark 2 *It can be seen that there are some parameters in the controller to be assigned. In the finite-time disturbance observer, the parameter L_i should be larger than the upper bounds of disturbances ω_i and their derivatives to ensure that the estimation errors converge to zero in a finite-time. According to [2], a possible choice of the differentiator parameters λ_j^i for $n \leq 5$ is $\lambda_0^i = 1.1, \lambda_1^i = 1.5, \lambda_2^i = 3, \lambda_3^i = 5, \lambda_4^i = 8, \lambda_5^i = 12$. Another possible choice is $\lambda_0^i = 1.1, \lambda_1^i = 1.5, \lambda_2^i = 2, \lambda_3^i = 3, \lambda_4^i = 5, \lambda_5^i = 8$. For other cases, the details are provided in [2] and [26]. The feedback control gains k_1 and k_2 should be positive. Large values of feedback control gains will reduce the settling time but the overshoot and the control efforts*

will be large. Generally, when choosing the parameters, one needs to balance the overshoot, the settling time, and the control efforts.

5 Simulations

In this section, a single-link manipulator system [34] including actuator dynamics is considered. The system mathematical model is given by

$$\begin{aligned} M\ddot{q} + B\dot{q} + N\sin(q) &= I + d_2, \\ D\dot{I} + HI &= V - K_m\dot{q} + d_3, \end{aligned} \quad (28)$$

where $q, \dot{q}$ and $\ddot{q}$ denote the link angular position, velocity and acceleration, respectively, I is the motor current, V denotes the voltage to be used in the control input, M denotes the lumped inertia, B is the friction coefficient, N denotes the lumped load term, D is the rotor inductance, H is the rotor resistance, K_m is the back-emf coefficient. d_2 and d_3 are unknown external disturbances. In simulation, the parameters are $M = 1\text{kgm}^2/\text{rad}$, $B = 1$, $N = 10\text{kgA}$, $D = 0.05\text{H}$, $H = 0.5\Omega$ and $K_m = 10\text{Nm/A}$. Let $x_1 = q, x_2 = \dot{q}, x_3 = I$ and $u = V$. The system model is written as

$$\begin{aligned} \dot{x}_1 &= x_2, \dot{x}_2 = x_3 + f_2(x_1, x_2) + d_2, \\ \dot{x}_3 &= \frac{u}{0.05} + f_3(x_1, x_2, x_3) + \bar{d}_3, y = x_1, \end{aligned} \quad (29)$$

where $f_2(x_1, x_2) = -10\sin(x_1) - x_2$, $f_3(x_1, x_2, x_3) = (-10x_2 - 0.5x_3)/0.05$, $\bar{d}_3 = d_3/0.05$ and y denotes the output of the manipulator system.

The initial conditions are $[x_1(0), x_2(0), x_3(0)]^T = [0.2, 0.2, 0.1]^T$. The reference signal to be tracked is $y_r = 0.5\sin(0.5t)$, and $d_2 = 10\sin(\pi t)$ for $t \leq 16\text{s}$, $d_2 = 10\sin(\frac{\pi}{2}t)$ for $t > 16\text{s}$, $\bar{d}_3 = -20\sin(\pi t)$ for $t \leq 12\text{s}$, $\bar{d}_3 = -20\sin(\frac{\pi}{3}t)$ for $t > 12\text{s}$.

For system (29), FTDO are designed based on (3). The steady-states of $x_i (i = 1, 2, 3)$ and the

Figure 1: Implementation block diagram of the proposed continuous NTSM controller.

control input are shown as follows:

$$\begin{aligned}
 x_{1s} &= y_r, \hat{x}_{2s} = \dot{y}_r, \\
 \hat{x}_{3s} &= \ddot{y}_r + 10\sin(y_r) + \dot{y}_r - z_1^2, \\
 \hat{u}_s &= y_r^{(3)} + \ddot{y}_r + 10\dot{y}_r\cos(y_r) \\
 &\quad + \frac{10\hat{x}_{2s} + 0.5\hat{x}_{3s}}{0.05} - z_2^2 - z_1^3.
 \end{aligned} \tag{30}$$

Denote $\eta_1 = x_1 - x_{1s}$, $\eta_2 = x_2 - \hat{x}_{2s}$, $\eta_3 = x_3 - \hat{x}_{3s}$. The sliding manifold is designed as

$$s = \eta_3(t) - \eta_3(0) + \int_0^t \lambda_3(\eta_1, \eta_2, \eta_3) [\sigma(\tau)]^{\frac{r_4}{\rho}} d\tau, \tag{31}$$

with $\sigma(t) = \lambda_1[\eta_1]^{\frac{\rho}{r_1}} + \lambda_2[\eta_2]^{\frac{\rho}{r_2}} + \lambda_3[\eta_3]^{\frac{\rho}{r_3}}$. The NTSM controller is

$$\begin{aligned}
 u &= -0.05 \left[k_1 |s|^{\frac{1}{2}} + k_2 \int_0^t |s|^0 d\tau \right. \\
 &\quad \left. - \lambda_3(\eta_1, \eta_2, \eta_3) [\sigma]^{\frac{r_4}{\rho}} + \frac{10x_2 + 0.5x_3}{0.05} \right. \\
 &\quad \left. - \frac{10\hat{x}_{2s} + 0.5\hat{x}_{3s}}{0.05} + 20\hat{u}_s \right],
 \end{aligned} \tag{32}$$

where $\rho = 1$, $r_1 = 1$, $\tau = -0.1$, $k_1 = 5$, $k_2 = 10$, $\lambda_1 = 10$, $\lambda_2 = 10$, $\lambda_3 = 100$.

Figs. 2 and 3 show the simulation results of the system states and the control input. It can be observed from Fig. 2 that the angular position

x_1 tracks the reference signal y_r well under the proposed NTSM controller in the presence of unknown time-varying external disturbances. Moreover, the angular velocity x_2 , the motor current x_3 and the control signal u reach their pre-calculated steady-states x_{2s} , x_{3s} and u_s after a transient period, respectively. Fig. 3 clearly shows that the steady-state estimations converge to their true values in a finite-time.

6 Conclusion

In this paper, a continuous NTSM control strategy is proposed to solve the problem of the finite-time exact tracking and disturbance rejection for nonlinear systems with matched and mismatched nonlinearities as well as matched and mismatched time-varying disturbances. The mismatched disturbances are exactly estimated by the disturbance observer and compensated in the designed nonlinear recursive sliding manifold. The Lyapunov theory is used to demonstrate and validate that the tracking error of the closed-loop sys-

Figure 2: Response curves of system states and control signal: (a) system output x_1 ; (b) Control signal u ; (c) Angular velocity x_2 ; (d) Motor current x_3 .

Figure 3: Response curves of estimations of the invariant variables: (a) x_{3s} and $\hat{x}_{3s}$; (b) u_s and $\hat{u}_s$.

tem converges to zero in a finite-time under the proposed controller. The simulation results verify the effectiveness of the proposed continuous NTSM controller. The finite-time sliding mode controller for multi-input-multi-output systems with unknown mismatched nonlinearities and mismatched disturbances will be considered in the future.

Appendix A: proof of the Proposition 1

Step 1. With the definition of V_i , one obtains

$$V_1(\eta_1) = \int_{\eta_1^*}^{\eta_1} \left[|q|^{\frac{\rho}{r_1}} - |\eta_1^*|^{\frac{\rho}{r_1}} \right]^{\frac{2\mu-r_2}{\rho}} dq, \quad (33)$$

By Lemma 2 and (15), the time derivative of $V_1(\eta_1)$ is

$$\begin{aligned}\dot{V}_1(\eta_1) &= \lfloor \eta_1 \rfloor^{\frac{2\mu-r_2}{r_1}} [\eta_2 + f_1(x_1) - f_1(\hat{\pi}_1) + \varepsilon_1(t)] \\ &\leq \lfloor \eta_1 \rfloor^{\frac{2\mu-r_2}{r_1}} (\eta_2 - \eta_2^* + \eta_2^*) + \frac{1}{\nu_1} \lfloor \varepsilon_1 \rfloor^{\frac{2\mu}{r_2}} \\ &\quad + [\gamma_1(\eta_1) + \varrho_1] \lfloor \eta_1 \rfloor^{\frac{2\mu}{r_1}},\end{aligned}\quad (34)$$

where $\nu_1 = n, \varrho_1 = \frac{2\mu-r_2}{2\mu} (\frac{2\mu}{r_2\nu_1})^{-\frac{2\mu r_2}{r_1(2\mu-r_2)}} > 0$ is a constant. By (2), it can be induced that the following inequality holds

$$|f_i(\bar{x}_i) - f_i(\hat{x}_{is})| \leq \gamma_i(\bar{\eta}_i) \sum_{j=1}^i |\eta_j|^{\frac{r_{i+1}}{r_j}}, \quad (35)$$

with $i \in \mathbb{N}_{1:n}$, $\gamma_i(\bar{\eta}_i) > 0$ is a $\mathcal{C}^0$ function.

The virtual control law η_2^* is designed as $\eta_2^* = -k_1(\eta_1) \lfloor \xi_1 \rfloor^{\frac{r_2}{\rho}}$, with $\xi_1 = \lfloor \eta_1 \rfloor^{\frac{\rho}{r_1}}$ and $k_1(\eta_1)$ is the control gain assigned such that $k_1(\eta_1) \geq \nu_1 + \gamma_1(\eta_1) + \varrho_1$. Substituting the virtual control law η_2^* into (34) gets

$$\dot{V}_1(\eta_1) \leq \lfloor \xi_1 \rfloor^{\frac{2\mu-r_2}{\rho}} (\eta_2 - \eta_2^*) - \nu_1 \lfloor \xi_1 \rfloor^{\frac{2\mu}{\rho}} + \frac{1}{\nu_1} \lfloor \varepsilon_1 \rfloor^{\frac{2\mu}{r_2}}. \quad (36)$$

Induction step. It is supposed that at step $i-1$, $V_{i-1}(\bar{\eta}_{i-1}) : \mathbb{R}^{i-1} \rightarrow \mathbb{R}$ is a positive definite and proper $\mathcal{C}^1$ Lyapunov function. Choose nonlinear control gains $\kappa_j(\bar{\eta}_j) > 0$ such that

$$\begin{aligned}\dot{V}_{i-1}(\bar{\eta}_{i-1}) &\leq \lfloor \xi_{i-1} \rfloor^{\frac{2\mu-r_i}{\rho}} (\eta_i - \eta_i^*) - \nu_{i-1} \sum_{j=1}^{i-1} \lfloor \xi_j \rfloor^{\frac{2\mu}{\rho}} \\ &\quad + \frac{1}{\nu_{i-1}} \sum_{j=1}^{i-1} \lfloor \varepsilon_j \rfloor^{\frac{2\mu}{r_{j+1}}},\end{aligned}\quad (37)$$

where $\nu_i = n + 1 - i$.

Note that $V_i(\bar{\eta}_i) = V_{i-1}(\bar{\eta}_{i-1}) + W_i(\bar{\eta}_i)$ with $W_i(\bar{\eta}_i) = \int_{\eta_i^*}^{\eta_i} \left[\lfloor q \rfloor^{\frac{\rho}{r_i}} - \lfloor \eta_i^* \rfloor^{\frac{\rho}{r_i}} \right]^{\frac{2\mu-r_{i+1}}{\rho}} dq$. As a re-

sult, the time derivative of $V_i(\bar{\eta}_i)$ along (15) is

$$\begin{aligned}\dot{V}_i(\bar{\eta}_i) &= \dot{V}_{i-1}(\bar{\eta}_{i-1}) + \lfloor \xi_i \rfloor^{\frac{2\mu-r_{i+1}}{\rho}} \dot{\eta}_i + \sum_{l=1}^{i-1} \frac{\partial W_i}{\partial \eta_l} \dot{\eta}_l, \\ &\leq \dot{V}_{i-1}(\bar{\eta}_{i-1}) + \lfloor \xi_i \rfloor^{\frac{2\mu-r_{i+1}}{\rho}} (\eta_{i+1} - \eta_{i+1}^*) \\ &\quad + \sum_{l=1}^{i-1} \frac{\partial W_i}{\partial \eta_l} \dot{\eta}_l + \lfloor \xi_i \rfloor^{\frac{2\mu-r_{i+1}}{\rho}} \left(\eta_{i+1}^* \right. \\ &\quad \left. + f_i(\bar{x}_i) - f_i(\hat{\pi}_i) + \varepsilon_i(t) \right),\end{aligned}\quad (38)$$

for a virtual control law η_{i+1}^* to be designed later. Next, each term in the right hand side of (38) is estimated.

Now that $\rho \geq r_1$ and $r_{i+1} = r_i + \tau$. By Lemma 1 and Lemma 2, there is

$$|\lfloor \xi_{i-1} \rfloor^{\frac{2\mu-r_i}{\rho}} (\eta_i - \eta_i^*)| \leq \frac{1}{3} \lfloor \xi_{i-1} \rfloor^{\frac{2\mu}{\rho}} + \bar{c}_i \lfloor \xi_i \rfloor^{\frac{2\mu}{\rho}}, \quad (39)$$

where $\bar{c}_i$ is a positive constant. Define $\varsigma_l(\bar{\eta}_l) = \kappa_l^{\frac{\rho}{r_{l+1}}}(\bar{\eta}_l)$ for $l \in \mathbb{N}_{1:i-1}$. By Lemma 1 and (35), one obtains [36]

$$\begin{aligned}& \left| \lfloor \xi_i \rfloor^{\frac{2\mu-r_{i+1}}{\rho}} (f_i(\bar{x}_i) - f_i(\hat{\pi}_i)) \right| \\ & \leq \left| \lfloor \xi_i \rfloor^{\frac{2\mu-r_{i+1}}{\rho}} \right| (\gamma_i(\bar{\eta}_i) \sum_{j=1}^i |\xi_i - \varsigma_{j-1} \xi_i|^{\frac{r_{i+1}}{\rho}} + |\varepsilon_i|) \\ & \leq \frac{1}{2} \sum_{j=1}^{i-2} \lfloor \xi_j \rfloor^{\frac{2\mu}{\rho}} + \frac{1}{3} \lfloor \xi_{i-1} \rfloor^{\frac{2\mu}{\rho}} + \frac{1}{\nu_i} \lfloor \varepsilon_i \rfloor^{\frac{2\mu}{r_{i+1}}} \\ & \quad + \hat{c}_i(\bar{\eta}_i) \lfloor \xi_i \rfloor^{\frac{2\mu}{\rho}}.\end{aligned}\quad (40)$$

By Lemma 1, Lemma 2 and (35), the following inequality can be obtained:

$$\begin{aligned}\sum_{l=1}^{i-1} \left| \frac{\partial W_i}{\partial \eta_l} \dot{\eta}_l \right| &\leq \frac{1}{2} \sum_{j=1}^{i-2} \lfloor \xi_j \rfloor^{\frac{2\mu}{\rho}} + \bar{c}_{i-1}(\bar{\eta}_{i-1}) \lfloor \xi_i \rfloor^{\frac{2\mu}{\rho}} \\ &\quad + \frac{1}{3} \lfloor \xi_{i-1} \rfloor^{\frac{2\mu}{\rho}} + \frac{1}{\nu_i \nu_{i-1}} \sum_{j=1}^{i-1} \lfloor \varepsilon_j \rfloor^{\frac{2\mu}{r_{j+1}}}.\end{aligned}\quad (41)$$

Substituting (37) and estimations (39)-(41) in-

to (38) gets

$$\begin{aligned}\dot{V}_i(\bar{\eta}_i) &\leq [\xi_i]^{\frac{2\mu-r_{i+1}}{\rho}}(\eta_{i+1} - \eta_{i+1}^*) - \nu_i \sum_{j=1}^{i-1} [\xi_j]^{\frac{2\mu}{\rho}} \\ &\quad + (\tilde{c}_{i-1}(\bar{\eta}_{i-1}) + \hat{c}_i(\bar{\eta}_i) + \bar{c}_i)[\xi_i]^{\frac{2\mu}{\rho}} \\ &\quad + \frac{1}{\nu_i} \sum_{j=1}^i [\varepsilon_j]^{\frac{2\mu}{r_{j+1}}} + [\xi_i]^{\frac{2\mu-r_{i+1}}{\rho}} \eta_{i+1}^*.\end{aligned}\quad (42)$$

Substituting a virtual control law

$$\eta_{i+1}^* = -k_i(\bar{\eta}_i)[\xi_i]^{\frac{r_{i+1}}{\rho}}$$

with $k_i(\bar{\eta}_i) \geq \nu_i + \tilde{c}_{i-1}(\bar{\eta}_{i-1}) + \bar{c}_i(\bar{\eta}_i) + \bar{c}_i$ into (42) yields

$$\begin{aligned}\dot{V}_i(\bar{\eta}_i) &\leq [\xi_i]^{\frac{2\mu-r_{i+1}}{\rho}}(\eta_{i+1} - \eta_{i+1}^*) - \nu_i \sum_{j=1}^i [\xi_j]^{\frac{2\mu}{\rho}} \\ &\quad + \frac{1}{\nu_i} \sum_{j=1}^i [\varepsilon_j]^{\frac{2\mu}{r_{j+1}}},\end{aligned}\quad (43)$$

with $i = 1, 2, \dots, n-1$. This completes the induction step.

With the definition of V_i in mind, combining (15) and (43), the following inequality is obtained

$$\begin{aligned}\dot{V}_n(\bar{\eta}_n) &\leq (\tilde{c}_{n-1}(\bar{\eta}_{n-1}) + \hat{c}_n(\bar{\eta}_n) + \bar{c}_n)[\xi_n]^{\frac{2\mu}{\rho}} \\ &\quad + \frac{1}{2} \sum_{j=1}^{n-1} [\varepsilon_j]^{\frac{2\mu}{r_{j+1}}} - \nu_n \sum_{j=1}^{n-1} [\xi_j]^{\frac{2\mu}{\rho}} \\ &\quad + [\xi_n]^{\frac{2\mu-r_{n+1}}{\rho}} \left(u - \hat{\pi}_u + f_n(\bar{x}_n) \right. \\ &\quad \left. - f_n(\hat{\pi}_n) + \varepsilon_n \right).\end{aligned}\quad (44)$$

This completes the proof.

Appendix B: proof of proposition 2

Define a composite Lyapunov function as follows:

$$V(\bar{\eta}_n, s, v) = V_n(\bar{\eta}_n) + \tilde{V}_s(s, v) \quad (45)$$

with $\tilde{V}_s(s, v) = \hbar V_s^{\frac{2\mu-\tau}{-2\tau}}(s, v)$, where

$$\hbar = \left(\frac{2\epsilon}{(1-2\mu/\tau)\gamma_s} \right)^{\frac{2\mu-\tau}{-2\tau}}, \epsilon = \left(\frac{r_1}{\rho} - 1 \right)^{\frac{2\mu}{2\mu-\tau}}.$$

Since $\tau \in (-\infty, 0)$, the function $V(\bar{\eta}_n, s, v)$ is positive definite. Combining (19) and (24), the derivative of $V(\bar{\eta}_n, s, v)$ yields

$$\begin{aligned}\dot{V}(\bar{\eta}_n, s, v) &\leq -\hbar\gamma_s \left(\frac{1}{2} - \frac{\mu}{\tau} \right) V_s^{\frac{\mu}{-2\tau}}(s, v) \\ &\quad - \sum_{i=1}^n [\xi_i]^{\frac{2\mu}{\rho}} + m.\end{aligned}\quad (46)$$

By definition of $V_n(\bar{\eta}_n)$, it follows from the properties of homogeneity that $\epsilon V_n^{\frac{2\mu}{2\mu-\tau}}(\bar{\eta}_n) \leq \sum_{i=1}^n [\xi_i]^{\frac{2\mu}{\rho}}$, we have

$$\begin{aligned}\dot{V}(\bar{\eta}_n, s, v) &\leq -\epsilon \left(V_n^{\frac{2\mu}{2\mu-\tau}}(\bar{\eta}_n) + \tilde{V}_s^{\frac{2\mu}{2\mu-\tau}}(s, v) \right) + m, \\ &\leq -\epsilon V^c(\bar{\eta}_n, s, v) + m,\end{aligned}\quad (47)$$

where $c = \frac{2\mu}{2\mu-\tau} \in (0, 1)$. Denote $V(t)$ as the solution of $V(\bar{\eta}_n, s, v)$ for neatness. By comparison lemma given in [35], the solution $V(t)$ satisfies

$$V(t) \leq \begin{cases} [V^{1-c}(0) - \epsilon(1-c)t]^{\frac{1}{1-c}} + V_u, & 0 \leq t \leq T_u \\ V_u, & t \geq T_u \end{cases} \quad (48)$$

where $V(0) = V(\bar{\eta}_n(0), s(0), v(0))$ denotes the initial value of $V(\bar{\eta}_n, s, v)$, and $V_u = \left(\frac{m}{\epsilon} \right)^{\frac{2\mu-\tau}{2\mu}}$, $T_u = \frac{(2\mu-\tau)V^{\frac{-\tau}{2\mu-\tau}}(0)}{-\epsilon\tau}$ represent the ultimate bound and convergence time of $V(t)$. This implies that the system states and control input are ultimate bound, i.e., $\eta_i \in \mathcal{L}_\infty, u \in \mathcal{L}_\infty$ and $s \in \mathcal{L}_\infty$.

The homogeneous degree of $V_n(\bar{\eta}_n)$ is $\frac{2\mu-\tau}{2}$. By homogeneity property, there exists a positive constant $\underline{c}$ such that $\underline{c} \|\bar{\eta}_n\|_{\Delta^r}^{\frac{2\mu-\tau}{2}} \leq V_n(\bar{\eta}_n) \leq V(\bar{\eta}_n, s, v)$. As a result, the boundness of system states is given by

$$|\eta_i(t)| \leq \|\bar{\eta}_n\|_{\Delta^r}^{T_i} \leq \delta_i(\bar{\eta}_n(0), \tau, \bar{r}), \quad (49)$$

where $\delta_i(\bar{\eta}_n(0), \tau, \bar{r}) = \left(\frac{V(0)+V_u}{\underline{c}} \right)^{\frac{2r_i}{2\mu-\tau}}$. This completes the proof.

Conflict of interest

The authors declare no potential conflict of interests.

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