Supplementary Material

Consistent testing for pairwise dependence in time series

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1 Some further remarks on applications

Choice of Kernel: The kernel functions chosen for the simulations are the following:

- The Daniell kernel (DAN), given by

$$k(z) = \frac{\sin(\pi z)}{\pi z}, \quad z \in \mathbb{R},$$

- The Parzen kernel (PAR), given by

$$k(z) = \begin{cases} 1 - 6(\pi z/6)^2 + 6|\pi z/6|^3, & |z| \leq 3/\pi, \\ 2(1 - |\pi z/6|)^3, & 3/\pi \leq |z| \leq 6/\pi, \\ 0, & \text{otherwise;} \end{cases}$$

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Table 1: Computational time (in seconds) for computing the test statistics T_n and $M_n^{(2)}$.

n :		100			200			500		
p :		3	7	16	3	9	25	4	13	42
T_n	BAR	0.13	0.14	0.18	1.20	1.12	1.12	18.39	18.56	17.94
	PAR	0.13	0.12	0.11	0.83	0.79	0.78	18.19	17.97	18.69
	DAN	0.11	0.12	0.11	0.80	0.80	0.78	18.93	18.48	18.47
$M_n^{(2)}$	BAR	1.20	1.21	1.20	4.79	4.74	4.72	29.61	29.58	29.58
	PAR	1.21	1.22	1.22	4.82	4.79	4.77	29.62	29.58	29.69
	DAN	1.21	1.24	1.24	4.82	4.77	4.81	29.61	29.60	29.61

- The Bartlett kernel (BAR), given by

$$k(z) = \begin{cases} 1 - |z|, & |z| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Computational Speed: Table 1 shows the computational time taken to obtain T_n and $M_n^{(2)}$, for various choices of sample size n and bandwidth p on a standard laptop with Intel Core i5 system and CPU 2.30 GHz. Clearly, $M_n^{(2)}$ is computationally more expensive than T_n , especially when n and p are large.

Power: Recall the NMA(2) model

$$X_t = \epsilon_t \epsilon_{t-1} \epsilon_{t-2}, \quad (1)$$

where $\{\epsilon_t\}$ is a sequence of i.i.d standard normal random variables. Figure 1 shows bootstrap p-values for various values of the bandwidth parameter and sample sizes. We note again that the performance of both test statistics, T_n and $M_n^{(2)}$, is superior to that of the other test statistics.

Figure 1: Bootstrap p-values of all test statistics as a function of the bandwidth p . The data are generated by the NMA(2) model given by (1). The results are based on $B = 499$ bootstrap replications and 100 simulations. The test statistics $T_n, M_n^{(1)}, M_n^{(2)}$ are calculated by employing the Daniell kernel.

Plot of ACF and ADCF of S&P 500 index: Figure 2 shows the ACF (upper plot) and the ADCF (lower plot) of the original series and the squared original series.

Figure 2: Upper plots: The ACF of the original series and the squared original series. Lower plots: The ADF of the original and the squared original series.

2 Proofs

In this section, we prove the main theorems of the paper. For following the presentation it is more convenient to recall the main assumptions.

Assumption 1 $\{X_t\}$ is a strictly stationary α -mixing process with mixing coefficients $\alpha(j)$, $j \geq 1$.

Assumption 2 $E|X_t| < \infty$.

Assumption 3 The mixing coefficients of $\{X_t\}$, $\alpha(j)$, satisfy (i) $\sum_{j=-\infty}^{\infty} \alpha(j) < \infty$, (ii) $\alpha(j) = O(1/j^2)$.

Assumption 4 Suppose that $k(\cdot)$ is a kernel function such that $k : \mathbb{R} \rightarrow [-1, 1]$, is symmetric and is continuous at 0 and at all but a finite number of points, with $k(0) = 1$, $\int_{-\infty}^{\infty} k^2(z) dz < \infty$ and $|k(z)| \leq C|z|^{-b}$ for large z and $b > 1/2$.

We first prove two Lemmas which will be employed in the proof of the main results of the paper. In what follows C , possibly with a subscript, denotes a generic constant. We define the pseudoestimator:

$$\bar{f}_n(\omega, u, v) = \frac{1}{2\pi} \sum_{j=-(n-1)}^{(n-1)} k(j/p)(1 - |j|/n)^{1/2} \tilde{\sigma}_j(u, v) e^{-ij\omega},$$

where

$$\tilde{\sigma}_j(u, v) = \frac{1}{n - |j|} \sum_{t=|j|+1}^n \psi_t(u) \psi_{t-|j|}(v) \quad (2)$$

and

$$\psi_t(u) \equiv e^{iuX_t} - \phi(u). \quad (3)$$

Lemma 1 Suppose that $\{X_t, t \geq 1\}$ satisfies Assumptions 1 and 3(ii). Then,

$$(n - j)^2 E |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 \leq C,$$

and

$$(n - |j|) E |\tilde{\sigma}_j(u, v)|^2 \leq C,$$

uniformly in $(u, v) \in \mathbb{R}^2$. The result of the Lemma is also true under independence.

Proof of Lemma 1 Note that

$$\begin{aligned} \tilde{\sigma}_j(u, v) &= \frac{1}{n - |j|} \sum_{t=|j|+1}^n e^{i(uX_t + vX_{t-|j|})} - \frac{\phi(v)}{n - |j|} \sum_{t=|j|+1}^n e^{iuX_t} \\ &\quad - \frac{\phi(u)}{n - |j|} \sum_{t=|j|+1}^n e^{ivX_{t-|j|}} + \phi(u)\phi(v) \end{aligned}$$

Therefore, by subtracting $\tilde{\sigma}_j(u, v)$ from $\hat{\sigma}_j(u, v)$ we get:

$$\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v) = -\frac{1}{(n - |j|)^2} \sum_{t=|j|+1}^n \psi_t(u) \sum_{t=|j|+1}^n \psi_{t-|j|}(v).$$

The Cauchy - Schwarz inequality implies that

$$E |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 \leq \frac{1}{(n - j)^4} \left\{ E \left| \sum_{t=|j|+1}^n \psi_t(u) \right|^4 E \left| \sum_{t=|j|+1}^n \psi_{t-|j|}(v) \right|^4 \right\}^{1/2} \leq \frac{C}{(n - j)^2}$$

uniformly because $E \left| \sum_{t=|j|+1}^n \psi_t(u) \right|^4 \leq C(n - j)^2$ (Doukhan and Louhichi, 1999, Lemma 6).

From the definition of $\tilde{\sigma}_j(u, v)$ in (2), it follows that:

$$E |\tilde{\sigma}_j(u, v)|^2 = E \left| \frac{1}{n - |j|} \sum_{t=|j|+1}^n \psi_t(u) \psi_{t-|j|}(v) \right|^2 \leq \frac{C}{n - |j|}$$

uniformly. □

Lemma 2 Suppose that $\{X_t, t \geq 1\}$ satisfies Assumptions 1 and 3(ii). For each $\gamma > 0$, denote by $D(\gamma)$ the region $D(\gamma) = \{(u, v) : \gamma \leq |u| \leq 1/\gamma, \gamma \leq |v| \leq 1/\gamma\}$. Then, under Assumption 4, for any fixed $\gamma > 0$,

$$\int_{D(\gamma)} \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left\{ |\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} = O_P(p/\sqrt{n}) = o_P(\sqrt{p})$$

as $p/n \rightarrow 0$. The result of the Lemma is also true under independence.

Proof of Lemma 2 From the properties of kernels we obtain that

$$\sum_{|j| < n} \frac{1}{n - |j|} k^2(j/p) = O(p/n)$$

for bandwidth $p = cn^\lambda$, $\lambda \in (0, 1)$. In addition,

$$\sum_{j=1}^{n-1} \frac{1}{n-j} k^2(j/p) = O(p/n) \quad (4)$$

and

$$\sum_{j=1}^{n-1} \frac{1}{\sqrt{n-j}} k^2(j/p) = O(p/\sqrt{n}). \quad (5)$$

Now, define

$$Z_{n;p}^\gamma = \int_{D(\gamma)} \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left\{ |\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 \right\} d\mathcal{W},$$

and observe that, for any fixed $\gamma > 0$, the chosen weight function $\mathcal{W}(u, v)$ is bounded on $D(\gamma)$. Thus, $\mathcal{W}(u, v)$ is an integrable function in this region.

Using

$$|\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 = |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 + 2\text{Re}\{|\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)| \tilde{\sigma}_j(u, v)^*\},$$

where $*$ denotes complex conjugate, we get the following:

$$\begin{aligned}
E |Z_{n;p}^\gamma| &\leq \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) E \left(|\hat{\sigma}_j(u, v)|^2 - |\tilde{\sigma}_j(u, v)|^2 \right) \right\} d\mathcal{W} \\
&= \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} \frac{k^2(j/p)}{(n-j)} (n-j)^2 E \left(|\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 \right) \right\} d\mathcal{W} \\
&\quad + 2 \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) \operatorname{Re} \{ E |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)| \tilde{\sigma}_j(u, v)^* \} \right\} d\mathcal{W}.
\end{aligned}$$

Now Lemma 1, the Cauchy - Schwarz inequality and the fact that $\int_{D(\gamma)} d\mathcal{W} < \infty$ show that,

$$\begin{aligned}
E |Z_{n;p}^\gamma| &\leq \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} \frac{k^2(j/p)}{(n-j)} (n-j)^2 E \left(|\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2 \right) \right\} d\mathcal{W} \\
&\quad + 2 \int_{D(\gamma)} \sum_{j=1}^{n-1} \frac{k^2(j/p)}{\sqrt{n-j}} \sqrt{(n-j)^2 E |\hat{\sigma}_j(u, v) - \tilde{\sigma}_j(u, v)|^2} \sqrt{(n-j) E |\tilde{\sigma}_j(u, v)|^2} d\mathcal{W} \\
&\leq C_1 \int_{D(\gamma)} d\mathcal{W} \sum_{j=1}^{n-1} \frac{k^2(j/p)}{n-j} + C_2 \int_{D(\gamma)} d\mathcal{W} \sum_{j=1}^{n-1} \frac{k^2(j/p)}{\sqrt{n-j}} \\
&= O(p/\sqrt{n})
\end{aligned}$$

as $p/n \rightarrow 0$. By Markov's inequality this implies the first result. The second result follows immediately. $\square$

Proof of Proposition 1 Recall the region $D(\gamma)$ defined in Lemma 2, and for each $\gamma > 0$, define the random variables

$$\hat{V}_{X;\gamma}^2(j) = \int_{D(\gamma)} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W}. \quad (6)$$

Because of the SLLN for α -mixing random variables and the fact that $\int_{D(\gamma)} d\mathcal{W} < \infty$, we obtain

$$\lim_{n \rightarrow \infty} \hat{V}_{X;\gamma}^2(j) = V_{X;\gamma}^2(j) = \int_{D(\gamma)} |\sigma_j(u, v)|^2 d\mathcal{W},$$

almost surely. Clearly, $V_{X;\gamma}^2(j) \rightarrow V_X^2(j)$ as γ tends to zero. So, it remains to prove that almost surely

$$\limsup_{\gamma \rightarrow 0} \limsup_{n \rightarrow \infty} \left| \hat{V}_{X;\gamma}^2(j) - \hat{V}_X^2(j) \right| = 0. \quad (7)$$

For each $\gamma > 0$, we obtain that

$$\begin{aligned} \left| \hat{V}_{X;\gamma}^2(j) - \hat{V}_X^2(j) \right| &\leq \int_{|u| < \gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} + \int_{|u| > 1/\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} \\ &+ \int_{|v| < \gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} + \int_{|v| > 1/\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W}. \end{aligned} \quad (8)$$

Now, for $z \in \mathbb{R}$, define $H(y) = \int_{|z| < y} (1 - \cos z)/|z|^2 dz$. This is bounded by $1/\pi^2$ and $\lim_{y \rightarrow 0} H(y) = 0$. Because of $|x + y|^2 \leq 2|x|^2 + 2|y|^2$ and the Cauchy-Schwarz inequality

$$|\hat{\sigma}_j(u, v)|^2 \leq 4 \left\{ \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n |\psi_t(u)|^2 \right\} \left\{ \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n |\psi_{t-|j|}(v)|^2 \right\}. \quad (9)$$

But, the first summand in (8) satisfies:

$$\begin{aligned} \int_{|u| < \gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} &\leq \left\{ \frac{4}{(n - |j|)} \sum_{t=|j|+1}^n \int_{|u| < \gamma} \frac{|\psi_t(u)|^2}{\pi |u|^2} du \right\} \\ &\times \left\{ \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n \int_{\mathbb{R}} \frac{|\psi_{t-|j|}(v)|^2}{\pi |v|^2} dv \right\}. \end{aligned} \quad (10)$$

However, (3) yields

$$|\psi_{t-|j|}(v)|^2 = 1 + |\phi(v)|^2 - \phi(v)^* e^{ivX_{t-|j|}} - \phi(v) (e^{ivX_{t-|j|}})^*$$

and similarly for $|\psi_t(u)|^2$. Now, letting $X_{t-|j|} \equiv Y_t$ and using Székely et al. (2007, Lemma 1) we get

$$\int_{\mathbb{R}} \frac{|\psi_{t-|j|}(v)|^2}{\pi |v|^2} dv = 2E_Y |Y_t - Y| - E |Y - Y'| \leq 2(|Y_t| + E |Y|) = 2(|X_{t-|j|}| + E |X_1|),$$

where the expectation E_Y is taken with respect to Y and we denote by Y' the random variable which is a copy of Y and independent of Y_t . Similarly

$$\int_{|u|<\gamma} \frac{|\psi_t(u)|^2}{\pi |u|^2} \leq 2E_X |X_t - X| H(|X_t - X| \gamma)$$

where the expectation E_X is taken with respect to X . Therefore, from (10)

$$\int_{|u|<\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} \leq 16 \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n (|X_{t-|j|}| + E |X_1|) \frac{1}{(n - |j|)} \sum_{t=|j|+1}^n E_X [|X_t - X| H(|X_t - X| \gamma)]$$

Because of Assumptions 1 and 2 and the ergodic theorem for α -mixing processes we obtain

$$\begin{aligned} \frac{1}{n - |j|} \sum_{t=|j|+1}^n (|X_{t-|j|}| + E |X_1|) &\rightarrow E |X_1| + E |X_1| = 2E |X_1|, \\ \frac{1}{n - |j|} \sum_{t=|j|+1}^n E_X |X_t - X| H(|X_t - X| \gamma) &\rightarrow E |X_0 - X_1| H(|X_0 - X_1| \gamma), \end{aligned}$$

as $n \rightarrow \infty$, almost surely. Therefore,

$$\limsup_{n \rightarrow \infty} \int_{|u|<\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} \leq 32E |X_1| E |X_0 - X_1| H(|X_0 - X_1| \gamma)$$

and by Lebesgue's dominated convergence theorem,

$$\limsup_{\gamma \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{|u|<\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} = 0.$$

For the second term of (8), (9) implies that $|\psi_t(u)|^2 \leq 4$ and $1/(n - |j|) \sum_{t=|j|+1}^n |\psi_t(u)|^2 \leq 4$.

Therefore

$$\begin{aligned} \int_{|u|>1/\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} &\leq 16 \int_{|u|>1/\gamma} \frac{du}{\pi |u|^2} \int_{\mathbb{R}} \frac{1}{n - |j|} \sum_{t=|j|+1}^n \frac{|\psi_{t-|j|}(v)|^2}{\pi |v|^2} dv \\ &\leq 16\gamma \frac{2}{n - |j|} \sum_{t=|j|+1}^n (|X_{t-|j|}| + E |X_1|). \end{aligned}$$

Then, almost surely

$$\limsup_{\gamma \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{|u|>1/\gamma} \int_{|u|>1/\gamma} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} = 0.$$

The other two summands of (8) can be dealt in a similar way to obtain (7). $\square$

Proof of Theorem 1 Arguing as in Hong (1999), we get the following result:

$$\sum_{j=1}^{n-1} k^2(j/p)(n-j) |\tilde{\sigma}_j(u, v)|^2 = \hat{C}(u, v) + \hat{V}(u, v), \quad (11)$$

where

$$\begin{aligned} \hat{C}(u, v) &= \sum_{j=1}^{n-1} \frac{k^2(j/p)}{n-j} \left[\sum_{t=j+1}^n C_{ttj}(u, v) \right], \\ \hat{V}(u, v) &= \sum_{j=1}^{n-1} \frac{k^2(j/p)}{n-j} \left[\sum_{t=j+2}^n \sum_{s=j+1}^{t-1} V_{tsj}(u, v) \right], \end{aligned}$$

with

$$V_{tsj}(u, v) = C_{tsj}(u, v) + C_{stj}(u, v)^*$$

and

$$C_{tsj}(u, v) = \psi_t(u) \psi_s(u)^* \psi_{t-j}(v) \psi_{s-j}(v)^*.$$

Consider the first term of (11). We observe that $\int_{D(\gamma)} C_{ttj}(u, v)d\mathcal{W}$ and $\int_{D(\gamma)} C_{ssj}(u, v)d\mathcal{W}$ are two independent integrals except from the cases where $t = s$ and $s = \pm j$. In addition

$$E \int_{D(\gamma)} C_{ttj}(u, v)d\mathcal{W} = C_0^\gamma \equiv \int_{D(\gamma)} \sigma_0(u, -u)\sigma_0(v, -v)d\mathcal{W} < \infty.$$

Therefore, $E \left\{ \sum_{t=j+1}^n \left[\int_{D(\gamma)} C_{ttj}(u, v)d\mathcal{W} - C_0^\gamma \right] \right\}^2 \leq C(n-j)$, under independence. Hence, by Markov's inequality, the Cauchy-Schwarz inequality and (4) we obtain that

$$\int_{D(\gamma)} \hat{C}(u, v)d\mathcal{W} - C_0^\gamma \sum_{j=1}^{n-1} k^2(j/p) = O_P(p/\sqrt{n}). \quad (12)$$

Lemma 2 and Equations (11) and (12) yield that

$$\begin{aligned} \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} &= \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\tilde{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} + O_P(p/\sqrt{n}) \\ &= \int_{D(\gamma)} \hat{C}(u, v)d\mathcal{W} + \int_{D(\gamma)} \hat{V}(u, v)d\mathcal{W} + O_P(p/\sqrt{n}) \\ &= C_0^\gamma \sum_{j=1}^{n-1} k^2(j/p) + \hat{V}_n^\gamma + O_P(p/\sqrt{n}), \end{aligned} \quad (13)$$

where $\hat{V}_n^\gamma \equiv \int_{D(\gamma)} \hat{V}(u, v)d\mathcal{W}$.

Because of Assumption 4, we obtain by applying Hong (1999, Thm. A3) on $D(\gamma)$ that

$$\hat{V}_n^\gamma = \hat{V}_{ng}^\gamma + o_P(\sqrt{p}) \quad (14)$$

where

$$\hat{V}_{ng}^\gamma = \sum_{t=g+2}^n \sum_{s=1}^{t-g-1} \sum_{j=1}^g \frac{k^2(j/p)}{n-j} \int_{D(\gamma)} V_{tsj}(u, v)d\mathcal{W}$$

and $g \equiv g(n)$ such that $g/p \rightarrow 0, g/n \rightarrow 0$. Now, by applying Hong (1999, Thm. A4) on $D(\gamma)$

we get the following:

$$\left[p D_0^\gamma \int_0^\infty k^4(z)dz \right]^{-1/2} \hat{V}_{ng}^\gamma \rightarrow N(0, 1) \quad (15)$$

as $n \rightarrow \infty$ in distribution, where

$$D_0^\gamma = 2 \left[\int_{D(\gamma)} |\sigma_0(u, u')|^2 d\mathcal{W} \right]^2.$$

Then, using (13), (14) and (15) we have the following:

$$\frac{\int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} - C_0^\gamma \sum_{j=1}^{n-1} k^2(j/p)}{\left[p D_0^\gamma \int_0^\infty k^4(z) dz \right]^{1/2}} \rightarrow N(0, 1). \quad (16)$$

as $n \rightarrow \infty$ in distribution.

Observe that $\hat{C}_0^\gamma - C_0^\gamma = O_P(1/\sqrt{n})$ and that $\sum_{j=1}^{n-1} k^2(j/p) = O(p)$. Hence C_0^γ can be replaced by $\hat{C}_0^\gamma$ asymptotically given $p/n \rightarrow 0$. Furthermore, $p^{-1} \sum_{j=1}^{n-2} k^4(j/p) \rightarrow \int_0^\infty k^4(z) dz$ and $\hat{D}_0^\gamma \rightarrow D_0^\gamma$, in probability. We conclude that the factor $p D_0^\gamma \int_0^\infty k^4(z) dz$ can be replaced by $\hat{D}_0^\gamma \sum_{j=1}^{n-2} k^4(j/p)$, by Slutsky's theorem. Thus (16) becomes

$$\frac{\int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} - \hat{C}_0^\gamma \sum_{j=1}^{n-1} k^2(j/p)}{\left[\hat{D}_0^\gamma \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2}} \rightarrow N(0, 1). \quad (17)$$

Write $T_{n;\gamma}$ as the test-statistic T_n defined on $D(\gamma)$ rather than on $\mathbb{R}^2$, i.e.

$$T_{n;\gamma} = \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u, v)|^2 \right\} d\mathcal{W}.$$

We note that

$$T_n - T = T_n - T_{n;\gamma} + T_{n;\gamma} - T_{\cdot,\gamma} + T_{\cdot,\gamma} - T$$

where T_{γ} is defined as an asymptotically distributed normal random variable such that

$$\frac{T_{\gamma} - \hat{C}_0^{\gamma} \sum_{j=1}^{n-1} k^2(j/p)}{\left[\hat{D}_0^{\gamma} \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2}} \rightarrow N(0, 1),$$

in distribution, as $n \rightarrow \infty$ such that $p/n \rightarrow 0$. In addition, T is an asymptotically normally distributed random variable with

$$\frac{T - \hat{C}_0 \sum_{j=1}^{n-1} k^2(j/p)}{\left[\hat{D}_0 \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2}} \rightarrow N(0, 1)$$

as $\gamma \rightarrow 0$. Obviously, (17) shows that

$$T_{n;\gamma} - T_{\gamma} = o_P(1),$$

as $n \rightarrow \infty$. But, it is also true that

$$T_{\gamma} - T = o_P(1),$$

as $\gamma \rightarrow 0$, because $\hat{C}_0^{\gamma} \rightarrow \hat{C}_0$ and $\hat{D}_0^{\gamma} \rightarrow \hat{D}_0$. If we show that

$$\limsup_{\gamma \rightarrow 0} \limsup_{n \rightarrow \infty} |T_n - T_{n;\gamma}| = 0, \quad (18)$$

almost surely, then the proof of Theorem 1 will be complete.

For each $\gamma > 0$:

$$\begin{aligned}
|T_n - T_{n;\gamma}| &= \left| \int_{\mathbb{R}^2} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} - \int_{D(\gamma)} \left\{ \sum_{j=1}^{n-1} k^2(j/p)(n-j) |\hat{\sigma}_j(u, v)|^2 \right\} d\mathcal{W} \right| \\
&= \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left| \int_{\mathbb{R}^2} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} - \int_{D(\gamma)} |\hat{\sigma}_j(u, v)|^2 d\mathcal{W} \right| \\
&= \sum_{j=1}^{n-1} k^2(j/p)(n-j) \left| \hat{V}_{X;\gamma}^2(j) - \hat{V}_X^2(j) \right| \tag{19}
\end{aligned}$$

where $\hat{V}_{X;\gamma}^2(j)$ is defined as in (6). Now recall result (7) and combine (19) and (7) to finally get the required relation (18). $\square$

Proof of Theorem 2 We need to show the following: (i) $1/p \sum_{j=1}^n k^4(j/p) \rightarrow \int_0^\infty k^4(j/p)$.

This follows from Assumption 4, $p \rightarrow \infty$ and $p/n \rightarrow 0$. In addition, we need that (ii)

$E \int_{D(\gamma)} \int_{-\pi}^\pi \left| \hat{f}_n(\omega, u, v) - f(\omega, u, v) \right|^2 d\omega d\mathcal{W}(u, v) \rightarrow 0$ where

$$\hat{f}_n(\omega, u, v) = \frac{1}{2\pi} \sum_{j=-(n-1)}^{(n-1)} (1 - |j|/n)^{1/2} k(j/p) \hat{\sigma}_j(u, v) e^{-ij\omega}.$$

This condition is established along the lines of Hong (1999, Thm. 2) on $D(\gamma)$ given Assumptions 1, 3(i) and 4. Furthermore, by applying Markov's inequality we get (iii) $\hat{C}_0^\gamma = O_P(1)$ and (iv) $\hat{D}_0^\gamma \rightarrow D_0^\gamma$ in probability.

Combining (i) and (iv) and by using Slutsky's theorem we get

$$\frac{1}{\sqrt{p}} \left[\hat{D}_0^\gamma \sum_{j=1}^{n-2} k^4(j/p) \right]^{1/2} \rightarrow \left[D_0^\gamma \int_0^\infty k^4(z) dz \right]^{1/2}, \tag{20}$$

in probability. Next (ii) and (iii) and after some calculations we get

$$\frac{1}{n} \left[T_{n;\gamma} - \hat{C}_0^\gamma \sum_{j=1}^{n-1} k^2(j/p) \right] \rightarrow \frac{\pi}{2} \int_{D(\gamma)} \int_{-\pi}^\pi |f(\omega, u, v) - f_0(\omega, u, v)|^2 d\omega d\mathcal{W}(u, v) \tag{21}$$

in probability. Combining (20) and (21) we get the required result on $D(\gamma)$. However, considering $\hat{C}_0^\gamma \rightarrow \hat{C}_0$ and $\hat{D}_0^\gamma \rightarrow \hat{D}_0$ as $\gamma \rightarrow 0$ and (18), the required result is now proved on $\mathbb{R}^2$. $\square$

References

- Doukhan, P. and S. Louhichi (1999). A new weak dependence condition and applications to moment inequalities. *Stochastic Processes and their Applications* 84, 313–342.
- Hong, Y. (1999). Hypothesis testing in time series via the empirical characteristic function: A generalized spectral density approach. *Journal of the American Statistical Association* 94, 1201–1220.
- Székely, G. J., M. L. Rizzo, and N. K. Bakirov (2007). Measuring and testing dependence by correlation of distances. *The Annals of Statistics* 35, 2769–2794.