Comparison of MacLane, Shukla and Hochschild cohomologies

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1. Introduction

Let k be a commutative ring and let A be a k -algebra and M a bimodule over A . Then there are three essential cohomology theories associated to the pair (A, M) due to Hochschild, Shukla and MacLane respectively. These theories are connected by natural maps

$$H^n(A/k, M) \rightarrow SH^n(A/k, M) \rightarrow HML^n(A, M).$$

A ring homomorphism $K \rightarrow k$ also induces the transformation

$$SH^n(A/k, M) \rightarrow SH^n(A/K, M).$$

If A is a k -algebra one also writes

$$H^*(A/k, M) = H^*(A, M) \quad \text{and} \quad SH^n(A/k, M) = SH^n(A, M).$$

It is known that MacLane cohomology coincides with topological Hochschild cohomology ([28]) and coincides also with Baues-Wirsching cohomology of the category of finitely generated free A -modules ([17]). We study the cohomologies mainly in dimensions ≤ 3 since in these dimensions the cohomologies classify various types of extensions. Moreover we are mainly interested in algebras A over a prime field $\mathbb{F}_p$ since we apply our results in particular to the Steenrod algebra A over $\mathbb{F}_p$. As one of our main results we prove (see Theorem 7.4.1):

1.0.1. Theorem. *For any $\mathbb{F}_p$ -algebra A and any bimodule M over A one has an exact sequence*

$$\begin{aligned} 0 \rightarrow H^2(A, M) \xrightarrow{\xi^2} SH^2(A/\mathbb{Z}, M) \xrightarrow{\xi^2} H^0(A, M) \xrightarrow{\gamma} H^3(A, M) \\ \xrightarrow{\xi^3} SH^3(A/\mathbb{Z}, M) \xrightarrow{\xi^3} H^1(A, M). \end{aligned}$$

The second author was supported by the Humboldt Foundation.

Moreover

$$\mathrm{SH}^3(A/\mathbb{Z}, M) \cong \mathrm{HML}^3(A, M)$$

and there is a naturally split short exact sequence

$$0 \rightarrow H^0(A, M) \rightarrow \mathrm{SH}^3(A/k, M) \rightarrow \mathrm{SH}^3(A/\mathbb{Z}, M) \rightarrow 0,$$

where $k = \mathbb{Z}/p^2\mathbb{Z}$.

1.0.2. Corollary. *There are exact sequences ($k = \mathbb{Z}/p^2\mathbb{Z}$)*

$$\begin{aligned} \mathrm{SH}^3(A/k, M) &\xrightarrow{i} \mathrm{HML}^3(A, M) \rightarrow 0, \\ H^3(A/\mathbb{F}_p, M) &\xrightarrow{j} \mathrm{HML}^3(A, M) \xrightarrow{\kappa} H^1(A/\mathbb{F}_p, M). \end{aligned}$$

This corollary corresponds to applications in topology in [3] for the Steenrod algebra A . In fact, let $\mathbb{K}_p$ be the homotopy category of finite products of Eilenberg-MacLane spectra over $\mathbb{F}_p$. Then the characteristic cohomology class $\langle \mathbb{K}_p \rangle$ is defined topologically in Baues-Wirsching cohomology, see [3], and one obtains

$$\langle \mathbb{K}_p \rangle \in \mathrm{HML}^3(A, \Sigma A)$$

since $\mathbb{K}_p$ is isomorphic to the category of finitely generated free A -modules, where Σ is a suspension of graded bimodules, $(\Sigma M)_n = M_{n-1}$. It is shown in [3] that the class $\langle \mathbb{K}_p \rangle$ can be described by the algebra B of secondary cohomology operations which represents an element

$$\langle B \rangle \in \mathrm{SH}^3(A/k, \Sigma A)$$

with $k = \mathbb{Z}/p^2\mathbb{Z}$. This corresponds to the surjectivity of the homomorphism i in the corollary. Also the homomorphism κ in the corollary was defined in [3] which carries the class $\langle \mathbb{K}_p \rangle$ to the element

$$\kappa(\langle \mathbb{K}_p \rangle) = \{\kappa_p\} \in H^1(A/\mathbb{F}_p, M)$$

where κ_p is the Kristensen derivation on the Steenrod algebra A [3]. It was observed in [3] that $\kappa \circ j = 0$ but exactness was not obtained in [3]. Since $\kappa(\langle \mathbb{K}_p \rangle)$ is nontrivial we see that $\langle \mathbb{K}_p \rangle$ cannot be described by an element in Hochschild cohomology $H^3(A/\mathbb{F}_p, M)$. These topological examples concerning the Steenrod algebra A motivated the study of cohomologies of $\mathbb{F}_p$ -algebras in this paper.

It is known that the Hochschild cohomology has good properties only for algebras which are projective modules over the ground ring. In the early 60-s Shukla [32] developed a cohomology theory for associative algebras with nicer properties than Hochschild theory. Quillen in [31] indicated that the Shukla cohomology fits in his general framework of homotopical algebra. The approach of Quillen is based on simplicial methods. One of the aims of this paper is to give the foundation of Shukla cohomology based on chain algebras. Another aim is to clarify the relationship between Hochschild and Shukla cohomology as

in the theorem above. Our main technical tool for doing this is the fact that elements of the Shukla cohomology can be described via extensions, in particular in the dimension 3 the Shukla cohomology classifies crossed bimodules (see Theorem 4.4.1). Shukla cohomology is related to MacLane cohomology via a spectral sequence [28], [26]. Our results on Shukla cohomology allow us to analyze some differentials in the spectral sequence.

The contents of the paper is as follows. In Section 2 we recall basics on Hochschild cohomology theory and especially relationship between abelian extensions which are split over the ground ring, and elements of the second Hochschild cohomology. In Section 3 we introduce crossed bimodules and crossed extensions. We recall the relationship between crossed extensions which are split over the ground ring, and elements of the third Hochschild cohomology. This section also contains a new interpretation of the classical obstruction theory in terms of crossed extensions (see Theorem 3.3.2). We also discuss a different generalization of the relationship between different sorts of extensions and higher cohomology. In Section 4 we define Shukla cohomology as a kind of derived Hochschild cohomology on the category of chain algebras and we prove basic properties of the Shukla cohomology including relationship with crossed bimodules. In the original paper Shukla used an explicit cochain complex for the definition of Shukla cohomology. Unfortunately this complex is very complicated to work with. Quillen instead used closed model category structure on the category of simplicial algebras. We use the closed model category structure on the category of chain algebras. The Section 5 is devoted to some computations of Shukla cohomology when the ground ring is the ring of integers or $\mathbb{Z}/p^2\mathbb{Z}$. In the next two sections we study the change of base rings and in particular we obtain Theorem 1.0.1. The Section 8 solves the problem of constructing a canonical cochain complex for computing the Shukla cohomology in the important case when the ground ring is an algebra over a field. Our cochain complex is built from tensor powers of the algebra under consideration, unlike the one proposed by Shukla. Section 9 recalls basics of MacLane cohomology [21] and relationship with Shukla cohomology. It is well known that these two theories are isomorphic up to dimension two. It turns out that for algebras over fields they are also isomorphic in dimension three. In higher dimensions there is a spectral sequence relating these theories [28], [26] and we prove that for commutative algebras over a prime field the first differential vanishes.

2. Hochschild cohomology

2.1. Definitions. Here we recall the basic notions around Hochschild cohomology theory, referring to [18] and [23] for more details. In this section k denotes a commutative ring with unit, which is considered as a ground ring. Thus all modules are defined over k and we assume that they are unitary, meaning that the unit 1 of k acts as the identity operator and all algebras are associative k -algebras with units. A linear map means a k -module homomorphism.

Let R be a k -algebra with unit and let M be a bimodule over R . Consider the module

$$C^n(R, M) := \text{Hom}(R^{\otimes n}, M)$$

(where $\otimes = \otimes_k$ and $\text{Hom} = \text{Hom}_k$). The *Hochschild coboundary* is the linear map $d : C^n(R; M) \rightarrow C^{n+1}(R; M)$ given by the formula

$$d(f)(r_1, \dots, r_{n+1}) = r_1 f(r_2, \dots, r_{n+1}) + \sum_{i=1}^n (-1)^i f(r_1, \dots, r_i r_{i+1}, \dots, r_{n+1}) \\ + (-1)^{n+1} f(r_1, \dots, r_n) r_{n+1}.$$

Here $f \in C^n(R; M)$ and $r_1, \dots, r_{n+1} \in R$. By definition the n -th Hochschild cohomology group of the algebra R with coefficients in the R -bimodule M is the n -th homology group of the Hochschild cochain complex $C^*(R; M)$ and it is denoted by $H^*(R; M)$. Sometimes these groups are denoted by $H^*(R/k, M)$ in order to make clear that the ground ring is k .

2.2. Basic properties. Let

$$(1) \quad 0 \rightarrow M_1 \xrightarrow{\mu} M \xrightarrow{\sigma} M_2 \rightarrow 0$$

be an exact sequence of bimodules over R . In general it does not yield a long cohomological exact sequence for Hochschild cohomology. However, it does so if the exact sequence (1) is k -split. In fact, in this case the sequence

$$0 \rightarrow C^*(R; M_1) \rightarrow C^*(R; M) \rightarrow C^*(R; M_2) \rightarrow 0$$

is exact in the category of cochain complexes and therefore yields the long cohomological exact sequence:

$$\dots \rightarrow H^n(R; M_1) \rightarrow H^n(R; M) \rightarrow H^n(R; M_2) \rightarrow H^{n+1}(R; M_1) \rightarrow \dots$$

The category of bimodules over R is equivalent to the category of left modules over the ring $R^e := R \otimes R^{\text{op}}$, where R^{op} is the opposite ring of R . The multiplication map $R \otimes R^{\text{op}} \rightarrow R$ is an algebra homomorphism. We always consider R as a bimodule over R via this homomorphism. For any associative algebra R the cohomology $H^*(R; R)$ is a graded commutative algebra under the cup-product, which is defined by

$$(f \cup g)(r_1, \dots, r_{n+m}) := f(r_1, \dots, r_n) g(r_{n+1}, \dots, r_{n+m}),$$

for $f \in C^n(R; R)$ and $g \in C^m(R; R)$ (see [15]). Actually for any bimodule M , the cup-product yields a well-defined pairing

$$\cup : H^n(R, R) \otimes H^m(R, M) \rightarrow H^{n+m}(R, M),$$

which makes $H^*(R, M)$ into a graded module over $H^*(R, R)$.

If A and B are left R -modules, then $\text{Hom}(A, B)$ is a bimodule over R by the following action

$$(rfs)(a) = rf(sa), \quad r, s \in R, a \in M, f \in \text{Hom}(A, B).$$

A bimodule is called *induced* if it is isomorphic to $\text{Hom}(R, A)$ for an R -module A . It is well known [23] that the Hochschild cohomology vanishes in positive dimensions on induced bimodules. In fact there is a natural isomorphism [23]

$$H^*(R; M) \cong \text{Ext}_{R^e, k}^*(R, M)$$

where subscript k indicates that Ext-groups in question are defined in the framework of relative homological algebra, with the proper class consisting of k -split exact sequences. If R is projective as a k -module, then one can take the usual Ext-groups $\text{Ext}_R^*(R, M)$ instead of the relative Ext-groups. In particular, Hochschild cohomology vanishes in positive dimensions on injective bimodules, provided R is projective as a k -module.

For the Hochschild cohomology in dimensions 0 and 1 one has the following exact sequence

$$0 \rightarrow H^0(R, M) \rightarrow M \xrightarrow{\text{ad}} \text{Der}(R, M) \rightarrow H^1(R, M) \rightarrow 0$$

where $\text{Der}(R, M)$ denotes the collection of all derivations from R to M and $\text{ad}(m)(r) = rm - mr$.

We also recall the well-known relationship between the second Hochschild cohomology and extensions of algebras [23].

An *abelian extension* (sometimes also called *singular extension*) of associative algebras is a short exact sequence

$$(E) \quad 0 \rightarrow M \rightarrow E \xrightarrow{\sigma} R \rightarrow 0$$

where R and E are associative algebras with unit, $\sigma : E \rightarrow R$ is a homomorphism of algebras with unit and $M^2 = 0$, in other words the product in E of any two elements from M is zero. For elements $m \in M$ and $r \in R$ we put $mr := me$ and $rm := em$. Here $e \in E$ is any element such that $\sigma(e) = r$. This definition does not depend on the choice of e . Therefore M has a bimodule structure over R .

An abelian extension (E) is called *k-split* if there exists a linear map $s : R \rightarrow E$ such that $\sigma s = \text{id}_R$.

Assume we have a bimodule M over an associative algebra R , then we let $\mathcal{E}(R; M)$ be the category, whose objects are the abelian extensions (E) such that the induced R -bimodule structure on M coincides with the given one. Morphisms $(E) \rightarrow (E')$ are commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & R \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \varphi & & \downarrow \text{id} \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & R \longrightarrow 0 \end{array}$$

where φ is a homomorphism of algebras with unit. Moreover, we let $\mathcal{E}_k(R; M)$ be the category, whose objects are k -split abelian extensions. It is clear that both $\mathcal{E}(R; M)$ and $\mathcal{E}_k(R; M)$ are groupoids, in other words all morphisms in $\mathcal{E}_k(R; M)$ and $\mathcal{E}(R; M)$ are isomorphisms. We let $\text{Extalg}(R; M)$ and $\text{Extalg}_k(R; M)$ be the sets of connected components of these categories. Clearly $\text{Extalg}_k(R; M) \subset \text{Extalg}(R; M)$. According to [23] there is a natural bijection

$$(2) \quad H^2(R; M) \cong \text{Extalg}_k(R; M).$$

We also recall that the map $\text{Extalg}_k(R; M) \rightarrow H^2(R; M)$ is defined as follows. Let (E) be an abelian extension. We choose a linear map $u: R \rightarrow E$ with the property $\sigma u = \text{id}_R$. Then we define the map $f_u: R \otimes R \rightarrow M$ by

$$f_u(r, s) := u(rs) - u(r)u(s).$$

One checks that f_u is a 2-cocycle, whose class in $H^2(R, M)$ is independent of the choice of u and yields the well-defined map $\text{Extalg}_k(R, M) \rightarrow H^2(R, M)$.

2.3. Cohomological obstruction for lifting of algebras. Assume we have a surjective morphism of commutative algebras $K \rightarrow k$. Let R be a k -algebra, which is free as a k -module. One says that R has a K -lifting if there exists a K -algebra S , which is free as a K -module and an isomorphism of k -algebras

$$S \otimes_K k \cong R.$$

In this case one says that S is a K -lifting of R .

In the rest of this section we will assume that there is given an abelian extension of commutative rings

$$(3) \quad 0 \rightarrow M \rightarrow K \rightarrow k \rightarrow 0.$$

The most important case for us will be the extension

$$(4) \quad 0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{F}_p \rightarrow 0.$$

Since K is commutative, the right and left action of k on M coincide and therefore M can be considered simply as a k -module.

One can ask for a given k -algebra R , which is free as a k -module, under what condition the k -algebra R has a K -lifting. It turns out that for any such R there is a class

$$o_K(R) \in H^3(R/k, M \otimes_k R),$$

which is also denoted by $o(R)$, such that $o(R) = 0$ if and only if R has a K -lifting (compare with [6], where the similar problem for Lie algebras is considered).

Now we give the construction of the class $o(R)$. Since R is free as a k -module, there is a free K -module V and an isomorphism of k -modules $V \otimes_K k \cong R$. The ring homomorphism $K \rightarrow k$ yields the surjective map

$$\text{Hom}_K(V \otimes_K V, V) \rightarrow \text{Hom}_k(R \otimes_k R, R)$$

(to see that this map is really surjective, one observes that both groups are non-canonically isomorphic to the group of $(n^2 \times n)$ -matrices over K and k respectively). In particular one can lift the product of R to a linear map $m: V \otimes_K V \rightarrow V$. Let us consider now the map $f: V \otimes_K V \otimes_K V \rightarrow V$ defined by $f(x, y, z) = m(x, m(y, z)) - m(m(x, y), z)$. Since we have an exact sequence

$$0 \rightarrow M \otimes_k R \rightarrow V \rightarrow R \rightarrow 0$$

it follows that the values of f lies in $M \otimes_k R$. The function f factors through $R \otimes_k R \otimes_k R$ and therefore defines a class $o(R) \in H^3(R, M \otimes_k R)$. If R has a lift S , then one can take $V = S$ and m to be the multiplication in S . In this case $f = 0$ and therefore $o(R) = 0$. Conversely, assume $o(R) = 0$. Then there is a map $g : R \otimes_k R \rightarrow M \otimes_k R$ such that $f(x, y, z) = \delta g(\bar{x}, \bar{y}, \bar{z})$. Here $\bar{x}$ denotes the image of $x \in V$ in R . One defines $n : V \otimes_k V \rightarrow V$ by $n(x, y) := m(x, y) - g(\bar{x}, \bar{y})$. It is clear that n yields an associative algebra structure on V and therefore the lifting exists.

The obstruction operator o_K has the following property: For any algebra homomorphism $f : R \rightarrow S$ one has the equality in the group $H^3(R, M \otimes_k S)$ (here S is considered as a bimodule over R via f)

$$f_*(o(R)) = f^*(o(S)).$$

Here $f_* : H^3(R, M \otimes_k R) \rightarrow H^3(R, M \otimes_k S)$ and $f^* : H^3(S, M \otimes_k S) \rightarrow H^3(R, M \otimes_k S)$ are induced homomorphisms in cohomology.

In particular, if we take $M = k$, we obtain the obstruction $o_K(R) \in H^3(R/k, R)$ defined for any abelian extension of commutative algebras

$$0 \rightarrow k \rightarrow K \rightarrow k \rightarrow 0.$$

Of special interest here is the extension in (4) above. Then the corresponding class is denoted by

$$o_{\mathbb{Z}/p^2\mathbb{Z}}(A) \in H^3(A/\mathbb{F}_p, A),$$

where A is any $\mathbb{F}_p$ -algebra and it is called the *lifting obstruction class* of A .

3. Crossed bimodules and Hochschild cohomology

3.1. Chain algebras and crossed bimodules. Let us recall that a *chain algebra* over k is a graded algebra $C_* = \bigoplus_{n \geq 0} C_n$ equipped with a boundary map $\partial : C_* \rightarrow C_*$ of degree -1 satisfying the Leibniz identity

$$\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial(y).$$

If C_* is a chain algebra and $c \in C_n$, then we write $|c| = n$. If C_* and C'_* are chain algebras, then a morphism $f : C_* \rightarrow C'_*$ of chain algebras is a chain map, which is compatible with the multiplicative structure. Let **DGA** be the category of chain algebras. A morphism of chain algebras is called *weak equivalence* if it induces an isomorphism in homology.

3.1.1. Definition. A *crossed bimodule* is a chain algebra C_* which is trivial in dimensions ≥ 2 , that is $C_n = 0$ for $n \geq 2$.

Thus a crossed bimodule consists of an algebra C_0 and a bimodule C_1 over C_0 together with a homomorphism of bimodules

$$C_1 \xrightarrow{\partial} C_0$$

such that

$$\partial(c)c' = c\partial(c'), \quad c, c' \in C_1.$$

Indeed, since $C_2 = 0$ the last condition is equivalent to the Leibniz identity $0 = \partial(cc') = \partial(c)c' - c\partial(c')$.

It follows that the product defined by

$$c * c' := \partial(c)c'$$

for $c, c' \in C_1$ gives an associative non-unital k -algebra structure on C_1 and $\partial : C_1 \rightarrow C_0$ is a homomorphism of non-unital k -algebras. This equivalent definition goes back at least to Dedecker and Lue [9]. The notion of crossed bimodule is an analogue of the notion of crossed module introduced by Whitehead [35] in the group theory framework, playing a rôle in homotopy theory of spaces with nontrivial fundamental groups [2], [20].

We denote by **Xmod** the category of crossed bimodules. We have also a category **Bim/Alg**, whose objects are triples (C_0, C_1, ∂) , where C_0 is an associative algebra, C_1 is a bimodule over C_0 and $\partial : C_1 \rightarrow C_0$ is a homomorphism of bimodules over C_0 . It is clear that **Xmod** is a full subcategory of **Bim/Alg** and the inclusion **Xmod** $\subset$ **Bim/Alg** has a left adjoint functor, which assigns $\partial : \tilde{C}_1 \rightarrow C_0$ to $C_1 \rightarrow C_0$. Here $\tilde{C}_1$ is the quotient of C_1 under the equivalence relation $x\partial(y) - \partial(x)y \sim 0$, $x, y \in C_1$.

We let **Mod/Alg** be the category whose objects are triples (V, C, ∂) , where C is an associative algebra, V is a k -module and $\partial : V \rightarrow C$ is a linear map. One has the forgetful functor **Bim/Alg** $\rightarrow$ **Mod/Alg**, which has a left adjoint functor sending (V, C, ∂) to the triple (M, d, C) , where $M = C \otimes V \otimes C$ and d is the unique homomorphism of bimodules which extends ∂ . As a consequence we see that the forgetful functor **Xmod** $\rightarrow$ **Mod/Alg** also has a left adjoint. Of special interest is the case when C is a free associative algebra and V is a free k -module on $X \subset V$. In this case the corresponding crossed bimodule is called a *totally free crossed bimodule*.

3.2. Hochschild cohomology in the dimension 3 and crossed extensions. Here we recall the relation between Hochschild cohomology and crossed bimodules (see [18], Exercise E.1.5.1 or [4]).

3.2.1. Lemma. *Let $\partial : C_1 \rightarrow C_0$ be a crossed bimodule. We put $M = \text{Ker}(\partial)$ and $R = \text{Coker}(\partial)$.*

(i) *The image of ∂ is an ideal of C_0 and R carries a natural associative algebra structure.*

(ii) *One has $M * C_1 = 0 = C_1 * M$ and M has a well-defined bimodule structure over R .*

Let R be an associative algebra with unit and let M be a bimodule over R . A *crossed extension* of R by M is an exact sequence

$$(5) \quad 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow R \rightarrow 0$$

where $\partial : C_1 \rightarrow C_0$ is a crossed bimodule, such that $C_0 \rightarrow R$ is a homomorphism of algebras with unit and $M \rightarrow C_1$ is a homomorphism of C_0 -bimodules. A crossed extension of R by M is *k-split*, if all arrows in the exact sequence (5) are *k-split*.

For fixed R and M one can consider the category $\mathcal{X}ext(R, M)$ whose objects are crossed extensions of R by M . Morphisms are maps between crossed modules which induce the identity on M and R .

3.2.2. Lemma. Assume (∂) is a crossed extension of R by M and a homomorphism $f : P_0 \rightarrow C_0$ of unital k -algebras is given. Let P_1 be the pull-back of the diagram

$$\begin{array}{ccc} & P_0 & \\ & \downarrow & \\ C_1 & \longrightarrow & C_0 \end{array}$$

Then there exists a unique crossed module structure on $P_1 \rightarrow P_0$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow f & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

defines a morphism of crossed extensions.

3.2.3. Corollary. In each connected component of $\mathcal{X}ext(R, M)$ there is a crossed extension

$$(P) \quad 0 \rightarrow M \rightarrow P_1 \rightarrow P_0 \rightarrow R \rightarrow 0$$

with free algebra P_0 and for any other object (∂) in this connected component there is a morphism $(P) \rightarrow (\partial)$. Thus (∂) and (∂') are in the same component of $\mathcal{X}ext(R, M)$ if and only if there exists a diagram of the form $(\partial) \leftarrow (P) \rightarrow (\partial')$.

We let $\mathcal{X}ext_k(R, M)$ be the subcategory of *k-split* crossed extensions. Morphisms are such morphisms from $\mathcal{X}ext(R, M)$ that all maps involved are *k-split*. Let $\text{Xext}(R, M)$ and $\text{Xext}_k(R, M)$ be the set of connected components of the category of crossed extensions and the category of *k-split* crossed extensions respectively. Then there is a canonical bijection:

$$(6) \quad H^3(R, M) \cong \text{Xext}_k(R, M)$$

(see for example [18], Exercise E.1.5.1, or [4]). A similar isomorphism for group cohomology goes back to [24], see also [19], [2]. For the proof (6) we associate a 3-cocycle to a *k-split* crossed extension

$$0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} R \rightarrow 0$$

of R by M . We put $V := \text{Im}(\partial)$ and consider k -linear sections $p : R \rightarrow C_0$ and $q : V \rightarrow C_1$ of $\pi : C_0 \rightarrow R$ and $\partial : C_1 \rightarrow V$ respectively. Now we define $m : R \otimes R \rightarrow V$ by $m(r, s) := q(p(r)p(s) - p(rs))$. Finally we define $f : R \otimes R \otimes R \rightarrow M$ by

$$f(r, s, t) := p(r)m(s, t) - m(rs, t) + m(r, st) - m(r, s)p(t).$$

Then f is a 3-cocycle of $C^*(R, M)$. The corresponding class in $H^3(R, M)$ depends only on the connected component of a given crossed extension and in this way one gets the expected isomorphism (see [4]).

3.3. Obstruction theory. Now we explain a variant of the classical obstruction theory in terms of crossed extensions (compare with [23], Sections IV.8 and IV.9). Let

$$0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow R \rightarrow 0$$

be a crossed extension of R by M . A ∂ -extension of R by C_1 is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\mu} & S & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \xi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R \longrightarrow 0 \end{array}$$

where S is a unital k -algebra and ξ is a homomorphism of unital k -algebras. Furthermore the equalities $\mu(x)s = \mu(x\xi(s))$ and $s\mu(x) = \mu(\xi(s)x)$ hold, where $x \in C_1$, $s \in S$. It follows then that the product in C_1 induced from S coincides with the $*$ -product: $x * y = \partial(x)y = x\partial(y)$.

The concept of ∂ -extension of R by C_1 is related with the following concept of ∂ -coextension of C_0 by M , which is by definition a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \mu & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & S & \xrightarrow{\xi} & C_0 & \longrightarrow & 0 \end{array}$$

where S is a unital k -algebra and ξ is a homomorphism of unital k -algebras. Furthermore the equalities $\mu(x)s = \mu(x\xi(s))$ and $s\mu(x) = \mu(\xi(s)x)$ hold, where $x \in C_1$, $s \in S$. It follows then that M is a square zero ideal in S and therefore $(0 \rightarrow M \rightarrow S \rightarrow C_0 \rightarrow 0) \in \text{Extalg}(C_0; M)$.

3.3.1. Lemma. *If*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\mu} & S & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \xi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \longrightarrow & R \longrightarrow 0 \end{array}$$

is a ∂ -extension of R by C_1 . Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \mu & & \downarrow \text{id} & & & & \\ 0 & \longrightarrow & M & \longrightarrow & S & \xrightarrow{\xi} & C_0 & \xrightarrow{\pi} & 0 \end{array}$$

is a ∂ -coextension of C_0 by M and vice versa.

It is clear that ∂ -extensions of R by C_1 form a groupoid, whose set of components will be denoted by ${}_{\partial}\text{Ext}(R, C_1)$.

Now we assume that ∂ is a k -split crossed extension. A ∂ -extension of C_1 by R is called k -split if ξ is a k -split epimorphism. Of course in this case ς is k -split as well. We let ${}_{\partial}\text{Ext}_k(R, C_1)$ be the subset of ${}_{\partial}\text{Ext}(R, C_1)$ consisting of k -split ∂ -extensions. A similar meaning has the notion of k -split ∂ -coextension.

3.3.2. Theorem. *The class of a k -split crossed extension*

$$(\partial) \quad 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow R \rightarrow 0$$

is zero in $H^3(R, M)$ if and only if ${}_{\partial}\text{Ext}_k(R, C_1)$ is nonempty. If this is the case then the group $H^2(R, M)$ acts transitively and effectively on ${}_{\partial}\text{Ext}_k(R, C_1)$.

Proof. For a crossed extension ∂ one considers k -linear sections $p : R \rightarrow C_0$ and $q : V \rightarrow C_1$, $V = \text{Im}(\partial)$ as above. We may and we will assume that $p(1) = 1$. Then the class of (∂) in H^3 is given by the cocycle

$$f(r, s, t) := p(r)m(s, t) - m(rs, t) + m(r, st) - m(r, s)p(t)$$

where $m(r, s) = q(p(r)p(s) - p(rs))$. Given a ∂ -extension of C_1 by R :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\mu} & S & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \xi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

choose a k -linear section $v : C_0 \rightarrow S$ such that $v(1) = 1$. One puts $u = vp : R \rightarrow S$. Then $\varsigma u = \text{id}_R$. One defines $n : R \otimes R \rightarrow C_1$ by $\mu(n(r, s)) = u(r)u(s) - u(rs)$. We claim that

$$(7) \quad p(r)n(s, t) - n(rs, t) + n(r, st) - n(r, s)p(t) = 0.$$

Indeed,

$$\mu(p(r)n(s, t)) = u(r)\mu(n(s, t)) = u(r)u(s)u(t) - u(r)u(st).$$

Similarly $\mu(n(r, s)p(t)) = u(r)u(s)u(t) - u(rs)u(t)$. Thus

$$\begin{aligned}
\mu(p(r)n(s,t) - n(rs,t) + n(r,st) - n(r,s)p(t)) &= u(r)u(s)u(t) - u(r)u(st) \\
&\quad - u(rs)u(t) + u(rst) + u(r)u(st) - u(rst) \\
&\quad - u(r)u(s)u(t) + u(rs)u(t) \\
&= 0.
\end{aligned}$$

Since $m(r,s) = q\partial n(r,s)$, it follows that $g(r,s) = m(r,s) - n(r,s)$ lies in M . Thus we obtain a well-defined linear map $g : R \otimes R \rightarrow M$. Then it follows from the equation (7) that

$$f(r,s,t) = rg(s,t) - g(rs,t) + g(r,st) - g(r,s)t,$$

which shows that the class of ∂ in H^3 is zero. Given any normalized 2-cocycle $h : R \otimes R \rightarrow M$, one can define a new ∂ -extension S_h of R by C_1 by putting $S_h = C_1 \oplus R$ with the following multiplication:

$$(x,r)(y,s) = (x * y + p(r)y + xp(s) + n(r,s) + h(r,s), rs).$$

This construction yields a transitive and effective action of $H^2(R, M)$ on ${}_{\partial}\text{Ext}_k(R, C_1)$.

Conversely, assume that the class of $0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow R \rightarrow 0$ is zero in $H^3(R, M)$. Thus there is a linear map $g : R \otimes R \rightarrow M$ such that

$$f(r,s,t) = rg(s,t) - g(rs,t) + g(r,st) - g(r,s)t.$$

One can define $n : R \otimes R \rightarrow C_1$ by $n(r,s) = m(r,s) - g(r,s)$. Then

$$p(r)n(s,t) - n(rs,t) + n(r,st) - n(r,s)p(t) = 0$$

and therefore $S = C_1 \oplus R$ with the product

$$(x,r)(y,s) = (x * y + p(r)y + xp(s) + n(r,s), rs)$$

defines a ∂ -extension. $\square$

3.3.3. Corollary. *The class of a k -split crossed extension*

$$(\partial) \quad 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow R \rightarrow 0$$

is zero in $H^3(R, M)$ if and only if there exists a k -split ∂ -coextension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow \sigma & & \downarrow \text{id} & & & & \\
0 & \longrightarrow & M & \longrightarrow & S & \xrightarrow{v} & C_0 & \longrightarrow & 0.
\end{array}$$

Proof. This is an immediate consequence of the previous result and Lemma 3.3.1. $\square$

3.3.4. Remark. We give now a brief summary how to describe classes in Hochschild cohomology via extensions in higher dimensions. For simplicity we assume that k is a field. We start with the following general construction. Let T be an additive functor from the category of bimodules over R to the category of k -modules. Objects of the category $\mathcal{E}^n(T)$ are pairs (E, x) , where

$$0 \rightarrow M \rightarrow E_0 \rightarrow \cdots \rightarrow E_n \rightarrow 0$$

is an n -fold extension of E_n by M in the category of R - R -bimodules and $x \in T(E_n)$. Morphisms in $\mathcal{E}^n(T)$ are defined in an obvious way. Let $\mathbf{E}^n(T)$ be the set of components of the category $\mathcal{E}^n(T)$. A result of Yoneda [34] asserts that one has a natural isomorphism:

$$\mathbf{E}^n(T) \cong S^n T(M)$$

where $S^n T$ is the n -th satellite of T [8]. Now we take $T = \text{Xext}(R, -)$. Then objects of the category $\mathcal{E}^{n-2}(T)$, $n \geq 4$ are nothing but crossed n -fold extensions of R by M in the sense of Baues and Minian [4]. They proved that there is a natural isomorphism $\mathbf{E}^{n-2}(T) \cong H^{n+1}(R, M)$, which can also be obtained using the Yoneda isomorphism:

$$\begin{aligned} \mathbf{E}^{n-2}(T) &= S^{n-2} T = S^{n-2} H^3(R, -)(M) \\ &= S^{n-2} S^3 H^0(R, -)(M) \cong S^{n+1} H^0(R, -)(M) = H^{n+1}(R, M). \end{aligned}$$

Here we used the isomorphism $T \cong H^3(R, -)$ and the classical fact that

$$H^n(R, M) = \text{Ext}_{R \otimes R^{\text{op}}}^n(R, M) = S^n H^0(R, -)(M)$$

see [8]. There is another variant of this isomorphism. We can take $T = \text{Extalg}(R, -)$. The objects of the category $\mathcal{E}^{n-1}(T)$ are known as abelian n -fold extensions. The same argument shows that $\mathbf{E}^{n-1}(T) \cong H^{n+1}(R, M)$. For example, an *abelian twofold extension* of an algebra R by an R - R -bimodule M is an exact sequence

$$0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\mu} S \xrightarrow{\pi} R \rightarrow 0$$

where N is a bimodule over R and α is a bimodule homomorphism. Moreover, S is an associative algebra with unit and π is a homomorphism of algebras with unit, such that $\text{Ker}(\pi)$ is a square zero ideal of S . Furthermore, for any $s \in S$ and $n \in N$ one has

$$\mu(n\pi(s)) = \mu(n)s, \quad \mu(\pi(s)n) = s\mu(n).$$

Let $\text{Extalg}^2(R, M)$ be the connected components of the category of abelian twofold extensions of R by M . Let us note that for any abelian twofold extension

$$0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\mu} S \xrightarrow{\pi} R \rightarrow 0$$

the morphism $\mu : N \rightarrow S$ is a crossed bimodule, where the action of S on N is given via π . Thus one obtains the natural map $\text{Extalg}^2(R, M) \rightarrow \text{Xext}(R, M)$. From the above discussion it is clear that it is in fact a bijection. As we can see these results strongly depend on the vanishing of Hochschild cohomology on injective bimodules.

4. Shukla cohomology

As we already saw the Hochschild cohomology in dimensions two and three classifies k -split abelian and crossed extensions respectively. There is a variant of Hochschild cohomology due to Shukla in [32], which classifies all abelian and crossed extensions. We will present these results. Our approach to Shukla cohomology is based on chain algebras and especially on the possibility of extension of Hochschild cohomology to chain algebras.

4.1. Hochschild cohomology for chain algebras. In this section we first give a naive extension of the Hochschild cohomology for chain algebras. Let $R_* = \bigoplus_{n \geq 0} R_n$ be a chain algebra. An R_* -bimodule is a chain complex M_* of k -modules, equipped with actions from both sides: $R_* \otimes M_* \rightarrow M_*$ and $M_* \otimes R_* \rightarrow M_*$, satisfying usual axioms. However, for our purposes we restrict ourselves to the case when M is concentrated in degree zero. In this case R_* -bimodule means simply a bimodule over $H_0(R_*)$. In particular $xm = 0 = mx$ as soon as $m \in M$ and $|x| \geq 1$. For a chain algebra R_* and a $H_0(R_*)$ -bimodule M we denote by $C^*(R_*, M)$ the total complex of the following cosimplicial cochain complex. The n -th component of this cosimplicial object is the cochain complex

$$C^n(R_*, M) := \text{Hom}(R_*^{\otimes n}, M).$$

Here $\otimes$ denotes the tensor product of chain complexes. The coface operations are given via Hochschild coboundary formula:

$$d^0(f)(r_1, \dots, r_{n+1}) = (-1)^{nk} r_1 f(r_2, \dots, r_{n+1}), \quad f: R_*^{\otimes n} \rightarrow M, \quad r_1 \in R_k$$

(actually this expression is zero provided $k > 0$)

$$d^i(f)(r_1, \dots, r_{n+1}) = f(r_1, \dots, r_i r_{i+1}, \dots, r_{n+1}), \quad 0 < i < n+1,$$

$$d^{n+1}(f)(r_1, \dots, r_{n+1}) = f(r_1, \dots, r_n) r_{n+1}.$$

The homology of $C^*(R_*, M)$ is denoted by $H^*(R_*, M)$.

The spectral sequences of a bicomplex in our situation have the following form:

$$E_{pq}^1 = H^q(\text{Hom}(R_*^{\otimes p}, M)) \Rightarrow H^{p+q}(R_*, M),$$

$$F_{pq}^1 = H^q(|R_*|, M) \Rightarrow H^{p+q}(R_*, M).$$

Here $|R_*|$ denotes the underlying graded algebra of the chain algebra (R_*, ∂) . A chain algebra R_* is called *quasi-free* if its underlying algebra $|R_*|$ is free.

4.1.1. Lemma. *Let $f: R_* \rightarrow S_*$ be a weak equivalence of chain algebras and let M be a bimodule over $H_0(S)$. Then the induced homomorphism*

$$H^*(S_*, M) \rightarrow H^*(R_*, M)$$

is an isomorphism provided R_ and S_* are projective k -modules.*

Proof. It is well known that any weak equivalence between degreewise projective bounded below chain complexes is a homotopy equivalence. Thus f is a homotopy equivalence in the category of chain complexes of k -modules. Therefore the induced map $R_*^{\otimes} \rightarrow S_*^{\otimes}$ is also a homotopy equivalence. It follows that the induced map of cosimplicial cochain complexes is a homotopy equivalence in each degree and therefore it induces a weak equivalence on the total complex level thanks to the spectral sequence associated to the bicomplex. $\square$

4.2. The complex $\text{Der}(|R_*|, M)$. Let R_* be a chain algebra and M be an $H_0(R_*)$ -bimodule. We can take the derivations $|R_*| \rightarrow M$ from the underlying graded algebra to M . Since $|R_*|$ is graded, the space of derivations $\text{Der}(|R_*|, M)$ is also graded. Since M is concentrated only in dimension zero, we see that the 0-th component is the space of all derivations $R_0 \rightarrow M$, while in dimensions $n > 0$ we get the space of linear maps $f : R_n \rightarrow M$ satisfying the conditions

$$\begin{aligned} f(xy) &= xf(y), \quad f(yx) = f(x)y, \quad x \in R_0, y \in R_n, n > 0, \\ f(uv) &= 0, \quad u \in R_i, v \in R_j, i + j = n, i > 0, j > 0. \end{aligned}$$

The boundary map $\partial : R_n \rightarrow R_{n-1}$ in R_* yields a cochain complex structure on $\text{Der}(|R_*|, M)$. In what follows $\text{Der}(|R_*|, M)$ is always considered with this cochain complex structure.

4.2.1. Lemma. *Let R_* be a chain algebra, which is a quasi-free algebra, meaning that the underlying algebra structure is free, and let M be an $H_0(A_*)$ -bimodule. Then the Hochschild cohomology $H^n(A_*, M)$ is isomorphic to the $(n - 1)$ -st homology of the cochain complex $\text{Der}(|R_*|, M)$ provided $n > 0$.*

Proof. This is a direct consequence of the spectral sequence related to the bicomplex $C^*(R_*, M)$ together with the fact that the Hochschild cohomology of a free algebra is zero in dimensions > 1 [18]. $\square$

4.3. Derived Hochschild cohomology and Shukla cohomology. Let $\mathbf{DGA}$ be the category of chain algebras over a commutative ring k .

4.3.1. Theorem. *Define a morphism of chain algebras to be*

- (i) *a weak equivalence if it induces an isomorphism in homology,*
- (ii) *a fibration if it is a surjection in positive dimensions,*
- (iii) *a cofibration if it has the left lifting property with respect to all maps which are fibrations and weak equivalences.*

Then with these choices $\mathbf{DGA}$ is a closed model category [30]. Moreover any cofibrant object is a retract of a quasi-free algebra.

This result is well-known and is a very particular case of the general results of Stanly [33], the case when k is a field was considered already by Munkholm in [25].

We use this model category structure to construct the derived Hochschild cohomology of chain algebras. We also need the following observation: If $f : R_* \rightarrow S_*$ is a weak equivalence of cofibrant chain algebras, then given an $H_0(S)$ -bimodule M the induced homomorphism $H^*(S_*, M) \rightarrow H^*(R_*, M)$ is an isomorphism. This follows from Lemma 4.1.1.

Let R_* be a chain algebra. Thanks to the properties of closed model categories there exists a chain algebra morphism $f : R_*^c \rightarrow R_*$ which is a weak equivalence and R_*^c is a cofibrant. For any R -bimodule M the naive Hochschild cohomology groups $H^*(R_*^c, M)$ do not depend on the cofibrant replacement and they are called *the derived Hochschild cohomology of R_* with coefficients in M* and are denoted by $H^*(R_*, M)$. Thus

$$H^*(R_*, M) := H^*(R_*^c, M).$$

This definition has expected functorial properties: for any morphism $f : R_* \rightarrow S_*$ of chain algebras and any $H_0(S)$ -bimodule M there is a well-defined homomorphism $H^*(S_*, M) \rightarrow H^*(R_*, M)$ which depends only on the homotopy class of f . Moreover it is an isomorphism provided f is a weak equivalence. One has also a natural homomorphism $H^*(R_*, M) \rightarrow H^*(R_*, M)$ which is induced by the chain algebra homomorphism $R_*^c \rightarrow R_*$. The following fact is a direct consequence of Lemma 4.1.1.

4.3.2. Lemma. *If R_* is projective as a k -module then $H^*(R_*, M) \rightarrow H^*(R_*, M)$ is an isomorphism.*

Since the category of algebras is the category of chain algebras concentrated in degree 0, we can consider the restriction of the derived Hochschild cohomology H^* on this subcategory. The resulting theory is called the *Shukla cohomology*. Thus for any algebra R and any R -bimodule M the Shukla cohomology of an algebra R with coefficients in M is defined by

$$SH^*(R, M) := H^*(R, M) \cong H^*(R_*^c, M) \cong H^*(\text{Der}(R_*^c, M)),$$

where $R_*^c \rightarrow R$ is a weak equivalence from a quasi-free chain algebra R_*^c to R . Sometimes these groups are denoted by $SH^*(R/k, M)$ in order to make clear the ground ring k . The natural transformation

$$H^n(R, M) \rightarrow SH^n(R, M), \quad n \geq 0$$

is an isomorphism in dimensions $n = 0, 1$ and is an isomorphism in all dimensions provided R is projective as a k -module. For example we have $SH^i(A, -) = 0$ provided A is a k -free algebra and $i \geq 2$.

The cup-product in Hochschild cohomology yields a (commutative graded) algebra structure on $SH^*(A, A)$.

4.4. Shukla cohomology and extensions. The following properties of Shukla cohomology are of special interests. They are analogues of the isomorphisms (2) and (6).

4.4.1. Theorem. *Let A be an associative algebra and let M be an A -bimodule. Then there are natural isomorphisms*

$$\mathrm{SH}^2(A, M) \cong \mathrm{Extalg}(A, M),$$

$$\mathrm{SH}^3(A, M) \cong \mathrm{Xext}(A, M).$$

The first isomorphism is well known (see [32], Theorem 4). However we give an independent proof.

Proof. (i) Let

$$(E) \quad 0 \rightarrow M \rightarrow E \rightarrow A \rightarrow 0$$

be a singular extension of algebras. Define the chain algebra E_* as follows:

$$E_0 = E, \quad E_1 = M, \quad E_n = 0, \quad n \geq 2.$$

The only nontrivial boundary map is induced by the inclusion $M \rightarrow E$. Then one has a map of chain algebras $E_* \rightarrow A$ which is an acyclic fibration. Let $A_* \rightarrow A$ be a weak equivalence with quasi-free A_* . Since A_* is cofibrant there exists a lifting $f_* : A_* \rightarrow E_*$. We consider now the first component $f_1 : A_* \rightarrow E_1 = M$ of f_* . Since f_* is a homomorphism of algebras it follows that $f_1 \in \mathrm{Der}(A_1, M)$ is a 1-cocycle of $\mathrm{Der}(A_*, M)$ and therefore it gives a class $e(E) \in \mathrm{SH}^2(A, M)$. If $g_* : A_* \rightarrow E_*$ is another lifting, then the values of $h = f_0 - g_0 : A_0 \rightarrow E$ lie in M . Thus $h \in \mathrm{Der}(A_0, M)$ and $f_1 - g_1 = \partial^*(h)$, which shows that the class $e(E)$ depends only on the isomorphism class of (E) . Conversely, if $f \in \mathrm{Der}(A_*, M)$ is a 1-cocycle, then one can form an abelian extension according to the following diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 \longrightarrow A \longrightarrow 0 \\ & & \downarrow & & \downarrow f & & \downarrow \mathrm{id} \\ & & 0 & \longrightarrow & M & \longrightarrow & E \longrightarrow A \longrightarrow 0. \end{array}$$

In this way we obtain the isomorphism (i).

(ii) Let

$$0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow A \rightarrow 0$$

be a crossed extension. The algebra C_0 acts on M via the projection to A . Moreover

$$C_* = (\cdots \rightarrow 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0)$$

can be considered as a chain algebra as follows. In dimensions 0 and 1 it is already defined. In the dimension two one puts $C_2 = M$, and $C_i = 0$ for $i > 2$. The pairing $C_i \otimes C_j \rightarrow C_{i+j}$ is the given one if $i = 0$ or $j = 0$, while the pairing $C_1 \otimes C_1 \rightarrow C_2$ as well as all other pairings are zero. Then $C_* \rightarrow A$ is an acyclic fibration. Therefore we have a lifting $f_* : A_* \rightarrow C_*$, where $A_* \rightarrow A$ is a weak equivalence with quasi-free A_* . It is clear that $f_2 \in \mathrm{Der}(A_*, M)$ is a 2-cocycle in $\mathrm{Der}(A_*, M)$ and therefore gives rise to an element in

$\mathrm{SH}^3(A, M)$. Conversely, starting with a 2-cocycle $f \in \mathrm{Der}(A_*, M)$ one can construct the corresponding crossed extension using the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow & & \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} & & \\ & & 0 & \longrightarrow & M & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & A & \longrightarrow & 0. \quad \square \end{array}$$

The following theorem is the analogue of Theorem 3.3.2.

4.4.2. Theorem. *The class of a crossed extension*

$$(\partial) \quad 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} R \rightarrow 0$$

is zero in $\mathrm{SH}^3(R, M)$ if and only if ${}_{\partial} \mathrm{Ext}(R, C_1)$ is nonempty. If this is the case then the group $\mathrm{SH}^2(R, M)$ acts transitively and effectively on ${}_{\partial} \mathrm{Ext}(R, C_1)$.

Proof. It is clear that the crossed extension

$$0 \rightarrow M \xrightarrow{\mathrm{id}} M \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \rightarrow 0$$

represents the zero element of $\mathrm{Xext}(R, M)$. Assume ${}_{\partial} \mathrm{Ext}(R, C_1)$ is nonempty and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\mu} & S & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow \xi & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \end{array}$$

be an object of the category ${}_{\partial} \mathrm{Ext}(R, C_1)$. Then $d: M \oplus C_1 \rightarrow S$ is a crossed bi-module, where $d(m, c_1) = \mu(c_1)$ and the actions of S on $M \oplus C_1$ are given by

$$s(m, c_1) = (\xi(s)m, \xi(s)c_1) \quad \text{and} \quad (m, c_1)s = (m\xi(s), c_1\xi(s)).$$

We also need the map $p: M \oplus C_1 \rightarrow C_1$ given by $p(m, c_1) = c_1 + i(m)$. Then one has the following commutative diagram in $\mathrm{Xext}(R, M)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\mathrm{id}} & M & \xrightarrow{0} & R & \xrightarrow{\mathrm{id}} & R & \longrightarrow & 0 \\ & & \uparrow \mathrm{id} & & \uparrow p_1 & & \uparrow \varsigma & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{i_1} & M \oplus C_1 & \xrightarrow{d} & S & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow p & & \downarrow \xi & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \end{array}$$

which shows that the class of (∂) in $\mathrm{SH}^3(R, M)$ is zero. Here $p_1(m, c) = m$ and $i_1(m) = (m, 0)$.

Conversely, assume the class of (∂) in $\mathrm{SH}^3(R, M)$ is zero. It follows from Corollary 3.2.3 that there exists a commutative diagram of crossed extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\text{id}} & M & \xrightarrow{0} & R & \xrightarrow{\text{id}} & R & \longrightarrow & 0 \\
& & \uparrow \text{id} & & \uparrow p & & \uparrow \varsigma & & \uparrow \text{id} & & \\
0 & \longrightarrow & M & \xrightarrow{i} & P_1 & \xrightarrow{\mu} & P_0 & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow \epsilon & & \downarrow \xi & & \downarrow \text{id} & & \\
0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0.
\end{array}$$

It follows that the restriction of μ to $\text{Ker}(p)$ is a monomorphism and therefore we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker}(p) & \xrightarrow{\mu} & P_0 & \xrightarrow{\varsigma} & R & \longrightarrow & 0 \\
& & \downarrow \epsilon & & \downarrow \xi & & \downarrow \text{id} & & \\
0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0.
\end{array}$$

One defines the k -algebra S via the exact sequence

$$0 \longrightarrow \text{Ker}(p) \xrightarrow{(-\epsilon, \mu)} C_1 \oplus P_0 \longrightarrow S \longrightarrow 0.$$

Here the product on $C_1 \oplus P_0$ is given by

$$(c, x)(c', x') := (c * c' + c\xi(x') + \xi(x)c', xx').$$

One easily checks that $\text{Ker}(p)$ is really an ideal of $C_1 \oplus P_0$ and therefore S is well-defined. Now it is clear that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & \\
0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0
\end{array}$$

is an object of ${}_{\partial}\text{Ext}(R, C_1)$ and the proof is finished. $\square$

4.4.3. Corollary. The class of a crossed extension

$$(\partial) \quad 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow R \rightarrow 0$$

is zero in $\text{SH}^3(R, M)$ if and only if there exists a ∂ -coextension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow \sigma & & \downarrow \text{id} & & & & \\
0 & \longrightarrow & M & \longrightarrow & S & \xrightarrow{\nu} & C_0 & \longrightarrow & 0.
\end{array}$$

Proof. This is an immediate consequence of the previous result and Lemma 3.3.1. $\square$

Remark. One cannot get non- k -split versions of results of Remark 3.3.4. This is because Shukla cohomology does not vanish on injective bimodules. Indeed, if $k = \mathbb{Z}$ and $R = \mathbb{F}_p$, then any bimodule over R is injective, while the computation in Section 4.4 shows that $\mathrm{SH}^{2i}(\mathbb{F}_p/\mathbb{Z}, \mathbb{F}_p) = \mathbb{F}_p$ for all i .

Remark. We have the interpretation of the higher Shukla cohomology using chain algebras. Further we use the following extension of Theorem 4.4.1 to higher dimensions. A chain algebra A_* is called of length $\leq n$ if $A_i = 0$ for all $i > n$. Let R be an algebra and M be a bimodule over R . For any $n \geq 1$ we let $\mathcal{X} \mathrm{ext}^n(R, M)$ be the category of triples (A_*, α, β) where A_* is a chain algebra of length $\leq n$ with property $H_i(A_*) = 0$ for all $0 < i < n$. Moreover $\alpha : H_0(A_*) \rightarrow R$ is an isomorphism of algebras and $\beta : M \rightarrow H_n(A_*)$ is an isomorphism of R -bimodules, where the R -bimodule structure on $H_n(A_*)$ is induced via α^{-1} . It is clear that for $n = 1$ the categories $\mathcal{X} \mathrm{ext}^1(R, M)$ and $\mathcal{X} \mathrm{ext}(R, M)$ are equivalent. The argument given in the proof of part (ii) of Theorem 4.4.1 shows that connected components of the category $\mathcal{X} \mathrm{ext}^n(R, M)$ are in one-to-one correspondence with elements of the group $\mathrm{SH}^{n+2}(R, M)$. Furthermore, for a given object $X = (A_*, \alpha, \beta)$ of the category $\mathcal{X} \mathrm{ext}^n(R, M)$ one can define the category ${}_X \mathrm{Ext}(R; A_n)$ of objects (C_*, γ, η) , where C_* is a chain algebra of length $\leq n$ with the property $H_i(C_*) = 0$ for all $i > 0$, $\gamma : H_0(C_*) \rightarrow R$ is an isomorphism of algebras and $\eta : C_* \rightarrow A_*$ is a chain algebra homomorphism such that the diagram

$$\begin{array}{ccc} H_0(C_*) & \xrightarrow{\gamma} & R \\ \downarrow \eta & & \downarrow \mathrm{id} \\ H_0(A_*) & \xrightarrow{\alpha} & R \end{array}$$

commutes and $\eta_n : C_n \rightarrow A_n$ is an isomorphism. Then the category ${}_X \mathrm{Ext}(R; A_n)$ is non-empty if and only if the class of X in $\mathrm{SH}^{n+2}(R, M)$ is zero. If this is so, then the group $\mathrm{SH}^{n+1}(R, M)$ acts transitively and effectively on the set of components of the category ${}_X \mathrm{Ext}(R; A_n)$.

Duskin in [11] introduced higher torsors to obtain an interpretation of elements of the cohomology groups in a very general context. For associative algebras his approach also gives the interpretation of H^3 via the crossed bimodules, but in higher dimensions his approach is totally different from the one indicated here.

4.5. Shukla cohomology via free crossed bimodules. Let R be an associative k -algebra. We claim that there is a totally free crossed bimodule $\partial : F_1 \rightarrow F_0$ with $\mathrm{Coker}(\partial) = R$. Indeed, first we take a surjective homomorphism of rings $\pi : F_0 \rightarrow R$, where F_0 is a free k -algebra. Then we choose a free k -module V together with an epimorphism $V \rightarrow \mathrm{Ker}(\pi)$. Finally we take $\partial : F_1 \rightarrow F_0$ to be the free crossed bimodule generated by $V \rightarrow F_0$. Then ∂ has the expected property.

4.5.1. Proposition. *Let R be an associative algebra and let M be a bimodule over R . Let*

$$(F) \quad 0 \rightarrow E \xrightarrow{j} F_1 \xrightarrow{\partial} F_0 \rightarrow R \rightarrow 0$$

be a crossed extension with totally free crossed bimodule $\partial : F_1 \rightarrow F_0$. Then there is an exact sequence

$$\mathrm{Hom}_{F_0^e}(F_1, M) \xrightarrow{j^*} \mathrm{Hom}_{R^e}(E, M) \rightarrow \mathrm{SH}^3(R, M) \rightarrow 0$$

where $j^*(h) = hj$, for $h \in \mathrm{Hom}_{F_0^e}(F_1, M)$.

Proof. The crossed extension (F) defines an element $e \in H^3(R, E)$. The homomorphism $e_* : \mathrm{Hom}_{R^e}(E, M) \rightarrow \mathrm{SH}^3(R, M)$ sends an element $f \in \mathrm{Hom}_{R^e}(E, M)$ to $f_*(e) \in \mathrm{SH}^3(R, M)$. Take any element in $\mathrm{SH}^3(R, M)$. Thanks to Theorem 4.4.1 and Corollary 3.2.3 it can be represented by a crossed extension of the form

$$0 \rightarrow M \rightarrow C \rightarrow F_0 \rightarrow R \rightarrow 0.$$

Since F_0 is a free algebra and $\partial : F_1 \rightarrow F_0$ is a free crossed bimodule, there exists a morphism of crossed extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & C & \longrightarrow & F_0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

which shows that $e_* : \mathrm{Hom}_{R^e}(E, M) \rightarrow \mathrm{SH}^3(R, M)$ is an epimorphism. We claim that $j_*(e) = 0$. Indeed, $j_*(e)$ is represented by the bottom crossed extension in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{j} & F_1 & \longrightarrow & F_0 & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow j & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & X & \longrightarrow & F_0 & \longrightarrow & R & \longrightarrow & 0. \end{array}$$

Obviously $F_1 \rightarrow X$ has a retraction, hence the claim. Take any $h \in \mathrm{Hom}_{F_0^e}(F_1, M)$. Then we have

$$e_* j^*(h) = (hj)_*(e) = h_* j_*(e) = 0.$$

Thus it remains to show that if $f \in \mathrm{Hom}_{R^e}(E, M)$ satisfies $f_*(e) = 0$, then $f = hj$ for some $h \in \mathrm{Hom}_{F_0^e}(F_1, M)$. If $f_*(e) = 0$, then we can use Theorem 4.4.2 to obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{j} & F_1 & \xrightarrow{\partial} & F_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{j'} & C & \xrightarrow{\delta} & F_0 & \xrightarrow{\pi} & R & \longrightarrow & 0 \\ & & & & \uparrow \mathrm{id} & & \uparrow t & & \uparrow \mathrm{id} & & \\ & & & & 0 & \longrightarrow & C & \xrightarrow{i} & S & \xrightarrow{p} & R & \longrightarrow & 0. \end{array}$$

Since F_0 is a free k -algebra, the homomorphism t has a section $s : F_0 \rightarrow S$. So we have $ts = \mathrm{id}_{F_0}$. Since $p = \pi t$, we obtain $ps = \pi ts = \pi$. It follows that $ps\partial = \pi\partial = 0$, thus there

exists a unique $r : F_1 \rightarrow C$ such that $s\partial = ir$. Then we have $irj = s\partial j = 0$ and therefore $rj = 0$. On the other hand

$$\delta(g - r) = \delta g - \delta r = \partial - tir = \partial - ts\partial = 0.$$

Therefore there exists a unique $h : F_1 \rightarrow M$ such that $g = r + j'h$. Since

$$j'f = gj = rj + j'hj = j'hj$$

we obtain $f = hj$ and we are done. $\square$

5. Shukla cohomology of liftable algebras

5.1. The case $k = \mathbb{Z}$. Let $k = \mathbb{Z}$ and $R = \mathbb{Z}/n\mathbb{Z}$, $n \geq 2$. Consider the exterior algebra $\Lambda_{\mathbb{Z}}^*(x)$ on a generator x of degree 1 over $\mathbb{Z}$. We put $\partial(x) = n$. Then $\Lambda_{\mathbb{Z}}^*(x)$ is a chain algebra, which is denoted by ${}_n\Lambda$. Thus as a chain complex ${}_n\Lambda$ looks as follows:

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z}.$$

Hence the chain algebra ${}_n\Lambda$ is weakly equivalent to the ring $\mathbb{Z}/n\mathbb{Z}$. It is clear that the normalized Hochschild cochain complex of the chain algebra ${}_n\Lambda$ with coefficients in $\mathbb{Z}/n\mathbb{Z}$ has a bicomplex structure, which is $\mathbb{Z}/n\mathbb{Z}$ in bidegree (i, i) , $i \geq 0$ and is zero elsewhere. Thus

$$\mathrm{SH}^*(R/\mathbb{Z}, R) = R[\xi], \quad R = \mathbb{Z}/n\mathbb{Z},$$

where $\xi \in \mathrm{SH}^2(R/\mathbb{Z}, R)$ has degree 2. Based on the interpretation of the second Shukla cohomology via abelian extensions (see Section 4.4) one easily sees that ξ represents the following extension:

$$\xi = (0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n^2\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0) \in \mathrm{Extalg}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}).$$

The following computation generalizes this example. For the definition of $\mathbb{Z}$ -lifting we refer the reader to Section 2.3.

5.1.1. Proposition. *Let A be an algebra over $R = \mathbb{Z}/n\mathbb{Z}$, which is free as an R -module. If A has a $\mathbb{Z}$ -lifting, then*

$$\mathrm{SH}^*(A/\mathbb{Z}, A) \cong \mathrm{H}^*(A/R, A)[\xi].$$

Proof. Let A_0 be a $\mathbb{Z}$ -lifting of A . We put $A_* = {}_n\Lambda \otimes A_0$. Since A_0 is free as an abelian group, it follows that $A_* \rightarrow A$ is a weak equivalence. Since

$$C^*(A_*, A) \cong C^*({}_n\Lambda \otimes A_0, A) \cong C^*(\Lambda^*(x) \otimes A, A),$$

where $\Lambda_R^*(x)$ is considered as a chain algebra with trivial differential. Hence the Künneth Theorem for Hochschild cohomology [23] implies

$$\mathrm{SH}^*(A/\mathbb{Z}, A) \cong \mathrm{H}^*(A/R, A) \otimes \mathrm{SH}^*(R/\mathbb{Z}, R) \cong \mathrm{H}^*(A/R, A)[\xi]. \quad \square$$

It is clear that group algebras (or more generally monoid algebras), truncated polynomial algebras have lifting to $\mathbb{Z}$. It is also known that any smooth commutative algebra has lifting to $\mathbb{Z}$ [1]. It is also clear that the class of algebras having lifting to $\mathbb{Z}$ is closed under finite tensor and cartesian products.

5.2. The case $k = \mathbb{Z}/p^2\mathbb{Z}$. Let p be a prime. We set $k = \mathbb{Z}/p^2\mathbb{Z}$. Consider the commutative chain algebra

$$\Lambda_{\mathbb{Z}/p^2\mathbb{Z}}^*(x) \otimes \Gamma_{\mathbb{Z}/p^2\mathbb{Z}}^*(y),$$

where x is of degree 1 and y is of degree 2. Here Γ^* denotes the divided power algebra. Now we put $\partial(x) = p$ and $\partial(y) = px$. One easily checks that in this way one obtains a chain algebra compatible with divided powers. Since the augmentation

$$\Lambda_{\mathbb{Z}/p^2\mathbb{Z}}^*(x) \otimes \Gamma_{\mathbb{Z}/p^2\mathbb{Z}}^*(y) \rightarrow \mathbb{F}_p$$

is a weak equivalence, one can use this chain algebra to compute the Shukla cohomology. It is clear that

$$C^*(\Lambda_{\mathbb{Z}/p^2\mathbb{Z}}^*(x) \otimes \Gamma_{\mathbb{Z}/p^2\mathbb{Z}}^*(y), \mathbb{F}_p) \cong C^*(\Lambda_{\mathbb{F}_p}^*(x) \otimes \Gamma_{\mathbb{F}_p}^*(y), \mathbb{F}_p)$$

where $\Lambda_{\mathbb{F}_p}^*(x) \otimes \Gamma_{\mathbb{F}_p}^*(y)$ is a chain algebra with zero boundary map. Then the Künneth theorem for Hochschild cohomology [23] implies that

$$\mathrm{SH}^*(\mathbb{F}_p/k, \mathbb{F}_p) \cong \mathbb{F}_p[\sigma x, \sigma y, \sigma y^{[2]}, \dots, \sigma y^{[2^n]}, \dots], \quad \text{if } p = 2$$

where $|\sigma z| = 1 + |z|$. Similarly, if p is odd, then

$$\mathrm{SH}^*(\mathbb{F}_p/k, \mathbb{F}_p) \cong \Lambda^*(\sigma y, \dots, \sigma y^{[p^n]}, \dots) \otimes \mathbb{F}_p[\sigma x, \sigma^2 y, \dots, \sigma^2 y^{[p^n]}, \dots].$$

Here we use the fact that one has an isomorphism of algebras:

$$\Gamma_{\mathbb{F}_p}^*(z) \cong \mathbb{F}_p[z]/(z^p) \otimes \mathbb{F}_p[z]/(z^{p^2}) \otimes \mathbb{F}_p[z]/(z^{p^3}) \otimes \dots$$

The element σx is represented by the following abelian extension of algebras:

$$(0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{F}_p \rightarrow 0) \in \mathrm{Extalg}_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p),$$

while σy is represented by the crossed extension of algebras:

$$(0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{F}_p \rightarrow 0) \in \mathrm{Xext}_{\mathbb{Z}/p^2\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p).$$

More generally, we get

5.2.1. Proposition. *Let A be an algebra over $\mathbb{F}_p$. If A has a lifting to $k = \mathbb{Z}/p^2\mathbb{Z}$, then*

$$\mathrm{SH}^*(A/k, A) \cong H^*(A/\mathbb{F}_p, A) \otimes \mathrm{SH}^*(\mathbb{F}_p/k, \mathbb{F}_p)$$

where

$$\mathrm{SH}^*(\mathbb{F}_2/k, \mathbb{F}_2) \cong \mathbb{F}_2[\sigma x, \sigma y, \sigma y^{[2]}, \dots, \sigma y^{[2^n]}, \dots]$$

and

$$\mathrm{SH}^*(\mathbb{F}_p/k, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}^*(\sigma y, \dots, \sigma y^{[p^n]}, \dots) \otimes \mathbb{F}_p[\sigma x, \sigma^2 y, \dots, \sigma^2 y^{[p^n]}, \dots]$$

if p is odd.

Proof. Let A_* be a chain algebra over $\mathbb{Z}/p^2$ given as the tensor product of chain algebras:

$$A_* = A_0 \otimes \Lambda_{\mathbb{Z}/p^2\mathbb{Z}}^*(x) \otimes \Gamma_{\mathbb{Z}/p^2\mathbb{Z}}^*(y).$$

By the Künneth theorem for Hochschild cohomology [23], $A_* \rightarrow A$ is a weak equivalence and hence

$$\mathrm{SH}_{\mathbb{Z}/p^2}^*(A, A) \cong H_{\mathbb{Z}/p^2}^*(A, A) \otimes \mathrm{SH}_{\mathbb{Z}/p^2}^*(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}). \quad \square$$

Let us observe that if an $\mathbb{F}_p$ -algebra A has a lifting to $\mathbb{Z}$ then it has also a lifting to $\mathbb{Z}/p^2\mathbb{Z}$. So, one can apply this result for monoid algebras, as well as for truncated polynomial algebras and smooth commutative algebras. We recall that there is a cohomological obstruction class $o(A) \in H^3(A, A)$ such that $o(A) = 0$ if and only if A has $\mathbb{Z}/p^2\mathbb{Z}$ -lifting (see Section 2.3). In particular such a lifting exists provided $H^3(A, A) = 0$.

6. Shukla cohomology over $\mathbb{Z}$ and $\mathbb{Z}/p^2\mathbb{Z}$ in low dimensions

6.1. The homomorphism b . In this section $k = \mathbb{Z}/p^2\mathbb{Z}$ and H^* denotes the Hochschild cohomology over $\mathbb{F}_p$.

Let M be a bimodule over an $\mathbb{F}_p$ -algebra A . Since A is also an algebra over $\mathbb{Z}$ and $k = \mathbb{Z}/p^2\mathbb{Z}$, we can consider not only the Hochschild cohomology $H^*(A, M)$, but also the Shukla cohomologies $\mathrm{SH}^*(A/k, M)$ and $\mathrm{SH}^*(A/\mathbb{Z}, M)$. The ring homomorphisms $\mathbb{Z} \rightarrow k \rightarrow \mathbb{F}_p$ yield the natural transformations

$$H^i(A, M) \rightarrow \mathrm{SH}^i(A/k, M)$$

and

$$b^i : \mathrm{SH}^i(A/k, M) \rightarrow \mathrm{SH}^i(A/\mathbb{Z}, M)$$

which are obviously isomorphisms for $i = 0, 1$. For $i = 2$, the groups in question classify abelian extensions of A by M , respectively in the category of algebras over $\mathbb{F}_p$, over k and over $\mathbb{Z}$. Let us observe that if $X \rightarrow Y \rightarrow Z$ is an exact sequence of abelian groups and $pX = 0 = pZ$, then $p^2Y = 0$. Thus any abelian extension of A by M in the category of all rings lies in the category of algebras over k . It follows that for $i = 2$, the first map $H^2(A, M) \rightarrow \mathrm{SH}^2(A/k, M)$ is a monomorphism, while the second homomorphism is an isomorphism:

$$b^2 : \mathrm{SH}^2(A/k, M) \cong \mathrm{SH}^2(A/\mathbb{Z}, M).$$

6.1.1. Lemma. *The homomorphism*

$$b^n : \mathrm{SH}^n(A/k, M) \rightarrow \mathrm{SH}^n(A/\mathbb{Z}, M)$$

is an isomorphism for $n = 0, 1, 2$. If $n \geq 3$, then b^n is an epimorphism and it has a natural splitting.

Proof. We have only to consider the case $n \geq 3$. We have to construct the homomorphism $d^n : \mathrm{SH}^n(A/\mathbb{Z}, M) \rightarrow \mathrm{SH}^n(A/k, M)$, which is a right inverse of b^n . We consider more carefully the case $n = 3$ and then we indicate how to modify the argument for $n > 3$. In terms of crossed extensions, $b = b^3 : \mathrm{SH}^3(A/k, M) \rightarrow \mathrm{SH}^3(A/\mathbb{Z}, M)$ sends the class of a crossed extension

$$0 \rightarrow M \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0$$

of $\mathbb{Z}/p^2\mathbb{Z}$ -algebras to the same crossed extension but now considered as algebras over $\mathbb{Z}$. Now we take any element from $\mathrm{SH}^3(A/\mathbb{Z}, M)$, which is represented by the following crossed extension of A by M in the category of rings:

$$0 \rightarrow M \rightarrow D_1 \xrightarrow{\partial} D_0 \rightarrow A \rightarrow 0.$$

Thanks to Lemma 3.2.2 and Corollary 3.2.3 without loss of generality one can assume that D_0 is free as an abelian group. Thus $V := \mathrm{Im}(\partial)$ is also free as an abelian group and $0 \rightarrow M \rightarrow D_1 \rightarrow V \rightarrow 0$ splits as a sequence of abelian groups. It follows that $0 \rightarrow M \rightarrow D_1/pD_1 \rightarrow V/pV \rightarrow 0$ is exact. On the other hand pV is a two-sided ideal in D_0 and therefore one has an exact sequence

$$0 \rightarrow V/pV \rightarrow D_0/pV \rightarrow A \rightarrow 0.$$

It follows that D_0/pV is a $\mathbb{Z}/p^2\mathbb{Z}$ -algebra. By gluing these data we get a crossed extension

$$0 \rightarrow M \rightarrow D_1/pD_1 \rightarrow D_0/pV \rightarrow A \rightarrow 0$$

and therefore an element in $\mathrm{SH}^3(A/k, M)$. In this way we obtain the homomorphism $d = d^3 : \mathrm{SH}^3(A/\mathbb{Z}, M) \rightarrow \mathrm{SH}^3(A/k, M)$. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & D_1 & \longrightarrow & D_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & D_1/pD_1 & \longrightarrow & D_0/pV & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

shows that $bd = \mathrm{id}$ and the case $n = 3$ follows. Assume now $n > 3$. According to Remark at the end of Section 4.4 we know that elements of $\mathrm{SH}^n(A/\mathbb{Z}, M)$ are equivalence classes of chain algebras X_* of length $\leq n - 2$ which are acyclic in all but the extreme dimensions:

$$0 \rightarrow M \rightarrow X_{n-2} \rightarrow \cdots \rightarrow X_0 \rightarrow A \rightarrow 0.$$

Without loss of generality one can assume that $X_0, \dots, X_{n-3}$ are free as abelian groups. By repeating the previous argument we can construct a diagram of the form

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & X_{n-2} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & A \longrightarrow 0 \\
& & \downarrow \text{id} & & \downarrow & & & & \downarrow & & \downarrow \text{id} \\
0 & \longrightarrow & M & \longrightarrow & X_{n-2}/pX_{n-2} & \longrightarrow & \cdots & \longrightarrow & X_0/pV & \longrightarrow & A \longrightarrow 0
\end{array}$$

where $V := \text{Ker}(X_0 \rightarrow A)$ and the proof is finished. $\square$

6.2. Dimension three. Now we analyze the kernel of the homomorphism

$$b = b^3 : \text{SH}^3(A/k, M) \rightarrow \text{SH}^3(A/\mathbb{Z}, M).$$

6.2.1. Proposition. *Let A be an algebra over $\mathbb{F}_p$ and let M be a bimodule over A . Then one has a natural isomorphism*

$$\text{SH}^3(A/k, M) \cong \text{SH}^3(A/\mathbb{Z}, M) \oplus H^0(A, M)$$

where $k = \mathbb{Z}/p^2\mathbb{Z}$.

Proof. It consists of several steps. We already defined the homomorphism

$$d = d^3 : \text{SH}^3(A/\mathbb{Z}, M) \rightarrow \text{SH}^3(A/k, M)$$

with $bd = \text{id}$. Now we define the homomorphisms

$$e : \text{SH}^3(A/k, M) \rightarrow H^0(A, M)$$

and

$$c : H^0(A, M) \rightarrow \text{SH}^3(A/k, M)$$

such that

$$ed = 0, \quad ec = \text{id}, \quad bc = 0,$$

and we prove that $(e, b) : \text{SH}^3(A/k, M) \rightarrow H^0(A, M) \oplus \text{SH}^3(A/\mathbb{Z}, M)$ is a monomorphism. From these assertions the result follows.

First step. The homomorphism $e : \text{SH}^3(A/k, M) \rightarrow H^0(A, M)$. Let

$$(\partial) \quad 0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} A \rightarrow 0$$

be a crossed extension, where C_0 and C_1 are k -algebras. Since A is an algebra over $\mathbb{F}_p$, one has $\pi(p1_{C_0}) = 0$, where $1_{C_0} \in C_0$ is the unit of C_0 . Therefore one can write $p1 = \partial([P])$ for a suitable $[P]$ in C_1 . Now we put:

$$e((\partial)) = p[P] \in M.$$

It is easy to check that e is a well-defined homomorphism. Let us observe that $e(\partial) = 0$ if $pC_1 = 0$. It follows that $ed = 0$.

Second step. The canonical class $(\sigma)_A \in \mathrm{SH}^3(A/k, A)$. Let X be an abelian group. We let $\bar{\mathbb{Z}}[X]$ be the free abelian group generated by X modulo the relation $[0] = 0$. Here $[x]$ denotes an element of $\bar{\mathbb{Z}}[X]$ corresponding to $x \in X$. Then we have a canonical epimorphism $\eta : \bar{\mathbb{Z}}[X] \rightarrow X$, $\eta([x]) = x$ which gives rise to the canonical free resolution of X :

$$0 \rightarrow R(X) \rightarrow \bar{\mathbb{Z}}[X] \rightarrow X \rightarrow 0.$$

For any $x, y \in X$ we put

$$\langle x, y \rangle := [x] + [y] - [x + y] \in R(X).$$

We now assume that $pX = 0$, that is X is a vector space over $\mathbb{F}_p$. By applying the functor $(-) \otimes \mathbb{Z}/p^2\mathbb{Z}$ to the canonical free resolution we obtain the following exact sequence:

$$(\sigma)_X \quad 0 \rightarrow X \xrightarrow{i} R(X)/p^2R(X) \xrightarrow{\sigma} \overline{\mathbb{Z}/p^2\mathbb{Z}}[X] \xrightarrow{\eta} X \rightarrow 0.$$

Here we used the well-known isomorphism $V \cong \mathrm{Tor}(V, \mathbb{Z}/p^2\mathbb{Z})$ for any $\mathbb{F}_p$ -vector space V considered as an abelian group (the Tor and $\otimes$ are taken of course over $\mathbb{Z}$ and not over $k = \mathbb{Z}/p^2\mathbb{Z}$). The homomorphism i has the following form:

$$i(x) = \sum_{j=1}^{p-1} p \langle jx, x \rangle \mod(p^2R(X)).$$

Let us turn back to our situation. We can take $X = A$. The multiplicative structure on A can be extended linearly to $\bar{\mathbb{Z}}[A]$ to get an associative algebra structure on it. Then not only η is a ring homomorphism, but the exact sequence $(\sigma)_A$ is a crossed extension and therefore we obtain an element

$$(\sigma)_A = (0 \rightarrow A \rightarrow R(A)/p^2R(A) \xrightarrow{\sigma} \overline{\mathbb{Z}/p^2\mathbb{Z}}[A] \rightarrow A \rightarrow 0) \in \mathrm{SH}^3(A/k, A).$$

It is clear that $A \mapsto (\sigma)_A$ is a functor from $\mathbb{F}_p$ -algebras to the category of crossed extensions of $\mathbb{Z}/p^2\mathbb{Z}$ -algebras. Since

$$\sigma\left(\sum_{j=1}^{j=p-1} \langle j, 1 \rangle\right) = p[1]$$

one has

$$e((\sigma)_A) = 1 \in H^0(A, A) \subset A.$$

On the other hand $p^2R(A)$ is an ideal of $\mathbb{Z}[A]$. Thus we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R(A)/p^2R(A) & \longrightarrow & \bar{\mathbb{Z}}[A]/p^2R(A) & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \mathrm{id} \\ 0 & \longrightarrow & A & \longrightarrow & R(A)/p^2R(A) & \longrightarrow & \bar{\mathbb{Z}/p^2\mathbb{Z}}[A] \longrightarrow A \longrightarrow 0. \end{array}$$

It follows from Theorem 4.4.2 that the class $(\sigma)_A$ has the following important property:

$$0 = b((\sigma)_A) \in \mathrm{SH}^3(A/\mathbb{Z}, A).$$

Third step. The homomorphism $c : H^0(A, M) \rightarrow SH^3(A/k, M)$. Using the class $(\sigma)_A$ we now define the homomorphism $c : H^0(A, M) \rightarrow SH^3(A/k, M)$ by

$$c(m) = f_m^*((\sigma)_A).$$

Here $m \in H^0(A, M)$ and $f_m : A \rightarrow M$ is the unique bimodule homomorphism with $f_m(1) = m$ and $f_m^* : SH^3(A/k, A) \rightarrow SH^3(A/k, M)$ is the induced homomorphism in cohomology. Since e and b are natural transformations of functors it follows that for any $m \in M$ we have

$$ec(m) = ef_m^*((\sigma)_A) = f_m^*e((\sigma)_A) = f_m(1) = m$$

and

$$bc(m) = bf_m^*((\sigma)_A) = f_m^*b((\sigma)_A) = 0.$$

Thus

$$ec = \text{id} \quad \text{and} \quad bc = 0.$$

Fourth step. It remains to show that

$$(e, b) : SH^3(A/k) \rightarrow H^0(A, M) \oplus SH^3(A/\mathbb{Z})$$

is a monomorphism. Let

$$0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \rightarrow A \rightarrow 0$$

be a crossed extension of $\mathbb{Z}/p^2\mathbb{Z}$ -algebras which lies in $\text{Ker}(e, b)$. Since it goes to zero in $SH^3(A/\mathbb{Z}, M)$ one has the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\mu} & S & \xrightarrow{\zeta} & R & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \xi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

where S is a ring. Since ξ is a homomorphism of algebras with unit we have $[P] = p1_S$, where 1_S is the unit of S . Therefore $e(\partial) = p^21_S = 0$, because (∂) goes also to zero under the map e . It follows that S is an algebra over $\mathbb{Z}/p^2\mathbb{Z}$. Theorem 4.4.2 shows that the class of $0 \rightarrow M \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0$ in $SH^3(A/k, M)$ is zero and the proof is finished. $\square$

7. Shukla cohomology over $\mathbb{Z}$ and Hochschild cohomology over $\mathbb{F}_p$ in low dimensions

7.1. The homomorphisms ξ^* and ζ^* . Let A be an $\mathbb{F}_p$ -algebra and M be a bimodule over A . In this section we let $H^*(A, M)$ be the Hochschild cohomology of A with coefficients in M considered over the ground ring $\mathbb{F}_p$, while $SH^*(A/\mathbb{Z}, M)$ denotes the Shukla cohomology of A considered over the ground ring $\mathbb{Z}$. By functoriality one has the following homomorphism:

$$\xi^i : H^i(A, M) \rightarrow SH^i(A/\mathbb{Z}, M), \quad i \geq 0,$$

which is an isomorphism provided $i = 0$ or $i = 1$. Let us recall that $\mathrm{SH}^2(A, M)$ classifies all abelian extensions of A by M in the category of rings, while $H^2(A, M)$ classifies all abelian extensions of A by M in the category of $\mathbb{F}_p$ -algebras. It follows that ζ^2 is a monomorphism. Our next aim is to get more information on ζ^2 and ζ^3 . One defines the homomorphisms

$$\zeta^2 : \mathrm{SH}^2(A/\mathbb{Z}, M) \rightarrow H^0(A, M)$$

as follows. Let

$$0 \rightarrow M \xrightarrow{\mu} E \rightarrow A \rightarrow 0$$

be an abelian extension. It represents an element $(E) \in \mathrm{SH}^2(A/\mathbb{Z}, M)$. Since A is an algebra over $\mathbb{F}_p$ we have $\mu^{-1}(p1_E) \in M$, where 1_E denotes the unit of E . Now we put

$$\zeta^2((E)) := \mu^{-1}(p1_E) \in H^0(A, M).$$

It is clear that $\zeta^2((E)) = 0$ if and only if E is an algebra over $\mathbb{F}_p$. Thus we proved the following easy:

7.1.1. Lemma. *For any associative $\mathbb{F}_p$ -algebra A and any bimodule M over A one has the exact sequence*

$$0 \rightarrow H^2(A, M) \xrightarrow{\zeta^2} \mathrm{SH}^2(A/\mathbb{Z}, M) \xrightarrow{\zeta^2} H^0(A, M).$$

In order to extend this exact sequence we need the homomorphism

$$\zeta^3 : \mathrm{SH}^3(A/\mathbb{Z}, M) \rightarrow H^1(A, M)$$

and the class $\sigma'(A)$. The definition of $\sigma'(A)$ is given in Section 7.3, while ζ^3 is defined as follows. Let

$$0 \rightarrow M \xrightarrow{\mu} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\eta} A \rightarrow 0$$

be a crossed extension in the category of rings. Since A is an algebra over $\mathbb{F}_p$ there exists an element $[P] \in C_1$ such that $\partial([P]) = p1_{C_0} \in C_0$. We let $D^\partial : A \rightarrow M$ be the derivation given by

$$D^\partial(a) = b[P] - [P]b \in M, \quad \eta(b) = a, \quad b \in C_0.$$

This definition is independent of the choice of b . Indeed if one replaces b by $b + \partial(c)$, $c \in C_1$, then

$$(b + \partial(c))[P] - [P](b + \partial(c)) - b[P] + [P]b = \partial(c)[P] - [P]\partial(c) = c\partial([P]) - \partial([P])c = 0$$

because $\partial([P]) = p1_{C_0}$. One easily sees that the class of the derivation D^∂ in $H^1(A, M)$ depends only on the class of the crossed extension (∂) in $\mathrm{SH}^3(A/\mathbb{Z}, M)$ and therefore defines the map $\zeta^3 : \mathrm{SH}^3(A/\mathbb{Z}, M) \rightarrow H^1(A, M)$.

7.2. Weak lifting. We will need the following version of lifting of $\mathbb{F}_p$ -algebras to $\mathbb{Z}/p^2\mathbb{Z}$.

7.2.1. Definition. Let A be an $\mathbb{F}_p$ -algebra. We will say that a $\mathbb{Z}/p^2\mathbb{Z}$ -algebra A_0 is a *weak lifting* of A if there exists an abelian extension of $\mathbb{Z}/p^2\mathbb{Z}$ -algebras

$$(8) \quad 0 \rightarrow A \xrightarrow{\mu} A_0 \xrightarrow{\sigma} A \rightarrow 0$$

such that $\mu(1) = p1_{A_0}$, where 1_{A_0} is the unit element of A_0 . One requires that the induced A - A -bimodule structure on A is the usual one. Thus we have the following identities:

$$\mu(a)\mu(b) = 0, \quad \mu(ab) = \mu(a)b_0 = a_0\mu(b)$$

where $a, b \in A$, $a_0, b_0 \in A_0$ and $\sigma(a_0) = a$, $\sigma(b_0) = 0$. So, one has $\mu(a) = pa_0$. If such extension exists then we say that A has *weak lifting* to $\mathbb{Z}/p^2\mathbb{Z}$.

7.2.2. Lemma. *An associative $\mathbb{F}_p$ -algebra has a weak lifting if and only if there exists an element $x \in \mathrm{SH}^2(A/\mathbb{Z}, A)$ such that $\zeta^2(x) = 1_A \in H^0(A, A)$, where 1_A is the unit element of A .*

Proof. If A_0 is a weak lifting, then the exact sequence (8) defines an element in $\mathrm{SH}^2(A/\mathbb{Z}, A)$ with expected properties and conversely, assume for $x \in \mathrm{SH}^2(A/\mathbb{Z}, A)$ the equality $\zeta^2(x) = 1_A$ holds. We take any abelian extension which represents x , then it is of the form (8) and A_0 is a weak lifting. $\square$

7.2.3. Lemma. *Any lift of A to $\mathbb{Z}/p^2\mathbb{Z}$ is also a weak lifting.*

Proof. Assume A_0 is a $\mathbb{Z}/p^2\mathbb{Z}$ -lifting of A . Consider the canonical exact sequence $0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{F}_p \rightarrow 0$. Since A_0 is free as $\mathbb{Z}/p^2\mathbb{Z}$ -module, by tensoring on A_0 we obtain the exact sequence

$$0 \rightarrow A \xrightarrow{\mu} A_0 \xrightarrow{\sigma} A \rightarrow 0,$$

because $A_0 \otimes \mathbb{F}_p = A$. It is clear that $\mu(a) = pa_0$, where $a \in A$ and $\sigma(a_0) = a$, where σ is the quotient map $A_0 \rightarrow A_0/pA_0 = A$ and all needed properties readily follow. $\square$

7.2.4. Lemma. *Any commutative algebra has a weak lifting.*

Proof. Actually there exist functorial weak lifting due to Witt. Let A be a commutative $\mathbb{F}_p$ -algebra. Consider the ring $W_2(A)$ of Witt vectors of length two. Let us recall that $W_2(A)$ as a set is $A \times A$, with the following operations:

$$\begin{aligned} (x, a) + (y, b) &= (x + y + S(a, b), a + b), \\ (x, a)(y, b) &= (xb^p + a^py, ab). \end{aligned}$$

Here $S(a, b)$ is the polynomial over $\mathbb{Z}$, satisfying

$$pS(a, b) = a^p + b^p - (a + b)^p.$$

For example, $S(a, b) = -ab$ if $p = 2$. One has the following well known exact sequence

$$0 \rightarrow A \xrightarrow{\mu} W_2(A) \xrightarrow{\sigma} A \rightarrow 0,$$

where $\mu(x) = (x, 0)$ and $\sigma(x, a) = a$. The unit of $W_2(A)$ is the element $(0, 1)$. Since $p(0, 1) = (1, 0)$ the result follows. $\square$

7.3. The invariant o' . In this section for any associative $\mathbb{F}_p$ -algebra A we construct a canonical element $o'(A) \in H^3(A, A)$. This class differs from the class $o(A)$ constructed in Section 2.3.

We choose a free $\mathbb{Z}/p^2\mathbb{Z}$ -module $\tilde{A}$ such that $\tilde{A} \otimes \mathbb{F}_p = A$. We have a canonical algebra homomorphism $\varepsilon : T(A) \rightarrow A$ from the tensor algebra over A , which is identity on A . We also consider the tensor algebra $T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})$ over $\mathbb{Z}/p^2\mathbb{Z}$ generated by $\tilde{A}$. Then we have an abelian extension of algebras:

$$0 \rightarrow T(A) \xrightarrow{\alpha} T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) \xrightarrow{\beta} T(A) \rightarrow 0.$$

Consider the push-forward construction:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(A) & \xrightarrow{\alpha} & T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) & \xrightarrow{\beta} & T(A) & \longrightarrow & 0 \\ & & \varepsilon \downarrow & & \downarrow \eta & & \downarrow \text{id} & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu} & B(A) & \xrightarrow{\nu} & T(A) & \longrightarrow & 0. \end{array}$$

We put $C(A) := \ker(\varepsilon \circ \nu : B(A) \rightarrow A)$. Then one has an exact sequence

$$(9) \quad 0 \rightarrow A \xrightarrow{\mu'} C(A) \xrightarrow{\nu'} T(A) \xrightarrow{\varepsilon} A \rightarrow 0,$$

where μ' and ν' are induced by μ and ν . We claim that the exact sequence (9) is a crossed extension in the category of $\mathbb{F}_p$ -algebras and therefore defines an element $o'(A) \in H^3(A, A)$. Since, obviously $C(A)$ is a module over $T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})$ it suffices to show that $pC = 0$. Thus for any $b \in B(A)$ with $\varepsilon\nu(b) = 0$ we have to prove that $pb = 0$. Since ε is surjective, there exists $x \in T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})$ such that $b = \eta(x)$. Since $\alpha(1) = p$ and $\alpha(T(A))$ is a square zero ideal in $T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})$ we have

$$pb = (\eta\alpha(1))b = (\eta\alpha(1))(\eta(x)) = \eta(\alpha(1)x) = \eta\alpha\beta(x) = \mu\varepsilon\nu\eta(x) = \mu\varepsilon\nu(b) = 0.$$

Hence (9) is really a crossed extension in the category of $\mathbb{F}_p$ -algebras.

We claim that although $A \mapsto B(A)$ and $A \mapsto C(A)$ do not extend as functors, the class $o'(A) \in H^3(A, A)$ is natural, in the sense that for any algebra homomorphism $f : A \rightarrow A_1$ one has the equality $f_*(o'(A)) = f^*(o'(A_1))$ in $H^3(A, A_1)$. To prove the claim we choose a lifting $\tilde{f} : \tilde{A} \rightarrow \tilde{A}_1$ such that $\tilde{f} \otimes \mathbb{F}_p = f$. Of course $\tilde{f}$ is only a $\mathbb{Z}/p^2\mathbb{Z}$ -module homomorphism. However, it still induces the homomorphism

$$T(\tilde{f}) : T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) \rightarrow T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}_1)$$

of algebras, such that the diagram

$$\begin{array}{ccccccc}
 & & & A & & & \\
 & & \nearrow \varepsilon & \downarrow f & & & \\
 0 & \longrightarrow & T(A) & \xrightarrow{\alpha} & T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) & \xrightarrow{\beta} & T(A) \longrightarrow 0 \\
 & & \downarrow T(f) & & \downarrow T(\tilde{f}) & & \downarrow T(f) \\
 & & A_1 & & & & \\
 & \nearrow \varepsilon_1 & & & & & \\
 0 & \longrightarrow & T(A_1) & \xrightarrow{\alpha_1} & T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}_1) & \xrightarrow{\beta_1} & T(A) \longrightarrow 0
 \end{array}$$

commutes. Apply the push-forward construction to get the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{\mu} & B(A) & \xrightarrow{v} & T(A) \longrightarrow 0 \\
 & & \downarrow f & & \downarrow B(\tilde{f}) & & \downarrow T(f) \\
 0 & \longrightarrow & A_1 & \xrightarrow{\mu_1} & B(A_1) & \xrightarrow{v_1} & T(A) \longrightarrow 0
 \end{array}$$

which yields the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{\mu'} & C(A) & \xrightarrow{v'} & T(A) \xrightarrow{\varepsilon} A \longrightarrow 0 \\
 & & \downarrow f & & \downarrow c(\tilde{f}) & & \downarrow T(f) \\
 0 & \longrightarrow & A_1 & \xrightarrow{\mu'_1} & C(A_1) & \xrightarrow{v'_1} & T(A_1) \xrightarrow{\varepsilon_1} A_1 \longrightarrow 0
 \end{array}$$

and the claim follows.

7.3.1. Theorem. For an associative $\mathbb{F}_p$ -algebra A the class $o'(A)$ vanishes if and only if A has a weak lifting to $\mathbb{Z}/p^2\mathbb{Z}$.

Proof. By Lemma 7.2.2, A has a weak lifting if and only if there exists an element $x \in \text{SH}^3(A/\mathbb{Z}, A)$ with $\zeta^2(x) = 1_A$. Now we put $M = A$ in Theorem 7.4.1 below, which shows that this happens if and only if $o'(A) = \gamma(1_A) = 0$. $\square$

7.3.2. Corollary. If A is a commutative $\mathbb{F}_p$ -algebra, then $o'(A) = 0$.

Proof. This follows from Lemma 7.2.4. $\square$

7.4. The fundamental exact sequence. Using the invariant o' we construct the homomorphism

$$\gamma^i : H^i(A, M) \rightarrow H^{i+3}(A, M)$$

by $\gamma^i(x) = x \cup o'(A)$. Of the special interest is the case $i = 0$. In this case we put $\gamma = \gamma^0 : H^0(A, M) \rightarrow H^3(A, M)$. Then we have $\gamma(x) = x_*(o'(A))$, where $x_* : A \rightarrow M$ is the unique homomorphism of A - A -bimodules such that $x_*(1) = x \in M$. It is also clear, that for $1_A \in H^0(A, A)$ one has $o'(A) = \gamma(1_A)$.

Now we are in position to prove the main result of this section, which should be compared with [3], Proposition 4.6.3, where the equality $\zeta^3 \xi^3 = 0$ is proved in a slightly different framework.

7.4.1. Theorem. *For any associative $\mathbb{F}_p$ -algebra A and any bimodule M over A one has the exact sequence*

$$0 \rightarrow H^2(A, M) \xrightarrow{\xi^2} SH^2(A/\mathbb{Z}, M) \xrightarrow{\xi^2} H^0(A, M) \xrightarrow{\gamma} H^3(A, M) \\ \xrightarrow{\xi^3} SH^3(A/\mathbb{Z}, M) \xrightarrow{\xi^3} H^1(A, M).$$

Proof. The beginning part was already considered in Lemma 7.1.1. The rest consists in several steps.

First step. $\gamma \circ \zeta^2 = 0$. Take any abelian extension of rings

$$0 \rightarrow M \xrightarrow{\mu} E \xrightarrow{\sigma} A \rightarrow 0.$$

It represents an element in $SH^2(A/\mathbb{Z}, M)$. We let $\eta: A \rightarrow M$ be the unique morphism of bimodules over A such that $\mu\eta(1_A) = p1_E$, where as usual 1_E and 1_A denotes the unit of E and A respectively. Since $T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})$ is a free associative $\mathbb{Z}/p^2\mathbb{Z}$ -algebra, the composite $T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) \xrightarrow{\beta} T(A) \xrightarrow{\epsilon} A$ has a lifting as an $\mathbb{Z}/p^2\mathbb{Z}$ -algebra homomorphism $\epsilon: T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) \rightarrow E$. We have $\epsilon\alpha(1_{TA}) = \epsilon(p1_{T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})}) = p1_E = \eta(1_A)$. It follows that the following is a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(A) & \xrightarrow{\alpha} & T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) & \xrightarrow{\beta} & T(A) \longrightarrow 0 \\ & & \downarrow \eta\epsilon & & \downarrow \epsilon & & \downarrow \epsilon \\ 0 & \longrightarrow & M & \xrightarrow{\mu} & E & \xrightarrow{\sigma} & A \longrightarrow 0. \end{array}$$

Using the push-forward construction the previous commutative diagram yields the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B(A) & \xrightarrow{\nu} & T(A) \longrightarrow 0 \\ & & \downarrow \eta & & \downarrow \epsilon' & & \downarrow \epsilon \\ 0 & \longrightarrow & M & \xrightarrow{\mu} & E & \xrightarrow{\sigma} & A \longrightarrow 0. \end{array}$$

If $\epsilon(\nu(b)) = 0$, then $\epsilon'(b) \in \text{Im}(\mu)$. Therefore there exists a unique $\rho: C(A) \rightarrow M$ with $\mu\rho = \epsilon'|_{C(A)}$. For any $c \in C(A)$ and $b \in B(A)$ we have

$$\mu\rho(cb) = \epsilon'(C)\epsilon'(b) = \mu\rho(c)\epsilon'(b) = \mu(\rho(c)\sigma\epsilon(b)) = \mu(\rho(c)\epsilon\nu(b)).$$

Let us recall that the restriction of ν on $C(A)$ yields a crossed bimodule $\nu': C(A) \rightarrow T(A)$, where the action of $x \in T(A)$ on $c \in C(A)$ is given by the multiplication of b and c in $B(A)$, where b is any lift of x , i.e. $\nu(b) = x$. Therefore the previous computation shows that for

any $c \in C(A)$ and $x \in T(A)$ one has $\rho(cx) = \rho(c)\varepsilon(x)$ and similarly $\rho(xc) = \varepsilon(x)\rho(c)$. This shows that one has a morphism of crossed extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu'} & C(A) & \xrightarrow{v'} & T(A) & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & & \downarrow \eta & & \downarrow \rho & & \downarrow \varepsilon & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{\text{id}} & M & \xrightarrow{0} & A & \xrightarrow{\text{id}} & A & \longrightarrow & 0 \end{array}$$

which shows that $\gamma \circ \zeta^2 = 0$.

Second step. Exactness on $H^0(A, M)$. Take any element $m \in H^0(A, M)$ such that $\gamma(m) = 0$. By Corollary 4.4.3 there is a v'' -coextension of $T(A)$ by M :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu'} & C(A) & \xrightarrow{v'} & T(A) & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & & \downarrow \eta & & \downarrow \theta' & & \downarrow \text{id} & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{\mu''} & D & \xrightarrow{v''} & T(A) & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \theta'' & & \downarrow \text{id} & & & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & F & \longrightarrow & T(A) & \longrightarrow & 0. \end{array}$$

We set $\theta = \theta'' \circ \theta' : C(A) \rightarrow F$. Since any abelian extension of the tensor algebra splits, it follows that there is a derivation $d : F \rightarrow M$ such that $d \circ i = \text{id}_M$. We let ρ be the composite $d \circ \theta : C(A) \rightarrow M$. One easily shows that for any $c \in C(A)$ and $x \in T(A)$ one has $\rho(cx) = \rho(c)\varepsilon(x)$ and $\rho(xc) = \varepsilon(x)\rho(c)$. Consider the split extension $M \rtimes B(M)$, where $B(M)$ acts on M via $\varepsilon \circ v$. Then $\{(\rho(c), -c) \mid c \in C\}$ is an ideal of $M \rtimes B(M)$. This allows us to make the "push-forward" construction

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(A) & \longrightarrow & B(A) & \xrightarrow{\varepsilon \circ v} & A & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

and easy computations show that the lower exact sequence is an abelian extension with $\zeta^2((E)) = m$.

Third step. $\xi^3\gamma = 0$. The construction of the crossed extension (9) shows that one has the following v' -coextension in the category of $\mathbb{Z}/p^2\mathbb{Z}$ -algebras:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu'} & C(A) & \xrightarrow{v'} & T(A) & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu} & B(A) & \xrightarrow{v} & T(A) & \longrightarrow & 0, \end{array}$$

and the result follows from Corollary 4.4.3.

Fourth step. Exactness at $H^3(A, M)$. Assume

$$0 \rightarrow M \xrightarrow{\mu} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\eta} A \rightarrow 0$$

is a crossed extension in the category of $\mathbb{F}_p$ -algebras which defines the trivial class in $SH^3(A/\mathbb{Z}, M)$. According to Corollary 4.4.3 there exists a ∂ -coextension in the category of $\mathbb{Z}/p^2\mathbb{Z}$ -algebras

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & M & \longrightarrow & C_1 & \xrightarrow{\partial} & C_0 \xrightarrow{\pi} A \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \sigma & & \downarrow \text{id} \\ 0 & \longrightarrow & M & \xrightarrow{\iota} & S & \xrightarrow{\kappa} & C_0 \longrightarrow 0 \\ & & & & \downarrow \pi\kappa & & \\ & & & & A & & \\ & & & & \downarrow & & \\ & & & & 0. & & \end{array}$$

Clearly $a = \iota^{-1}(p1_S) \in H^0(A, M)$. We have to show that $\gamma(a) \in H^3(A, M)$ coincides with the class represented by the cross extension (∂) . Since $T(A)$ is a free $\mathbb{F}_p$ -algebra, there is an algebra homomorphism $\psi : T(A) \rightarrow C_0$, such that $\pi\psi = \varepsilon$. Since $T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A})$ is a free $\mathbb{Z}/p^2\mathbb{Z}$ -algebra, there is a $\mathbb{Z}/p^2\mathbb{Z}$ -algebra homomorphism $\varphi : T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) \rightarrow S$, such that $\kappa\varphi = \psi\beta$, where $\beta : T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) \rightarrow T(A)$ is the canonical surjective algebra homomorphism. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(A) & \xrightarrow{\alpha} & T_{\mathbb{Z}/p^2\mathbb{Z}}(\tilde{A}) & \xrightarrow{\beta} & T(A) \longrightarrow 0 \\ & & \downarrow a, \varepsilon & & \downarrow \varphi & & \downarrow \psi \\ 0 & \longrightarrow & M & \xrightarrow{\iota} & S & \xrightarrow{\kappa} & C_0 \longrightarrow 0 \end{array}$$

where $a_* : A \rightarrow M$ is the unique bimodule homomorphism with $a_*(1) = a$. This diagram commutes because $\iota(a) = p1_S$. It follows from the construction $B(A)$ that one has the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B(A) & \xrightarrow{\nu} & T(A) \longrightarrow 0 \\ & & \downarrow a_* & & \downarrow \phi & & \downarrow \psi \\ 0 & \longrightarrow & M & \xrightarrow{\iota} & S & \xrightarrow{\kappa} & C_0 \longrightarrow 0. \end{array}$$

Take any element $c \in C(A) = \ker(B(A) \xrightarrow{\varepsilon\nu} A)$. Then $\pi\kappa(\phi(c)) = \pi\psi\nu(a) = \varepsilon\nu(a) = 0$. Thus ϕ yields the well defined homomorphism $\phi' : C(A) \rightarrow C_1$. It is immediate that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\mu'} & C(A) & \xrightarrow{\nu'} & T(A) & \xrightarrow{\varepsilon} & A \longrightarrow 0 \\
& & \downarrow a. & & \downarrow \phi' & & \downarrow \psi & & \downarrow \text{id} \\
0 & \longrightarrow & M & \xrightarrow{\mu} & C_1 & \xrightarrow{\nu} & C_0 & \longrightarrow & A \longrightarrow 0
\end{array}$$

commutes and the result follows.

Fifth step. $\zeta^3 \circ \xi^3 = 0$. Indeed, if

$$0 \rightarrow M \xrightarrow{\mu} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\eta} A \rightarrow 0$$

is a crossed extension in the category of $\mathbb{F}_p$ -algebras, then one can take $[P] = 0$ in the definition of ζ^3 given in Section 7.1 and therefore $D^\partial = 0$, which implies the result.

Sixth step. *Exactness at $\text{SH}^3(A/\mathbb{Z}, M)$.* Assume

$$0 \rightarrow M \xrightarrow{\mu} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\eta} A \rightarrow 0$$

is a crossed extension in the category of rings such that $\zeta^3((\partial)) = 0 \in H^1(A, M)$. We have to prove that the crossed extension (∂) is equivalent to a crossed extension of $\mathbb{F}_p$ -algebras. Without loss of generality one can also assume that C_0 is torsion free as an abelian group. By assumption $D^\partial(a) = am - ma$. Now replacing $[P]$ by $[P] - \mu(m)$ in the definition of ζ^3 given in Section 7.1 one can assume that $b[P] = [P]b$ holds for all $b \in C_0$. Thus the left C_0 -submodule C' of C_1 generated by $[P]$ is actually a sub-bimodule. We have $C' \cap M = 0$, indeed, if $b[P] = \mu(m)$, then $pb = 0$ and hence $b = 0$ by our assumption. It follows that $C' \cong \partial(C') \subset C_0$. We set $C'_1 := C_1/C'$ and $C'_0 = C_0/\partial(C')$. Then we have a morphism of crossed extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & A \longrightarrow 0 \\
& & \downarrow \text{Id} & & \downarrow & & \downarrow & & \downarrow \text{Id} \\
0 & \longrightarrow & M & \longrightarrow & C'_1 & \longrightarrow & C'_0 & \longrightarrow & A \longrightarrow 0.
\end{array}$$

By our construction $p \in \partial(C')$ therefore C'_0 is an $\mathbb{F}_p$ -algebra and the result follows. $\square$

7.4.2. Corollary. *For any commutative $\mathbb{F}_p$ -algebra A and any A -module M one has the exact sequences*

$$0 \rightarrow H^2(A, M) \xrightarrow{\xi^2} \text{SH}^2(A/\mathbb{Z}, M) \xrightarrow{\xi^2} H^0(A, M) \rightarrow 0$$

and

$$0 \rightarrow H^3(A, M) \xrightarrow{\xi^3} \text{SH}^3(A/\mathbb{Z}, M) \xrightarrow{\xi^3} H^1(A, M) \rightarrow 0.$$

Proof. This is clear, because for commutative algebras we have $\sigma'(A) = 0$ (see Corollary 7.3.2) and therefore $\gamma = 0$. $\square$

8. A bicomplex computing Shukla cohomology

8.1. Construction of a bicomplex. In this section following [29] we construct a canonical bicomplex which computes the Shukla cohomology in the special case, when the ground ring is an algebra over a field.

In this section we assume that k is a field, and K is a commutative k -algebra. Let R be a K -algebra and let M be a bimodule over R , which is symmetric as a bimodule over K , that is $km = mk$, for all $k \in K$. We are going to construct a bicomplex which computes the Shukla cohomology $\mathrm{SH}^*(R/K, M)$.

We let $C^*(R, M)$ be the Hochschild cochain complex of R considered as an algebra over k . Similarly, we let $C^*(R/K, M)$ be the Hochschild cochain complex of R considered as an algebra over K . Accordingly $H^*(R, M)$ and $H^*(R/K, M)$ denotes the Hochschild cohomology of R with coefficients in M over k and K respectively.

We let $K^{**}(K, R, M)$ be the following bicosimplicial vector space:

$$K^{pq}(K, R, M) = \mathrm{Hom}(K^{\otimes pq} \otimes R^{\otimes q}, M).$$

The q -th horizontal cosimplicial vector space structure comes from the identification

$$K^{*q}(K, R, M) = C^*(K^{\otimes q}, C^q(R, M)),$$

where $C^q(R, M) = \mathrm{Hom}(R^{\otimes q}, M)$ is considered as a bimodule over $K^{\otimes q}$ via

$$((a_1, \dots, a_q)f(b_1, \dots, b_q))(r_1, \dots, r_q) := a_1 \cdots a_q f(b_1 r_1, \dots, b_q r_q).$$

Here $f \in \mathrm{Hom}(R^q, M)$ and $a_i, b_j \in K$, $r_k \in R$. The p -th vertical cosimplicial vector space structure comes from the identification

$$K^{p*}(K, R, M) = C^*(K^{\otimes p} \otimes R, M)$$

where M is considered as a bimodule over $K^{\otimes p} \otimes R$ via

$$(a_1 \otimes \cdots \otimes a_p \otimes r)m(b_1 \otimes \cdots \otimes b_p \otimes s) := (a_1 \cdots a_p r)m(b_1 \cdots b_p s).$$

We allow ourselves to denote the corresponding bicomplex by $K^{**}(K, R, M)$ as well. Thus $K^{**}(K, R, M)$ looks as follows:

$$\begin{array}{ccccccc} M & \xrightarrow{0} & M & \xrightarrow{\mathrm{Id}} & M & \xrightarrow{0} & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \mathrm{Hom}(R, M) & \xrightarrow{d} & \mathrm{Hom}(K \otimes R, M) & \xrightarrow{d} & \mathrm{Hom}(K \otimes K \otimes R, M) & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \mathrm{Hom}(R \otimes R, M) & \xrightarrow{d} & \mathrm{Hom}(K^{\otimes 2} \otimes R^{\otimes 2}, M) & \xrightarrow{d} & \mathrm{Hom}(K^{\otimes 4} \otimes R^{\otimes 2}, M) & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \vdots & & \vdots & & \vdots & & \end{array}$$

Therefore for $f : K^{\otimes pq} \otimes R^{\otimes q} \rightarrow M$ the corresponding linear maps

$$d(f) : K^{\otimes (p+1)q} \otimes R^{\otimes q} \rightarrow M \quad \text{and} \quad \delta(f) : K^{\otimes p(q+1)} \otimes R^{\otimes (q+1)} \rightarrow M$$

are given by

$$\begin{aligned} df(a_{01}, \dots, a_{0q}, a_{11}, \dots, a_{1q}, \dots, a_{p1}, \dots, a_{pq}, r_1, \dots, r_q) \\ = a_{01} \cdots a_{0q} f(a_{11}, \dots, a_{1q}, \dots, a_{p1}, \dots, a_{pq}, r_1, \dots, r_q) \\ + \sum_{0 \leq i < p} (-1)^{i+1} f(a_{01}, \dots, a_{0q}, \dots, a_{i1} a_{i+1,1}, \dots, a_{iq} a_{i+1,q}, \dots, a_{p1}, \dots, a_{pq}, r_1, \dots, r_q) \\ + (-1)^{p+1} f(a_{01}, \dots, a_{0q}, \dots, a_{p-1,1}, \dots, a_{p-1,q}, a_{p1} r_1, \dots, a_{pq} r_q) \end{aligned}$$

and

$$\begin{aligned} \delta(f)(a_{10}, \dots, a_{1q}, \dots, a_{p0}, \dots, a_{pq}, r_0, \dots, r_q) \\ = (-1)^p a_{10} \cdots a_{p0} r_0 f(a_{11}, \dots, a_{1q}, a_{p1}, \dots, a_{pq}, r_1, \dots, r_q) \\ + \sum_{0 \leq i < q} (-1)^{i+p+1} f(a_{10}, \dots, a_{1i} a_{1,i+1}, \dots, a_{pi} a_{p,i+1}, \dots, a_{pq}, r_0, \dots, r_i r_{i+1}, \dots, r_q) \\ + (-1)^{q+p+1} f(a_{10}, \dots, a_{1q-1}, \dots, a_{p0}, \dots, a_{p,q-1}, r_0, \dots, r_{q-1}) a_{1q} \cdots a_{pq} r_q. \end{aligned}$$

We let $H^*(K, R, M)$ be the homology of the bicomplex $K^{**}(K, R, M)$. We also consider the following subbicomplex $\bar{K}^{**}(K, R, M)$ of $K^{**}(K, R, M)$:

$$\begin{array}{ccccccc} M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow & & \downarrow & & \\ \text{Hom}(R, M) & \xrightarrow{d} & \text{Hom}(K \otimes R, M) & \xrightarrow{d} & \text{Hom}(K \otimes K \otimes R, M) & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \text{Hom}(R \otimes R, M) & \xrightarrow{d} & \text{Hom}(K^{\otimes 2} \otimes R^{\otimes 2}, M) & \xrightarrow{d} & \text{Hom}(K^{\otimes 4} \otimes R^{\otimes 2}, M) & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \vdots & & \vdots & & \vdots & & \end{array}$$

It is clear that $H^*(K, R, M) \cong H^*(\bar{K}^{**}(K, R, M))$.

8.2. The homomorphism α . It follows from the definition that

$$\text{Ker}(d : K^{*0} \rightarrow K^{*1}) \cong C^*(R/K, M).$$

Therefore one has the canonical homomorphism

$$\alpha^n : H^n(R/K, M) \rightarrow H^n(K, R, M), \quad n \geq 0.$$

8.2.1. Theorem. (i) *The homomorphisms α^0 and α^1 are isomorphisms. The homomorphism α^2 is a monomorphism.*

(ii) *If R is projective over K , then*

$$\alpha^n : H^n(R/K, M) \rightarrow H^n(K, R, M)$$

is an isomorphism for all $n \geq 0$.

(iii) *The groups $H^*(K, R, M)$ are canonically isomorphic to $SH^*(R/K, M)$.*

Proof. (i) is an immediate consequence of the definition of the bicomplex $\bar{K}^{**}(K, R, M)$.

(ii) The bicomplex gives rise to the following spectral sequence:

$$E_1^{pq} = H^q(K^{\otimes p}, C^p(R, M)) \Rightarrow H^{p+q}(K, R, M).$$

Let us recall that if X and Y are left modules over an associative algebra S , then $\text{Ext}_S^*(X, Y) \cong H^*(S, \text{Hom}(X, Y))$ ([8]), where $\text{Hom}(X, Y)$ is considered as a bimodule over S via $(sft)(x) = sf(tx)$. Here $x \in X$, $s, t \in S$ and $f : X \rightarrow Y$ is a linear map. Having this isomorphism in mind, we can rewrite $E_1^{pq} \cong \text{Ext}_{K^{\otimes p}}^q(R^{\otimes p}, M)$. By our assumptions $R^{\otimes p}$ is projective over $K^{\otimes p}$. Therefore the spectral sequence degenerates and we get $H^*(K, R, M) \cong H^*(C(R/K, M)) = H^*(R/K, M)$. Here we used the obvious isomorphism

$$\text{Hom}_{K \otimes K \otimes \dots \otimes K}(R \otimes R \otimes \dots \otimes R, M) = \text{Hom}_K(R \otimes_K R \otimes_K \dots \otimes_K R, M).$$

(iii) We let $\bar{K}^*(K, R, M)$ denote the total cochain complex associated to the bicomplex $\bar{K}^{**}(K, R, M)$. Then this construction has an obvious extension to the category of chain K -algebras. Unlike Lemma 4.1.1, for any weak equivalence $R_* \rightarrow S_*$ of chain K -algebras the induced map $\bar{K}^*(K, S_*, M) \rightarrow \bar{K}^*(K, R_*, M)$ is a weak equivalence. This is because the definition of $\bar{K}^*(K, R, M)$ involves the tensor products and hom's over the field k and not over K . Furthermore, by (ii) $H^*(K, R_*, M)$ is isomorphic to the Hochschild cohomology, provided R_* is degreewise projective over K . In particular this happens when R_* is cofibrant. Now we take any K -algebra R and a cofibrant replacement R_*^c of R . Then one has

$$H^*(K, R, M) \cong H^*(K, R_*^c, M) \cong H^*(R_*^c/K, M) \cong SH^*(R/K, M).$$

8.2.2. Corollary. (i) *There is a natural bijection*

$$\text{Extalg}(K, R, M) \cong H^2(K, R, M).$$

(ii) *There is a natural bijection*

$$\text{Xext}(K, R, M) \cong H^3(K, R, M).$$

Our next aim is to describe directly the cocycles of $H^*(K, R, M)$ corresponding to abelian and crossed extensions.

We have

$$H^2(K, R, M) = Z^2(K, R, M)/B^2(K, R, M)$$

where $Z^2(K, R, M)$ consists of pairs (f, g) such that $f : R \otimes R \rightarrow M$ and $g : K \otimes R \rightarrow M$ are linear maps and the equalities

$$\begin{aligned} ag(b, r) - g(ab, r) + g(a, br) &= 0, \\ abf(r, s) - f(ar, bs) &= arg(b, s) - g(ab, rs) + g(a, r)bs, \\ rf(s, t) - f(rs, t) + f(r, st) - f(r, s)t &= 0 \end{aligned}$$

hold. Here $a, b \in K$ and $r, s, t \in R$. Moreover, (f, g) belongs to $B^2(K, R, M)$ if and only if there exists a linear map $h : R \rightarrow M$ such that $f(r, s) = rh(s) - h(rs) + h(r)s$ and $g(a, r) = ah(r) - h(ar)$. Starting with $(f, g) \in Z^2(K, R, M)$ we construct an abelian extension of R by M by putting $S = M \oplus R$ as a vector space. A K -module structure on S is given by $a(m, r) = (am + g(a, r), ar)$, while the multiplication on S is given by $(m, r)(n, s) = (ms + rn + f(r, s), rs)$. Conversely, given an abelian extension

$$0 \rightarrow M \rightarrow S \rightarrow R \rightarrow 0$$

we choose a k -linear section $h : R \rightarrow S$ and then we put $f(r, s) := h(r)h(s) - h(rs)$ and $g(a, r) := ah(r) - h(ar)$. One easily checks that $(f, g) \in Z^2(K, R, M)$ and one gets (i). Similarly, we have $H^3(K, R, M) = Z^3(K, R, M)/B^3(K, R, M)$. Here $Z^3(K, R, M)$ consists of triples (f, g, h) such that $f : R \otimes R \otimes R \rightarrow M$, $g : K \otimes K \otimes R \rightarrow M$ and $h : K \otimes K \otimes R \rightarrow M$ are linear maps and the following relations hold:

$$\begin{aligned} r_1 f(r_2, r_3, r_4) - f(r_1 r_2, r_3, r_4) + f(r_1, r_2 r_3, r_4) - f(r_1, r_2, r_3 r_4) + f(r_1, r_2, r_3) r_4 &= 0, \\ abcf(r, s, t) - f(ar, bs, ct) &= arg(b, c, y, z) - g(ab, c, xy, z) + g(a, bc, x, yz) - g(a, b, x, y)cz, \\ abg(c, d, x, y) - g(ac, bd, x, y) + g(a, b, cx, dy) \\ &= acxh(b, d, y) - h(ab, cd, xy) + h(a, c, x)bdy, \\ ah(b, c, x) - h(ab, c, x) + h(a, bc, x) - h(a, b, cx) &= 0. \end{aligned}$$

Moreover, (f, g, h) belongs to $B^3(K, R, M)$ if and only if there exist linear maps $m : R \otimes R \rightarrow M$ and $n : K \otimes R \rightarrow M$ such that

$$\begin{aligned} f(r, s, t) &= rm(s, t) - m(rs, t) + m(r, st) - m(r, s)t, \\ g(a, b, r, s) &= abm(r, s) - m(ar, bs) - arn(b, s) + n(ab, rs) - n(a, x)bs, \\ h(a, b, r) &= an(b, r) - n(ab, r) + n(a, br). \end{aligned}$$

Let

$$0 \rightarrow M \rightarrow C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} R \rightarrow 0$$

be a crossed extension. We put $V := \text{Im}(\partial)$ and consider k -linear sections $p : R \rightarrow C_0$ and $q : V \rightarrow C_1$ of $\pi : C_0 \rightarrow R$ and $\partial : C_1 \rightarrow V$ respectively. Now we define

$$m : R \otimes R \rightarrow V \quad \text{and} \quad n : A \otimes R \rightarrow V$$

by $m(r, s) := q(p(r)p(s) - p(rs))$ and $n(a, r) := q(ap(r) - p(ar))$. Finally we define $f : R \otimes R \otimes R \rightarrow M$, $g : K^{\otimes 3} \otimes R \otimes R \rightarrow M$ and $h : K \otimes K \otimes R \rightarrow M$ by

$$f(r, s, t) := p(r)m(s, t) - m(rs, t) + m(r, st) - m(r, s)p(t),$$

$$g(a, b, r, s) := p(as)n(b, s) - n(ab, rs) + bn(a, x)p(y) - abm(r, s) + m(ax, by),$$

$$h(a, b, r) := an(b, r) - n(ab, r) + n(a, bx).$$

Then $(f, g, h) \in Z^3(A, R, M)$ and the corresponding class in $H^3(A, R, M)$ depends only on the connected component of a given crossed extension. Thus we obtain a well-defined map $\text{Xext}(A, R, M) \rightarrow H^3(K, R, M)$ and a standard argument (see [4]) shows that it is an isomorphism.

9. Applications to MacLane cohomology

In this section we are working with rings. So our ground ring is the ring of integers $k = \mathbb{Z}$.

For any abelian group A , Eilenberg and MacLane constructed in [12] a chain complex $Q_*(A)$ whose homology is the stable homology of Eilenberg-MacLane spaces

$$H_q(Q_*(A)) \cong H_{n+q}(K(A, n)), \quad n > q.$$

Moreover for any abelian groups A and B there is a natural pairing

$$Q_*(A) \otimes Q_*(B) \rightarrow Q_*(A \otimes B)$$

(see for example [21], [17] or [18]). For any ring R , this pairing allows us to put a chain algebra structure on $Q_*(R)$.

By definition the MacLane cohomology $\text{HML}^*(R, M)$ is defined as the Hochschild cohomology of $Q_*(R)$ with coefficients in M . One can also introduce the dual objects—MacLane homology. It was proved in [28] that MacLane homology is isomorphic to the topological Hochschild homology of Bökstedt [5]. It is also isomorphic to the stable K -theory thanks to a result of Dundas and McCarthy [10]. We refer to [18], Chapter 13 for more information on MacLane cohomology.

9.1. Relation with Shukla cohomology in low dimensions. Since $H_0(Q_*(R)) \cong R$ we have a natural augmentation $\epsilon : Q_*(R) \rightarrow R$. Since $Q_*(R)$ is free as an abelian group the chain algebra

$$V_*(R) = (\cdots \rightarrow 0 \rightarrow \text{Ker}(\epsilon) \rightarrow Q_0(R))$$

is $\mathbb{Z}$ -free and $V_*(R) \rightarrow R$ is a weak equivalence. Hence $V_*(R)$ can be used to compute the Shukla cohomology. Therefore the morphism of chain algebras

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & Q_2(R) & \longrightarrow & Q_1(R) & \longrightarrow & Q_0(R) \longrightarrow R \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} \\
 & & 0 & \longrightarrow & \text{Ker}(\epsilon) & \longrightarrow & Q_0(R) \longrightarrow R \longrightarrow 0
 \end{array}$$

yields the natural transformation

$$(10) \quad \text{SH}^i(R/\mathbb{Z}, M) \rightarrow \text{HML}^i(R, M)$$

which is an isomorphism in dimensions 0, 1 and 2. Thus $\text{HML}^2(R, M)$ classifies singular extensions of R by M in the category of rings (see also [21]). According to [22], Theorem 9, in the dimension 3 one has the following exact sequence (see also part (i) of Theorem 9.2.1)

$$(11) \quad 0 \rightarrow \text{SH}^3(R/\mathbb{Z}; M) \rightarrow \text{HML}^3(R; M) \rightarrow H^0(R; {}_2M).$$

9.1.1. Proposition. *Let R be an algebra over $\mathbb{F}_p$ and M be an R -bimodule. Then the natural map*

$$\text{SH}^3(R/\mathbb{Z}, M) \rightarrow \text{HML}^3(R, M)$$

is an isomorphism.

Proof. If $p \neq 2$ then this is an immediate consequence of the exact sequence (11), because ${}_2M = 0$. So we have to consider only the case $p = 2$. For any $\mathbb{F}_2$ -algebra R we have the canonical homomorphism $\mathbb{F}_2 \rightarrow R$, which yields the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{SH}^3(R/\mathbb{Z}, M) & \longrightarrow & \text{HML}^3(R, M) & \longrightarrow & H^0(R, M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{SH}^3(\mathbb{F}_2/\mathbb{Z}, M) & \longrightarrow & \text{HML}^3(\mathbb{F}_2, M) & \longrightarrow & H^0(\mathbb{F}_2, M) = M.
 \end{array}$$

It is well known that $\text{HML}^3(\mathbb{F}_2, M) = 0$ see for example [13] or [5]. Since the last vertical arrow is a monomorphism we are done.

Based on Proposition 9.1.1 and Proposition 6.2.1 we obtain

9.1.2. Corollary. *Let R be an algebra over $\mathbb{F}_p$ and let M be an R -bimodule. Then one has a split exact sequence*

$$0 \rightarrow H^0(R, M) \rightarrow \text{SH}^3(R/k, M) \rightarrow \text{HML}^3(R, M) \rightarrow 0$$

where $k = \mathbb{Z}/p^2\mathbb{Z}$.

Remark. The homomorphism $\text{SH}^3(R/\mathbb{Z}, M) \rightarrow \text{HML}^3(R, M)$ in general is not an isomorphism. For example, if $R = \mathbb{Z}$, then $\text{SH}^i(\mathbb{Z}, -) = 0$ for all $i \geq 1$, thanks to Lemma

4.1.1. On the other hand $\mathrm{HML}^*(\mathbb{Z}, -)$ is quite nontrivial (see [5], [14]) and in particular $\mathrm{HML}^3(\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2$.

9.2. Relation with Shukla cohomology in higher dimensions. The relationship between Shukla cohomology $\mathrm{SH}^*(A/\mathbb{Z}, M)$ and MacLane cohomology $\mathrm{HML}^*(A, M)$ in higher dimensions is more complicated. Let us first consider the crucial case $A = \mathbb{Z}/p^k\mathbb{Z}$. We already saw $\mathrm{SH}^i(A/\mathbb{Z}, M)$, $A = \mathbb{Z}/p^k\mathbb{Z}$, is M if i is even and is zero otherwise. Unlike the Shukla cohomology, the behavior of $\mathrm{HML}^*(A, M)$ depends on whether $k = 1$ or $k > 1$. If $k = 1$, then similarly to Shukla cohomology the group $\mathrm{HML}^i(A, M)$ is M if i is even and is zero otherwise. However the natural map

$$\mathrm{SH}^i(\mathbb{F}_p/\mathbb{Z}, M) \rightarrow \mathrm{HML}^i(\mathbb{F}_p, M)$$

is an isomorphism only for $i = 0, \dots, 2p - 1$, and it is zero for $i > 2p - 2$. This follows from the fact that $\mathrm{SH}^*(\mathbb{F}_p, \mathbb{F}_p)$ is a polynomial algebra on the generator x of dimension two and $\mathrm{HML}^*(\mathbb{F}_p, \mathbb{F}_p)$ is a divided power algebra on the same generator x [13]. If $k > 1$, then the situation with MacLane cohomology is more complicated. A computation made in [27] shows that

$$\mathrm{HML}^{2n}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{F}_p) = (\mathbb{F}_p)^t, \quad \mathrm{HML}^{2n-1}(\mathbb{Z}/p^k\mathbb{Z}, \mathbb{F}_p) = (\mathbb{F}_p)^s,$$

where $t = 1 + \left\lfloor \frac{n}{p} \right\rfloor$ and $s = \left\lfloor \frac{n+1}{p} \right\rfloor$. The full computation of $\mathrm{HML}_*(\mathbb{Z}/p^k\mathbb{Z}, -)$ was obtained by Brun [7].

The relationship between MacLane cohomology and Shukla cohomology for general rings in all dimensions is given by the following theorem proved in [28] (see also [26]).

9.2.1. Theorem. (i) *Let A be a ring. Then for any A -bimodule M there is a spectral sequence*

$$E_2^{pq}(\mathbb{Z}) = \mathrm{SH}^p(A/\mathbb{Z}, \mathrm{HML}^q(\mathbb{Z}, M)) \Rightarrow \mathrm{HML}^{p+q}(A, M)$$

which is natural on A and M . The spectral sequence in low dimensions gives rise to the exact sequence

$$0 \rightarrow \mathrm{SH}^3(A/\mathbb{Z}, M) \rightarrow \mathrm{HML}^3(A, M) \rightarrow \mathrm{H}^0(A, {}_2M) \rightarrow \mathrm{SH}^4(A/\mathbb{Z}, M) \rightarrow \mathrm{HML}^4(A, M).$$

(ii) *Let A be an algebra over $\mathbb{F}_p$ and M be a A -bimodule. Then for any A -bimodule M there is a spectral sequence*

$$E_2^{pq}(\mathbb{F}_p) = \mathrm{H}^p(A, \mathrm{HML}^q(\mathbb{F}_p, M)) \Rightarrow \mathrm{HML}^{p+q}(A, M)$$

which is natural on A and M . The spectral sequence in low dimensions gives rise to the exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{H}^2(A, M) \rightarrow \mathrm{HML}^2(A, M) \rightarrow \mathrm{H}^0(A, M) \xrightarrow{d} \mathrm{H}^3(A, M) \rightarrow \mathrm{HML}^3(A, M) \\ \rightarrow \mathrm{H}^1(A, M) \rightarrow \mathrm{H}^4(A, M) \rightarrow \mathrm{HML}^4(A, M). \end{aligned}$$

Proof. For the existence of spectral sequences we refer to [28] (in case (i)) and [26]. The exact sequence in part (i) first was constructed in [16]. Both exact sequences are consequence of the existence of the spectral sequence together with the following computation due to Bökstedt [5] (see also [13] and [14]).

$$\begin{aligned} \mathrm{HML}^{2n}(\mathbb{Z}; M) &= M/nM, & \mathrm{HML}^{2n-1}(\mathbb{Z}; M) &= {}_nM, & n > 0. \\ \mathrm{HML}^{2n}(\mathbb{F}_p, M) &= M, & \mathrm{HML}^{2n-1}(\mathbb{F}_p, M) &= 0. & \square \end{aligned}$$

Now comparing the exact sequences of Corollary 7.4.2 with part (ii) of Theorem 9.2.1 one obtains

9.2.2. Corollary. *The differential*

$$d : E_2^{02}(\mathbb{F}_p) = E_3^{02}(\mathbb{F}_p) \rightarrow E_3^{30}(\mathbb{F}_p) = E_2^{30}(\mathbb{F}_p)$$

of the spectral sequence vanishes provided A is a commutative $\mathbb{F}_p$ -algebra and M is an A -module.

Acknowledgements. The second author is indebted to Mamuka Jibladze for the idea to modify classical obstruction theory in terms of crossed bimodules. He gratefully acknowledges the hospitality of University of Bielefeld during the preparation of this paper.

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Eingegangen 9. Juni 2005