This item was submitted to [Loughborough's Research Repository](#) by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

Closed-form stochastic response of linear building structures to spectrum-consistent seismic excitations

PLEASE CITE THE PUBLISHED VERSION

PUBLISHER

Elsevier

VERSION

AM (Accepted Manuscript)

PUBLISHER STATEMENT

This paper was accepted for publication in the journal Soil Dynamics and Earthquake Engineering and the definitive published version is available at <https://doi.org/10.1016/j.soildyn.2019.105724>.

LICENCE

CC BY-NC-ND 4.0

REPOSITORY RECORD

Barone, Giorgio, Francesco Lo Iacono, Giacomo Navarra, and Alessandro Palmeri. 2019. "Closed-form Stochastic Response of Linear Building Structures to Spectrum-consistent Seismic Excitations". figshare. <https://hdl.handle.net/2134/9642596.v1>.

Closed-form stochastic response of linear building structures to spectrum-consistent seismic excitations

Giorgio Barone^a, Francesco Lo Iacono^b, Giacomo Navarra^{b,*}, Alessandro Palmeri^a

^a*School of Architecture, Building and Civil Engineering, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom.*

^b*Facoltà di Ingegneria ed Architettura, Università degli Studi di Enna "Kore", Cittadella Universitaria, Enna Bassa I-94100 Enna, Italy.*

Abstract

Design codes typically define the seismic action in terms of pseudo-acceleration response spectra, encouraging the use of the modal superposition method for the evaluation of the structural response. For linear structural systems, provided that the ground shaking has been appropriately modelled, the full characterization of the response processes of interest can be achieved with the application of the random vibration theory. An analytical model for the power spectral density (PSD) functions consistent with seismic response spectra has been recently proposed. In this paper, taking advantage of this novel PSD model, closed-form approximate expressions of the spectral moments of the structural response are derived and numerically validated for single- and multi-degree of freedom systems. The proposed formulation is applied to the case of base isolated multi-story buildings, aimed at overcoming the difficulties associated with the non-classical nature of their damping. The paper shows how the proposed approach can be used for an accurate and computationally efficient evaluation of the probabilistic distribution of the structural response maxima.

Keywords: Base-isolated building, First-passage problem, Response spectrum, Spectrum-consistent PSD, Stochastic analysis.

1. Introduction

The seismic response of a structural system depends on both its dynamical properties and the characteristics of the ground motion time histories. Indeed, the evaluation of the response is strongly dependent on the mathematical model chosen to describe the seismic input and often needs to be carried out in a probabilistic framework. In several design scenarios, earthquakes can be modelled as stationary zero-mean Gaussian processes [1]. In this case, the seismic input can be probabilistically defined by its power spectral density (PSD) function [2]. The latter carries several advantages: for linear systems, a full characterization of the response process can be easily achieved by means of spectral analysis; for nonlinear systems, it enables the artificial generation of ground motion time-history samples, matching the design response spectra (RS), in order to perform Monte-Carlo simulations (MCS).

In the last decades, great attention has been devoted to the development of analytical and numerical models for RS-consistent PSD functions. Earlier contributions can be found in the review paper by Ahmadi [3]. A common approach to model earthquake time histories is to use time-varying envelope functions that modulates realizations of a stationary PSD expressed either by Tajimi-Kanai [4, 5] or Clough-Penzien [1] models. As an example, Falsone *et al.* [6] developed a technique to obtain the filter coefficients for the design RS of Eurocode 8 [7]. Several other approaches have been proposed to numerically generate PSD functions consistent with non-smooth RS (e.g. for the design of special structures, such as nuclear power plants). However, these techniques generally require numerical iterative procedures, either in the time or in the frequency domain [8–15]. Further methods have been developed to obtain sets of RS-consistent acceleration time histories, which are also able to mimic the non-stationary characteristics of field-recorded accelerograms [16–25] by means of the definition of evolutionary PSD functions [26–28].

*Corresponding author

Recently, a novel formulation for the PSD function consistent with the design RS of Eurocode 8 and the former Italian National Code has been proposed [29–31], using an analytical model, whose parameters are expressed as functions of the target RS parameters [32].

For linear structural systems, once the PSD function of the input ground motion is known, stochastic analysis allows determining the PSD function of the structural response. Then, various indicators of the structural safety can be computed by using the spectral moments (SMs) of the dynamic response [33–35]. Closed-form solutions of SMs have been determined only in some particular cases [36, 37]. In general, they are evaluated by means of numerical integration, resulting in a significant computational effort, especially for large multi-degree-of-freedom (MDOF) systems [38].

Furthermore, RS-consistent PSD functions have been recently used also for non-linear stochastic dynamic analyses of yielding structures through the application of the statistical linearisation technique [39, 40]. In the case of multi-storey inelastic building structures, the approach in Ref. [41] has been proposed to estimate the response peak values without the integration of the non-linear equations of motion and operating directly on the physical domain. In this way, no further approximations related to modal decomposition are needed. Coupling such technique with an analytical model of the PSD function of the seismic input could reduce the overall computational effort.

In this paper, exploiting the analytical PSD model proposed in Ref. [32], the authors propose novel closed-form expressions for the SMs of linear structural systems excited by RS-consistent seismic input. SMs are first derived as functions of the parameters defining both the SDOF (single-degree-of-freedom) oscillators and the RS; then, direct and cross SMs of order up to two are obtained in closed-form for MDOF systems. Finally, these expressions are applied to the case of multi-story base-isolated buildings by applying a convenient double modal transformation of coordinates [42]. The proposed formulae are validated against MCS. The authors also show how using the analytical SMs allows an accurate reproduction of the target RS, with reference to Eurocode 8 [7] and other international codes, including 2-, 3- and 4-branch design RS. Finally, the paper shows the significant advantage of SMs closed-form expression for the probabilistic assessment of the structural response by analytically computing the cumulative distribution functions (CDFs) of the response maxima. In the latter case, the authors show how the proposed method achieves improved accuracy and significantly reduces the computational effort.

2. Power Spectral Density functions consistent to Response Spectra

RS are widely used in earthquake engineering to represent the peak values of the dynamic response (e.g. in terms of displacements, velocities, accelerations) of linear and nonlinear SDOF oscillators to a specified ground motion as a function of its natural period of vibration and its damping capabilities [43]. For instance, assuming linear viscous damping, the RS in terms of displacements can be expressed as:

$$S_d(T, \zeta) = \max \{ |u(t)| \} , \quad (1)$$

in which T is the natural period of vibration, ζ is the viscous damping ratio and $u(t)$ is the time history of the response displacement.

Most building codes define the design value of the seismic action at a given site by means of the expected pseudo-acceleration RS, depending on the seismicity of the site, the soil conditions, the importance of the structure, the assumed ductility of the structure and the limit state under consideration. This approach is strictly valid only for SDOF systems, as the evaluation of the response for MDOF structures implies approximations in the determination of the modal response correlation coefficients [44, 45]. Despite this limitation, using the RS to predict the peak values of structural displacements and internal forces has the significant advantage to avoid the direct integration of the equations of motion.

Based on the current international seismic codes, a general class of pseudo-acceleration design RS can be described

Figure 1: Response spectrum in terms of pseudo-accelerations.

using the following piecewise function:

$$S_{pa}(T) = \begin{cases} S_0 \left[1 + (\alpha - 1) \frac{T}{T_3} \right], & 0 \leq T < T_3; \\ \alpha S_0, & T_3 \leq T \leq T_2; \\ \alpha S_0 \left(\frac{T_2}{T} \right)^{k_1}, & T_2 < T \leq T_1; \\ \alpha S_0 \left(\frac{T_2}{T_1} \right)^{k_1} \left(\frac{T_1}{T} \right)^{k_2}, & T > T_1, \end{cases} \quad (2)$$

where S_0 is the peak ground acceleration (PGA), α is a dynamic amplification factor, $T_1 > T_2 > T_3$ are periods delimiting the various RS branches, and k_1 and k_2 are shape factors. In the following, the authors explicitly refer to design RS having $k_1 = 1$ and $k_2 = 2$ (Figure 1), so that the second, third and fourth branches correspond to constant spectral pseudo-accelerations, pseudo-velocities and displacements, respectively, while the first branch linearly connects the point at $T = 0$, i.e. the PGA to the second branch. Several major seismic codes (e.g., see Table 1) adopt similar RS models, some only using three or two branches, that can be derived as special cases of Eq. (2).

Advanced seismic codes allow using alternative representations of seismic actions in terms of recorded, artificial or simulated accelerograms. The latter can be generated as samples of a zero-mean Gaussian stationary process, provided that the RS computed from the ensemble of the artificial accelerograms closely matches the design RS. An advantage of this approach is that the PSD function of this process is related only to the ground acceleration, being independent of the dynamic characteristics of the superimposed structure (e.g. viscous damping ratio, nonlinearities, etc.).

2.1. Numerical approach

While evaluating a RS for a given PSD function is straightforward, the inverse problem (i.e. determining the PSD function consistent to an assigned RS) is not. An estimate of the one-sided PSD function $G_{\ddot{U}_g}(\omega)$ consistent with an assigned RS can be obtained by considering the following implicit equation [14]:

$$G_{\ddot{U}_g}(\omega) = \frac{\gamma}{\omega} \left[\left(\frac{S_{pa}(\omega, \zeta)}{\eta_U(\omega, \zeta)} \right)^2 - \int_0^\omega G_{\ddot{U}_g}(\hat{\omega}) d\hat{\omega} \right], \quad (3)$$

where $\gamma = 4\zeta/(\pi - 4\zeta)$, $S_{pa}(\omega, \zeta)$ is the target pseudo-acceleration RS and $\eta_U(\omega, \zeta)$ is the so-called Vanmarcke's peak factor for the displacement response process $U(t)$ [34].

Table 1: Classification of seismic codes with design RS that can be expressed by Eq. (2) ($k_1 = 1$ and $k_2 = 2$).

Country	Seismic Code	Year
Four-branch design RS		
Australia	AS1170.4	2007
Colombia	NRS10	2010
European Union	Eurocode 8 [7]	2004
Three-branch design RS		
Dominican Republic	Decreto 201/11 MOPC	2011
India	Indian Standard 1893	2002
Indonesia	SNI-1726-2002	2002
Japan	Building Standard Law	2000
Korea	Korea Building Code	2005
Philippines	NSCP	2010
Taiwan	SDCB	2005
USA	International Building Code [46]	2015
Two-branch design RS		
Albania	KTP-N.2-89	1989
Ecuador	INEN-5	2001
Macedonia	Tech. Reg. Code	1990
Peru	E.030 [47]	2003
Serbia	Tech. Reg. Code	1990
Uganda	US 319:2003	2003

80 An approximate evaluation of $G_{\ddot{U}_g}(\omega)$ can be determined by assuming the PSD function to be piecewise constant
81 at all arbitrary selected frequency intervals $\Delta\omega$, and replacing the integral in Eq. (3) with a summation:

$$G_{\ddot{U}_g}(\omega_i) = \frac{4\zeta}{\pi\omega_i - 4\zeta\omega_{i-1}} \left[\left(\frac{S_{pa}(\omega_i, \zeta)}{\eta_U(\omega_i, \zeta)} \right)^2 - \Delta\omega \sum_{j=1}^{i-1} G_{\ddot{U}_g}(\omega_j) \right]. \quad (4)$$

82 Exploiting Eq. (4), Cacciola *et al.* [14] proposed a numerical procedure to estimate the PSD function consistent
83 with an assigned RS. This technique proved to be effective also for non-smooth RS obtained from natural earthquake
84 ground motion time histories [16], and its accuracy could be further improved by using an iterative corrective method,
85 as demonstrated in [39]. However, the recursive nature of this numerical method entails the entire process must be
86 repeated every time a new RS is selected. An alternative iterative corrective method to best-fit, in the mean sense, a
87 PSD function to a given response spectrum, offering improved accuracy, has been proposed in Ref. [41].

88 2.2. Analytical model

89 As part of the work reported in Ref. [32], an extensive numerical campaign has been executed, aimed at calculating
90 and comparing the spectrum-consistent PSD functions which satisfy Eq. (4) for assigned RS belonging to the class
91 defined in Eq. (2). That is, a large number of spectrum-consistent PSD functions have been obtained numerically by
92 adopting the method proposed in Ref. [14], varying the parameters α , S_0 , T_1 , T_2 and T_3 into Eq. (2). Specifically,
93 the parameters have been varied in the following intervals: $\alpha \in [2.0, 3.5]$; $T_3/s \in [0.05, 0.20]$; $T_2/s \in [0.20, 0.60]$;
94 $T_1/s \in [1.50, 3.50]$; while S_0 has been kept equal to 1, as its value does not affect the shape of RS and PSD function,
95 and the duration T_S of the stationary part of the ground shaking, affecting the parameter η_U in Eqs. (3) and (4), has
96 been varied between 10 and 30 s. The analyses have been repeated for 2-, 3- and 4-branch RS, covering all the building
97 codes listed in Table 1. It has been observed that, for all these cases, Eq. (4) returns PSD functions having the shape
98 qualitatively depicted in Figure 2, irrespectively of the soil type, i.e. from very stiff to very soft; this has also been

Figure 2: Power Spectral Density function consistent with Response Spectrum.

99 shown by further studies (see, among others, Refs. [39–41]). Importantly, the numerical investigations have proved
 100 that each branch of the spectrum-consistent PSD function can be analytically fitted by an exponential law. This in turn
 101 led to the development of the analytical model defined by the following piecewise expression:

$$G_{U_g}(\omega) = \begin{cases} G_0 \left(\frac{\omega_1}{\omega_2} \right)^{e_2} \left(\frac{\omega}{\omega_1} \right)^{e_1}, & 0 \leq \omega < \omega_1; \\ G_0 \left(\frac{\omega}{\omega_2} \right)^{e_2}, & \omega_1 \leq \omega < \omega_2; \\ G_0 \left(\frac{\omega}{\omega_2} \right)^{e_3}, & \omega_2 \leq \omega \leq \omega_3; \\ G_0 \left(\frac{\omega_3}{\omega_2} \right)^{e_3} \left(\frac{\omega}{\omega_3} \right)^{e_4}, & \omega > \omega_3, \end{cases} \quad (5)$$

102 where G_0 represents the peak value at the frequency ω_2 (see Figure 2). The proposed model depends on 5 parameters,
 103 namely G_0 and the four exponents e_i ($i = 1, \dots, 4$), whose closed-form expressions have been derived by manipulating
 104 Eq. (3):

$$e_1 = 2k_2 - 1 - L(\omega_1); \quad (6)$$

$$e_2 = 2k_1 - 1 - L(\omega_2); \quad (7)$$

$$e_3 = -1 - \gamma - \beta_2 L(\omega_2); \quad (8)$$

$$e_4 = -1 - \gamma - \beta_3 \left(L(\omega_3) + 2 \frac{\alpha - 1}{\alpha} \right); \quad (9)$$

$$G_0 = \frac{\gamma}{\beta_2 \omega_2} \left(\frac{\alpha S_0}{\eta_U(\omega_2, \zeta)} \right)^2. \quad (10)$$

109 In Eqs. (6) to (9), the function $L(\omega)$ is defined as:

$$L(\omega) = 2\omega \frac{d}{d\omega} \ln(\eta_U(\omega, \zeta)), \quad (11)$$

110 and the following quantities have been introduced:

$$\beta_2 = \left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} \frac{\gamma + e_1 + 1}{e_1 + 1} + \left(1 - \left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} \right) \frac{\gamma + e_2 + 1}{e_2 + 1}; \quad (12)$$

111

$$\beta_3 = \left(\frac{\omega_2}{\omega_3}\right)^{e_3+1} \beta_2 + \left(1 - \left(\frac{\omega_2}{\omega_3}\right)^{e_3+1}\right) \frac{\gamma + e_3 + 1}{e_3 + 1}. \quad (13)$$

112 For the sake of completeness, the mathematical process leading to Eqs. (6) to (13) is detailed in Appendix A. It is
 113 worth stressing here that Eqs. (6) to (13) map the 5 RS parameters α , S_0 , $T_1 = 2\pi/\omega_1$, $T_2 = 2\pi/\omega_2$ and $T_3 = 2\pi/\omega_3$
 114 onto the 5 PSD parameters G_0 , e_1 , e_2 , e_3 and e_4 through explicit analytical relationships. Accordingly, the definition
 115 of the seismic action can be equivalently done in terms of RS or PSD function.

116 The PSD function of Eq. (5) can be easily adapted to 3- or 2-branch RS. In the first case, the constant displacements
 117 branch should simply be removed, obtaining the corresponding RS by letting $T_1 \rightarrow \infty$ in Eq. (2). Hence, the RS-
 118 consistent PSD function can be derived by letting $\omega_1 \rightarrow 0$ in Eq. (5):

$$S_{\text{pa}}(T) = \begin{cases} S_0 \left[1 + (\alpha - 1) \frac{T}{T_3} \right], & 0 \leq T < T_3; \\ \alpha S_0, & T_3 \leq T \leq T_2; \\ \alpha S_0 \left(\frac{T_2}{T} \right)^{k_1}, & T > T_2; \end{cases} \quad (14a)$$

119

$$G_{\dot{U}_g}(\omega) = \begin{cases} G_0 \left(\frac{\omega}{\omega_2} \right)^{e_2}, & 0 \leq \omega < \omega_2; \\ G_0 \left(\frac{\omega}{\omega_2} \right)^{e_3}, & \omega_2 \leq \omega \leq \omega_3; \\ G_0 \left(\frac{\omega_3}{\omega_2} \right)^{e_3} \left(\frac{\omega}{\omega_3} \right)^{e_4}, & \omega > \omega_3. \end{cases} \quad (14b)$$

120 For 2-branch RS, the initial linear branch is removed as well, letting $T_3 \rightarrow 0$ and $\omega_3 \rightarrow \infty$ in Eq. (14), and returning
 121 the expressions:

$$S_{\text{pa}}(T) = \begin{cases} \alpha S_0, & T \leq T_2; \\ \alpha S_0 \left(\frac{T_2}{T} \right)^{k_1}, & T > T_2; \end{cases} \quad (15a)$$

122

$$G_{\dot{U}_g}(\omega) = \begin{cases} G_0 \left(\frac{\omega}{\omega_2} \right)^{e_2}, & 0 \leq \omega < \omega_2; \\ G_0 \left(\frac{\omega}{\omega_2} \right)^{e_3}, & \omega \geq \omega_2. \end{cases} \quad (15b)$$

123 Also in these cases, the exponents e_i and G_0 can be derived using Eqs. (7) to (10).

124 3. Analytical spectral moments of the response process

125 Spectral moments play an important role in random vibration problems, as they are related to various statistics of
 126 stochastic processes (e.g. variance of its time derivatives, central frequency, bandwidth parameter, etcetera) and allow
 127 deriving approximate solutions for the first-passage problem [33–38, 48, 49]. Mathematically defined as the geometric
 128 moments of the one-sided PSD function $G_X(\omega)$ with respect to the axis $\omega = 0$, the generic m th order spectral moment
 129 of a stochastic process $X(t)$ is given by:

$$\lambda_{m,X} = \int_0^{+\infty} \omega^m G_X(\omega) d\omega. \quad (16)$$

130 In general, the integral in Eq. (16) can only be computed numerically. In this paper, however, closed-form expres-
 131 sions of the spectral moments for structures excited by RS-consistent earthquake acceleration processes are proposed,

in approximate form, for order m up to three for SDOF oscillators and up to two for MDOF structures. To the best of the authors' knowledge, spectral moments in analytical form have never been proposed before for RS-consistent seismic accelerations.

3.1. Spectral moments for SDOF systems

Let us consider a linear SDOF oscillator driven by a base acceleration time history $\ddot{u}_g(t)$, sample of a process $\ddot{U}_g(t)$, fully characterized by the PSD function $G_{\ddot{U}_g}(\omega)$, consistent with an assigned RS. Based on Eq. (16), the m th-order spectral moment of the response process can be written as:

$$\lambda_{m,U} = \int_0^{+\infty} \omega^m |H(\omega)|^2 G_{\ddot{U}_g}(\omega) d\omega, \quad (17)$$

where $H(\omega)$ is the complex-valued transfer function of the SDOF oscillator:

$$H(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2i\zeta\omega_0\omega}, \quad (18)$$

in which ω_0 and ζ are the natural circular frequency and the viscous damping ratio of the system, while $i = \sqrt{-1}$ is the imaginary unit.

A first approximation of the spectral moments can be obtained if the function $|H(\omega)|^2$ exhibits a very sharp peak in ω_0 (e.g., damping ratio $\zeta \ll 1$) and the PSD function of the excitation is approximately flat in the neighbourhood of ω_0 . In this case, the spectral moments of order $m \leq 3$ can be determined in closed-form considering the input as a white noise process with constant PSD function $G_{\ddot{U}_g}(\omega_0)$ (see, for instance, Ref. [50]):

$$\lambda_{0,U}^{(1)} = \frac{\pi}{4\zeta\omega_0^3} G_{\ddot{U}_g}(\omega_0); \quad (19a)$$

$$\lambda_{1,U}^{(1)} = \frac{\pi - 2\zeta}{4\zeta\omega_0^2} G_{\ddot{U}_g}(\omega_0); \quad (19b)$$

$$\lambda_{2,U}^{(1)} = \omega_0^2 \lambda_{0,U}^{(1)}; \quad (19c)$$

$$\lambda_{3,U}^{(1)} = \omega_0^2 \lambda_{1,U}^{(1)}, \quad (19d)$$

where the superscripted (1) serves as a reminder of being the “first” approximation for the spectral moments.

An improvement to this simplified approach [35] entails accounting for contributions in the frequency range $[0, \omega_0]$, where the function $|H(\omega)|^2$ can be reasonably approximated as $1/\omega_0^4$. In this case, the spectral moments of order $m \leq 3$ can be calculated as:

$$\lambda_{m,U}^{(2)} = \lambda_{m,U}^{(1)} + \frac{1}{\omega_0^4} \int_0^{\omega_0} G_{\ddot{U}_g}(\omega) d\omega - \frac{G_{\ddot{U}_g}(\omega_0)}{(m+1)\omega_0^{3-m}}, \quad (20)$$

where $\lambda_{m,U}^{(1)}$ are the expressions in Eq. (19), the negative terms in the right hand side takes away the contribution around $\omega = \omega_0$, already included in $\lambda_{m,U}^{(1)}$, and the superscripted (2) means “second” approximation. Taking advantage of the analytical PSD model proposed in Section 2.2, these expressions can be rewritten as:

$$\lambda_{m,U,k}^{(2)} = \frac{G_{\ddot{U}_g}(\omega_0)}{4\zeta\omega_0^{3-m}} \varphi_{m,k} + \frac{1}{\omega_0^4} \sum_{j=1}^{k-1} G_{\ddot{U}_g}(\omega_j) \omega_j^{m+1} \gamma_{m,j}, \quad (21)$$

where $\lambda_{m,U,k}^{(2)}$ is the expression of the m -th spectral moment of the response process $U(t)$ when ω_0 falls within the k th branch of Eq. (5), while the dimensionless quantities $\varphi_{m,k}$ and $\gamma_{m,j}$ are defined as follows:

$$\varphi_{m,k} = \pi - \frac{4\zeta}{(1+m)(1+m+e_k)} \begin{cases} e_k, & m = 0, 2; \\ 2(1+e_k), & m = 1; \\ 8+3e_k, & m = 3; \end{cases} \quad (22)$$

$$\gamma_{m,j} = \frac{e_{j+1} - e_j}{(1 + m + e_{j+1})(1 + m + e_j)}. \quad (23)$$

Eq. (21) is valid for 4-branch RS, and can be easily extended to the case of 2 and 3 branches cases. In fact, following the notation adopted in Eqs. (14) and (15), for the 3-branch RS ($2 < k < 4$), the lower bound in the summation must be changed to $j = 2$, while for the 2-branch case ($2 < k < 3$) lower and upper bounds should be modified to $j = 2$ and $k = 2$, respectively.

Eq. (21) constitutes an accurate estimate of the spectral moments as a closed-form function of the RS parameters. It is worth noting that, by setting all the exponents e_i in Eq. (5) to zero, the PSD function assumes constant value G_0 and Eq. (21) returns the well-known spectral moments for white noise input.

3.2. Spectral moments for MDOF systems

The equations of motion of a n -degree-of-freedom system under seismic action can be expressed as:

$$\mathbf{M} \cdot \ddot{\mathbf{u}}(t) + \mathbf{C} \cdot \dot{\mathbf{u}}(t) + \mathbf{K} \cdot \mathbf{u}(t) = -\mathbf{M} \cdot \boldsymbol{\tau} \ddot{u}_g(t), \quad (24)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $(n \times n)$ time-invariant mass, damping and stiffness matrices, respectively; $\mathbf{u}(t)$ is the $(n \times 1)$ vector listing the degrees of freedom of the structure; and $\boldsymbol{\tau}$ is the $(n \times 1)$ location vector.

Closed-form direct and cross spectral moments for classically-damped MDOF system can be determined by first introducing the modal matrix $\boldsymbol{\Phi}$ of the system, solution of the classical eigen-problem $\mathbf{K} \cdot \boldsymbol{\Phi} = \mathbf{M} \cdot \boldsymbol{\Phi} \cdot \boldsymbol{\Omega}^2$, with the ortho-normal condition $\boldsymbol{\Phi}^T \cdot \mathbf{M} \cdot \boldsymbol{\Phi} = \mathbf{I}_n$, $\mathbf{I}_n$ being the identity matrix of size n , and then applying the modal transformation $\mathbf{u}(t) = \boldsymbol{\Phi} \cdot \mathbf{y}(t)$, leading to the following equation:

$$\ddot{\mathbf{y}}(t) + \boldsymbol{\Gamma} \cdot \dot{\mathbf{y}}(t) + \boldsymbol{\Omega}^2 \cdot \mathbf{y}(t) = \mathbf{p} \ddot{u}_g(t), \quad (25)$$

where $\mathbf{y}(t)$ is the vector collecting the $\ell \leq n$ modal coordinates retained in the analysis; $\boldsymbol{\Gamma} = \text{diag} \{ 2 \zeta_1 \omega_1 \quad \cdots \quad 2 \zeta_\ell \omega_\ell \}$ and $\boldsymbol{\Omega} = \text{diag} \{ \omega_1 \quad \cdots \quad \omega_\ell \}$ are the modal damping matrix and spectral matrix, respectively, both of dimensions $(\ell \times \ell)$; and $\mathbf{p} = -\boldsymbol{\Phi}^T \cdot \mathbf{M} \cdot \boldsymbol{\tau} = \{ p_1 \quad \cdots \quad p_\ell \}^T$ is the vector of modal participation factors. Under the assumptions of Gaussian excitation, cross-spectral moments of m th order related to the j th and k th modal oscillators (with $j = 1, \dots, \ell$ and $k = 1, \dots, \ell$) can be defined as:

$$\lambda_{m,j,k} = \int_0^{+\infty} \omega^m p_j p_k H_j^*(\omega) H_k(\omega) G_{\ddot{u}_g}(\omega) d\omega, \quad (26)$$

where the asterisk means complex conjugate and $H_j(\omega)$ is the transfer function of the j th modal oscillator, so defined:

$$H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + 2i \zeta_j \omega_j \omega}. \quad (27)$$

For a selected RS, the direct spectral moments ($j = k$) can be estimated using Eq. (21). Then, as demonstrated by Di Paola and Muscolino [38], the cross-spectral moments of any order, if they exist for a given PSD function of the excitation, can be obtained recursively as linear combinations of the direct spectral moments. In particular, real and imaginary parts of even-order cross-spectral moments ($m = 0, 2$) can be determined as:

$$\text{Re} [\lambda_{m,j,k}] = \frac{(-1)^{m/2}}{2} (\lambda_{0,j,j} \gamma_{m,k,j} \omega_j^2 + \lambda_{2,j,j} \delta_{m,k,j} + \lambda_{0,k,k} \gamma_{m,j,k} \omega_k^2 + \lambda_{2,k,k} \delta_{m,j,k}); \quad (28a)$$

$$\text{Im} [\lambda_{m,j,k}] = \frac{(-1)^{m/2}}{2} (\lambda_{1,j,j} \varepsilon_{m,k,j} - \lambda_{1,k,k} \varepsilon_{m,j,k} + \lambda_{3,j,j} \alpha_{m,j,k} - \lambda_{3,k,k} \alpha_{m,k,j}), \quad (28b)$$

while for the first-order cross-spectral moments:

$$\text{Re} [\lambda_{1,j,k}] = -\frac{1}{2} (\lambda_{1,j,j} \varepsilon_{1,k,j} + \lambda_{1,k,k} \varepsilon_{1,j,k} + \lambda_{3,j,j} \alpha_{1,k,j} + \lambda_{3,k,k} \alpha_{1,j,k}); \quad (29a)$$

$$\text{Im}[\lambda_{1,j,k}] = \frac{1}{2} (\lambda_{0,j,j} \gamma_{1,k,j} \omega_j^2 + \lambda_{2,j,j} \delta_{1,k,j} - \lambda_{0,k,k} \gamma_{1,j,k} \omega_k^2 - \lambda_{2,k,k} \delta_{1,j,k}) . \quad (29b)$$

In Eqs. (28) and (29), the following positions have been made:

$$\alpha_{m,j,k} = \alpha_{m-1,j,k} + \bar{\omega}_k \beta_{m-1,j,k} ; \quad (30a)$$

$$\beta_{m,j,k} = -\zeta_k \omega_k \beta_{m-1,j,k} - \bar{\omega}_k \alpha_{m-1,j,k} ; \quad (30b)$$

$$\gamma_{m,j,k} = \zeta_k \omega_k \alpha_{m,j,k} + \bar{\omega}_k \beta_{m,j,k} ; \quad (30c)$$

$$\delta_{m,j,k} = \zeta_k \omega_k \alpha_{m,j,k} - \bar{\omega}_k \beta_{m,j,k} ; \quad (30d)$$

$$\varepsilon_{m,j,k} = 2 \zeta_k \omega_k \bar{\omega}_k \beta_{m,j,k} + \alpha_{m,j,k} (\omega_k^2 - 2 \bar{\omega}_k^2) ; \quad (30e)$$

being:

$$\alpha_{0,j,k} = 4 (\zeta_j \omega_j + \zeta_k \omega_k) / K_{jk} \quad (31a)$$

$$\beta_{0,j,k} = 2 (\omega_j^2 - \omega_k^2 + 2 \zeta_j \zeta_k \omega_j \omega_k + 2 \zeta_k^2 \omega_k^2) / (\bar{\omega}_k K_{jk}) \quad (31b)$$

$$K_{j,k} = (\omega_j^2 - \omega_k^2)^2 + 4 \zeta_j \zeta_k (\omega_j^2 + \omega_k^2) \omega_j \omega_k + 4 (\zeta_j^2 + \zeta_k^2) \omega_j^2 \omega_k^2 \quad (31c)$$

$$\bar{\omega}_j = \omega_j \sqrt{1 - \zeta_j^2} \quad (31d)$$

Once the matrix of the cross-spectral moments of order m of the modal oscillators, $\mathbf{\Lambda}_{m,\mathbf{Y}}$, is obtained, the corresponding quantities in the geometric space can be computed as:

$$\mathbf{\Lambda}_{m,\mathbf{X}} = \mathbf{\Phi} \cdot \mathbf{\Lambda}_{m,\mathbf{Y}} \cdot \mathbf{\Phi}^T . \quad (32)$$

It is worth noting here that, when the seismic peak response of MDOF structures is of interest, as for instance in the case-study structure discussed in the Section 5, cross-spectral moments up to second order are required, and thus the computation of direct spectral moments up to third order is needed. Importantly, for the proposed spectrum-consistent PSD function of the ground shaking, they all exist and can be determined using the expression reported above.

4. Numerical validation for SDOF oscillators

The proposed approach provides closed-form expressions for the statistics of the seismic response of linear structures, inevitably within the approximations associated with the analytical model for the PSD function of the input and the hypotheses underpinning Eqs. (19) and (20). Therefore, the accuracy of the method is first tested in this section for SDOF oscillators, through: *i*) comparison with the results of numerical integration for the first three spectral moments λ_m, U for different values of the natural period of vibration and viscous damping ratio; *ii*) consistency analysis for three different international building codes, adopting RS with different numbers of branches; *iii*) seismic reliability analysis for SDOF oscillators with different natural periods of vibration.

4.1. Spectral moments for SDOF oscillators

In this section, a number of linear SDOF oscillators are analyzed, varying their natural periods of vibration and viscous damping ratios. A pseudo-acceleration RS has been selected according to the Italian Code [51] (which uses the same 3-branch shape as Eurocode 8, and whose parameters are defined through seismic maps and vary continuously with latitude and longitude), considering a return period of 475 years (i.e., the so-called “no-collapse requirement”, associated with 10% probability of exceedance in 50 years) and a site in Messina (Sicily), with “type A” soil (hard rock). Table 2 reports the parameters of both the RS and the RS-consistent PSD obtained through the application of Eqs. (5) to (13).

In Figure 3, closed-form spectral moments of order $m = 0, 1$ and 2 are compared with their numerical counterpart (computed by numerical integration of the unilateral PSD of the dynamic response, $G_U(\omega) = |H(\omega)|^2 G_{\ddot{u}_g}(\omega)$) for a range of natural periods of vibration and viscous damping ratios. Results are in excellent agreement, with only

Table 2: Parameters for the definition of the four-branch RS and PSD.

RS		PSD	
parameter	value	parameter	value
S_0	2.432 m/s ²	G_0	$1.4513 \cdot 10^{-2}(\text{m/s}^2)^2/(\text{rad/s})$
α	2.411	e_1	2.5094
T_3	0.120 s	e_2	0.7594
T_2	0.359 s	e_3	-1.3177
T_1	2.592 s	e_4	-2.6209

relatively small discrepancies observed for high-order moments at large natural periods and damping ratios. For instance, for damping ratios less than 10% and periods of vibration less than 2 s, deviations are less than 10%. For larger damping ratios and periods of vibration (i.e. relatively high-damped flexible structures) the proposed approach tends to underestimate the spectral moments of the dynamic response, mainly due to an insufficiently long duration assumed for the stationary excitation. In order to achieve higher accuracy, this well-known problem can be addressed by adopting Vanmarcke's frequency-dependent corrective damping factor [35]. Alternatively, depending on the natural period of vibration and viscous damping ratio of the structure being analyzed, a sufficiently long duration of the stationary excitation could be assumed. The latter can be selected on the basis of the numerical results reported in Ref. [40].

4.2. Reproduction of the target response spectra

Eurocode 8 [7] provides the requirements for alternative representations of the seismic action. In particular, according to clause § 3.2.3.1.2, a suite of artificial accelerograms can be utilized if, in the range of periods between $0.2 T_0$ and $2 T_0$ (where T_0 is the fundamental period of the structure in the direction to which the accelerogram is applied), no values of the mean 5%-damping RS $\hat{S}_{\text{pa}}(T)$, calculated from all the response time histories, is less than 90% of the corresponding value of the targeted 5%-damping design RS $S_{\text{pa}}(T)$; that is, $\hat{S}_{\text{pa}}(T) \geq 0.90 S_{\text{pa}}(T)$. In a stochastic framework, this corresponds to a consistency criterion between PSD and RS where the 50th percentile (i.e. the median) of the peak dynamic response of any SDOF oscillator with natural period $T_0 = 2\pi/\omega_0$, excited by a stationary random process with given PSD function, exceeds 90% of the targeted 5%-damping design RS.

For validation purposes, three PSD functions consistent with RS defined by three different building codes have been considered. For each period of vibration T , $\hat{S}_{\text{pa}}(T)$ can be estimated as the median of the maximum response in terms of pseudo-accelerations:

$$\hat{S}_{\text{pa}}(T) = \left(\frac{2\pi}{T}\right)^2 \sqrt{\lambda_{0,U}(T, \zeta)} \eta_U\left(\frac{2\pi}{T}, \zeta\right), \quad (33)$$

where $\lambda_{0,X}(T, \zeta)$ is the zero-order spectral moment of the response process $U(t)$ and $\eta_U(\omega, \zeta)$ is the corresponding Vanmarcke's peak factor:

$$\eta_U(\omega, \zeta) = \sqrt{2 \ln \left\{ 2 N_U(\omega, \zeta) \left[1 - \exp \left(-q_U^2(\omega, \zeta) \sqrt{\pi \ln (2 N_U(\omega, \zeta))} \right) \right] \right\}} \quad (34)$$

being:

$$N_U(\omega, \zeta) = \frac{T_S \nu_U(\omega, \zeta)}{-\ln(p)}; \quad (35a)$$

$$\nu_U(\omega, \zeta) = \frac{1}{2\pi} \sqrt{\frac{\lambda_{2,U}(2\pi/\omega, \zeta)}{\lambda_{0,U}(2\pi/\omega, \zeta)}}; \quad (35b)$$

$$q_U(\omega, \zeta) = \sqrt{1 - \frac{\lambda_{1,U}^2(2\pi/\omega, \zeta)}{\lambda_{0,U}(2\pi/\omega, \zeta) \lambda_{2,U}(2\pi/\omega, \zeta)}}, \quad (35c)$$

where $\nu_U(\omega, \zeta)$ and $q_U(\omega, \zeta)$ are the mean up-crossing rate and the bandwidth factor of the random process $U(t)$.

Figure 3: Comparison between numerical evaluation (solid lines) and proposed approximate analytic expressions (dots) of spectral moments against natural period; a) zero order moments; b) first order moments; c) second order moments.

Figure 4: (a) PSD function consistent with the four-branch RS (parameters listed in Table 2; (b) Pseudo-acceleration RS: target (black solid line), 10% tolerance (black dashed line) and generated RS obtained with proposed approach (red dots).

It should be noted at this point that the quantities of Eqs. (35) (and, ultimately, the peak factor of Eq. (34)) depend on: *i*) the spectral moments $\lambda_{m,U}$ up to the order $m = 2$, which can be calculated with of Eq. (21); *ii*) the fractile p of the maxima distribution, set $p = 0.50$ (i.e. median of the distribution); *iii*) the duration T_S of the stationary part of the input, which in the following has been arbitrarily assumed as $T_S = 20$ s. Obviously, the larger T_S , the lower the ordinates of the PSD function of the seismic input, but the same peak response is obtained for linear structural systems.

The 4-branch pseudo-acceleration RS, with parameters reported in Table 2, has been considered first. Figure 4(a) shows the RS-consistent PSD obtained with the application of Eqs. (5) to (13), while Figure 4(b) compares the code and reconstructed RS in terms of pseudo-accelerations, showing that the latter lies within the allowed 10% tolerance.

A 3-branch pseudo-acceleration RS has then been considered, following the International Building Code [46] and assuming that the SDOF oscillator was representative of structure located in Memphis (Missouri, USA), on a type A soil (hard rock). The RS-consistent PSD function is shown in Figure 5(a), whose parameters are reported in Table 3. Also in this case, the discrepancy between the reconstructed and target RS does not exceed 10% for all periods of vibration, as illustrated in Figure 5(b).

Finally, the two-branch RS defined by the Peruvian building code [47], with reference to an ordinary building located in the country's second seismic zone, on soil "type S1" (hard rock). The parameters of RS and PSD are reported in Table 4, and the RS-consistent PSD function is depicted in Figure 6(a), and the results in terms of pseudo-acceleration RS are shown in Figure 6(b).

Although not reported herein for sake of shortness, similar results have been obtained for all the international codes listed in Table 1, confirming thus the validity of the proposed approach.

4.3. Reliability analysis

The statistical distribution of the peak structural response can play a fundamental role in the seismic analysis and design of multi-story frames. Herein, the distribution of the maxima of a response process is determined by coupling the proposed formulation with the approach presented by Vanmarcke in Ref. [34], since the two techniques are based on common underpinning assumptions. This allows exploiting the significant computational advantages of the closed-form spectral moments of the response process. Results for SDOF oscillators are presented in this section, and compared with the outcomes of Monte Carlo simulations. The extension to MDOF structures is specifically addressed in Section 5.

Based on Ref. [34], for a stationary random process $U(t)$ (e.g. the displacements of a structure, relative to the ground), the probability associated to the non-occurrence of crossings of a generic barrier b into the time window T_S

Table 3: Parameters for the definition of the three-branch RS and PSD.

RS		PSD	
parameter	value	parameter	value
S_0	1.902 m/s ²	G_0	$9.2629 \cdot 10^{-3}(\text{m/s}^2)^2/(\text{rad/s})$
α	2.500	e_2	0.7609
T_3	0.070 s	e_3	-1.3164
T_2	0.351 s	e_4	-2.7216

Figure 5: (a) PSD function consistent with the three-branch RS (parameters listed in Table 3; (b) Pseudo-acceleration RS: target (black solid line), 10% tolerance (black dashed line) and generated RS obtained with proposed approach (red dots).

can be evaluated as:

$$P_S(b, T_S) = 1 - \exp\left(-\frac{b^2}{\lambda_{0,U}}\right) \exp(-\tilde{\alpha}_U(b) T_S), \quad (36)$$

where $\tilde{\alpha}_U$ is the so-called “risk function” for the random process $U(t)$, defined as:

$$\tilde{\alpha}_U(b) = 2\nu_U \frac{1 - \exp\left(-\sqrt{\frac{\pi}{2\lambda_{0,U}}} b q_U^{1.2}\right)}{\exp\left(\frac{b^2}{2\lambda_{0,U}} - 1\right)}, \quad (37)$$

in which the mean zero-crossing rate $\nu_U = \nu_U(\omega, \zeta)$ and the bandwidth factor $q_U = q_U(\omega, \zeta)$ have been defined in Eqs. (35b) and (35c), respectively. Note that, to simplify the notation, the explicit dependency on the properties of the SDOF oscillator, namely $T = 2\pi/\omega$ and ζ , has been omitted in Eqs. (36) and (37).

Importantly, since only the spectral moments of order up to two are involved, full statistical distribution of the peak values of the process can be determined in closed-form by using the analytical expressions proposed in Section 3.

For the case of SDOF oscillators and RS-consistent PSD functions depicted in Figure 3 (and parameters listed in Table 2), the proposed approach has been applied to determine the cumulative distribution function (CDF) of the peak values of the displacement process $U(t)$, varying the natural period of vibration of the system. Results, computed considering an observation time $T_S = 20$ s, are compared in Figure 7 with the outcomes of Monte Carlo simulations (MCS). The latter have been performed with 20,000 samples of ground acceleration, generated as realizations of a stationary, zero-mean Gaussian process, fully defined by the proposed PSD function. In order to analyze stationary response processes within the time observation window T_S , the time modulation function proposed by Jennings

Table 4: Parameters for the definition of the two-branch RS and PSD.

RS		PSD	
parameter	value	parameter	value
S_0	2.941 m/s ²	G_0	$2.6098 \cdot 10^{-2}(\text{m/s}^2)^2/(\text{rad/s})$
α	2.500	e_2	0.7526
T_2	0.400 s	e_3	-1.3250

Figure 6: (a) PSD function consistent with the two-branch RS (parameters listed in Table 4; (b) Pseudo-acceleration RS: target (black solid line), 10% tolerance (black dashed line) and generated RS obtained with proposed approach (red dots).

292 *et al.* [52] has been adopted:

$$\varphi_{\text{JHT}}(t) = \begin{cases} (t/t_1)^2, & t < t_1; \\ 1, & t_1 \leq t \leq t_2; \\ \exp(-(t-t_2)), & t > t_2, \end{cases} \quad (38)$$

293 where $t_1 = 5$ s and $t_2 = t_1 + T_S$. For each ground acceleration sample, the equation of motion of a linear SDOF system
 294 with assigned period and damping ratio has been numerically integrated with the 4th-order Runge-Kutta method.
 295 Finally, the relevant response statistics and the peak values of displacements, velocities and accelerations have been
 296 computed in the pseudo-stationary range $[t_1, t_2]$.

297 5. A case study: reliability analysis of a base isolated MDOF structure

298 Based on the formulation developed on the previous sections, a very wide range of structural problems can be tackled.
 299 In this section, for demonstration purposes, a base-isolated building taken from Ref. [42] is analyzed as case study.
 300 This structure is a five-story base-isolated building with irregular distribution of mass and stiffness, both in plan and
 301 in elevation. The damping ratio of the superstructure and isolated base are assumed to be $\zeta_s = 0.02$ and $\zeta_b = 0.12$,
 302 respectively. Detailed information on the spatial distribution of mass and stiffness can be found in Ref. [42] and, as
 303 much as possible, a notation consistent with Ref. [42] has been adopted in what follows.

304 The case study has been selected due to two main reasons: *i*) first, the classical RS method may lead in this case
 305 to an inaccurate evaluation of the dynamic response, mainly due to the empirical scaling of the spectral ordinates for
 306 different values of the viscous damping ratio; and *ii*) second, the case study allows highlighting the computational
 307 advantages of the proposed approach, in comparison with traditional numerical techniques.

Figure 7: CDF of peak displacements of SDOF systems excited by seismic acceleration characterized by the PSD having parameters in Table 2. Analytic approach vs Monte Carlo simulations.

In the following, the governing equation of the seismic response are determined by using a sub-structure modelling approach and a double transformation of coordinates, in order to reduce the computational burden. Then, the proposed formulation is applied for the determination of the spectral moments required for the reliability analysis.

5.1. Governing equations

Considering the isolated base and the superstructure as two dynamically interconnected systems, and assuming an in-plan rigid behavior of the 5 stories in elevation, the building possesses $n = 18$ degrees of freedom, i.e. three for each story in elevation plus three for the isolated base, collected in the super array $\mathbf{u}(t) = \left\{ \mathbf{u}_s^T(t) \quad \mathbf{u}_b^T(t) \right\}^T$.

Following the same reasoning as in Ref. [42], two independent transformations of coordinates are applied to the superstructure and the base:

$$\mathbf{u}_s(t) = \Phi_s \cdot \mathbf{q}_s(t); \quad \mathbf{u}_b(t) = \Phi_b \cdot \mathbf{q}_b(t), \quad (39)$$

where Φ_s and Φ_b are the $((n-3) \times \ell_s)$ and (3×3) real-valued modal matrices associated to superstructure and base, respectively, and $\mathbf{q}(t) = \left\{ \mathbf{q}_s^T(t) \quad \mathbf{q}_b^T(t) \right\}^T$ is an array collecting the $\ell = \ell_s + 3$ modal coordinates of the coupled dynamic system, ℓ_s being the number of modes retained for the superstructure only.

As the equations of motion are still coupled in the transformed vector space, a second transformation of coordinates is applied:

$$\mathbf{q}(t) = \psi(\alpha_g) \cdot \mathbf{y}(t), \quad (40)$$

where $\psi(\alpha_g)$ is a convenient square matrix of size ℓ , which depends on the epicentral direction of the seismic action, α_g , and requires the solution of a third real-valued eigen-problem, this time of a reduced size ℓ .

By neglecting any coupling in terms of damping forces for the resulting generalized modal coordinates in the vector $\mathbf{y}(t)$, the equations governing the seismic motion of this type of structures take the same form as Eq. (25), in which $\mathbf{p} = \{1, \dots, 1\}^T$ is conveniently defined as a unit vector, while the modal transformation matrix Φ can be expressed as:

$$\Phi = \begin{bmatrix} \Phi_s & \mathbf{0} \\ \mathbf{0} & \Phi_b \end{bmatrix} \cdot \psi(\alpha_g). \quad (41)$$

The matrices of spectral moments of order m in the geometrical space are, hence, determined by applying the procedure outlined in the previous section, i.e. Eqs. (26) to (31).

It should be stressed here that the two-stage modal transformation for base-isolated building structures returns a dynamic model which is classically damped, i.e. the damping forces do not couple the generalized modal coordinates, but the viscous damping ratio in the first three modes of vibration (associated with relatively large deformation in the seismic isolators) tends to be much higher than that in the higher modes of vibration.

Table 5: Modal participation masses for the analyzed base-isolated building.

Mode	Period [s]	Modal participation mass [%]	
		$\alpha_g = 0^\circ$	$\alpha_g = 90^\circ$
1	2.086	7.32	91.80
2	2.077	91.93	7.65
3	1.806	0.73	0.50
4	0.256	0.00	0.04
5	0.240	0.01	0.00
6	0.211	0.01	0.00

In the numerical application, the seismic analysis of the base-isolated building has been performed with reference to the RS already defined in Table 2 and for two epicentral directions, namely x and y , orthogonal to each other and parallel to the main structural frames in the superstructure (say, $\alpha_g = 0^\circ$ and $\alpha_g = 90^\circ$). The modal participation masses for the first six modes of vibration are reported in Table 5, considering both epicentral directions; then modes associated with relatively large displacements of the isolated base are embolden.

5.2. Reliability analysis

The statistical distributions of the peak values of the stories' horizontal displacements and rotations collected in the vector $\mathbf{u}(t)$ have been analytically determined with the approach presented in the previous sections. Results are first presented in Table 6 in terms of median peak values, where the degree of freedom $u_e^{(s)}$ stands for the horizontal displacement of the center of mass of the s th story in the direction e and $\theta_z^{(s)}$ is the rotation of the s th story about the vertical axis. The results offered by the proposed procedure, based on the calculation of the analytical spectral moments (ASM), are compared therein with the values obtained by the response spectrum method (RSM) using the CQC (complete quadratic combination) of the modal responses. As benchmark solutions, Monte Carlo simulations (MCS) have also been performed considering 20,000 samples with stationary observation time $T_S = 20$ s.

The proposed analytical method provides results which are much closer to MCS than the RSMn does; while in the first case the largest percentage errors are in the order of 17% maximum, the RSM percentage can exceed 300%. Furthermore, the computational effort required using the analytical formulae proposed in the previous sections is negligible when compared with the computational time needed for MCS.

Figures 8 and 9 plot the CDFs (cumulative distribution functions) of the peak horizontal displacements of the center of mass for all the stories of the building, including the isolated base, in both x and y direction and for both epicentral directions. The CDFs analytically determined with the proposed approach (ASM, dashed lines) are compared against the numerical ones, obtained through MCS (solid lines). As shown, the maximum percentage discrepancy between the two methods occurs for the displacement along the y direction due to a seismic excitation in the x direction. Furthermore, the percentage inaccuracies of the ASM are generally lower than those of the RSM (see Table 6). On the other hand, since the proposed method only requires the use of closed-form expressions, the computational effort required to determine the full statistical distribution of the peak response by ASM is negligible when compared to both MCS. From a practical perspective, the complete knowledge of the maximum peak displacements can be used for structural assessment purposes (e.g. comparing actual base displacements with isolator capacities), meaning that the rather significant improvement in computational efficiency makes the proposed method particular convenient for preliminary response analyses and parametric investigations requiring a large number of iterations.

6. Conclusions

In this paper, closed-form expressions for the spectral moments of the stationary Gaussian response of linear structures excited by spectrum-consistent seismic excitations have been determined. In particular, spectral moments up to the third order have been determined for SDOF (single-degree-of-freedom) oscillators and up to the second order for MDOF (multi-degree-of-freedom) structures. Noticeably, these are the spectral moment needed to estimate the peak

Figure 8: CDF of the peak displacements of the base-isolated building for epicentral direction x ; comparison between values by analytical spectral moments (ASM) and Monte Carlo simulations (MCS).

Figure 9: CDF of the peak displacements of the base-isolated building for epicentral direction y ; comparison between values by analytical spectral moments (ASM) and Monte Carlo simulations (MCS).

Table 6: Mean peak displacements [mm] and rotations [10^{-3} rad] for two orthogonal epicentral directions, obtained by Monte Carlo simulations (MCS), response spectral method (RSM) and proposed approach (ASM).

DOF	Epicentral direction x			Error [%]		Epicentral direction y			Error [%]	
	MCS	RSM	ASM	RSM	ASM	MCS	RSM	ASM	RSM	ASM
$u_1 = u_x^{(1)}$	1.37	1.40	1.25	2.1	-9.2	0.12	0.29	0.11	136.6	-6.7
$u_2 = u_y^{(1)}$	0.41	0.58	0.37	42.5	-8.7	1.77	2.70	1.60	52.6	-9.5
$u_3 = \theta_z^{(1)}$	0.04	0.05	0.04	34.2	-8.9	0.03	0.07	0.03	120.5	-4.8
$u_4 = u_x^{(2)}$	2.36	2.37	2.15	0.7	-8.8	0.18	0.44	0.17	142.6	-8.2
$u_5 = u_y^{(2)}$	0.75	1.00	0.66	34.2	-12.0	3.15	4.74	2.87	50.4	-9.1
$u_6 = \theta_z^{(2)}$	0.07	0.09	0.06	25.8	-12.2	0.05	0.10	0.04	118.4	-7.1
$u_7 = u_x^{(3)}$	2.63	2.64	2.40	0.3	-8.7	0.18	0.45	0.16	154.1	-8.6
$u_8 = u_y^{(3)}$	0.87	1.14	0.75	31.4	-13.3	3.59	5.38	3.27	49.7	-9.1
$u_9 = \theta_z^{(3)}$	0.08	0.10	0.07	23.6	-13.2	0.05	0.10	0.04	123.7	-7.9
$u_{10} = u_x^{(4)}$	3.00	2.98	2.74	-0.5	-8.7	0.14	0.41	0.13	198.6	-9.4
$u_{11} = u_y^{(4)}$	1.04	1.32	0.88	26.2	-15.8	4.22	6.26	3.84	48.4	-9.0
$u_{12} = \theta_z^{(4)}$	0.10	0.12	0.08	19.7	-14.3	0.04	0.09	0.03	141.8	-9.7
$u_{13} = u_x^{(5)}$	3.19	3.16	2.91	-1.0	-8.7	0.13	0.40	0.12	207.8	-11.6
$u_{14} = u_y^{(5)}$	1.15	1.41	0.95	23.2	-17.3	4.54	6.70	4.13	47.6	-9.1
$u_{15} = \theta_z^{(5)}$	0.11	0.12	0.09	17.2	-15.3	0.03	0.08	0.03	141.9	-11.4
$u_{16} = u_x^{(b)}$	70.31	78.43	66.17	11.6	-5.9	2.20	9.51	2.25	332.9	2.3
$u_{17} = u_y^{(b)}$	5.86	14.16	6.01	141.6	2.5	69.13	115.61	64.68	67.2	-6.4
$u_{18} = \theta_z^{(b)}$	0.51	1.13	0.52	124.7	3.4	0.44	1.53	0.46	245.5	4.1

response of both SDOF oscillators and MDOF structures using the classical Vanmarke's peak factor. While conventionally numerical integrations are required for this type of analysis, the proposed formulation delivers closed-form expressions for the probability distribution of the structural peak response.

Numerical validation of the proposed approach has been achieved by comparing the derived analytical expressions with the results of MCS (Monte Carlo simulations), studying several cases of SDOF oscillators with varying natural periods of vibration and viscous damping ratios, and considering also the case of a multi-story base-isolated building taken from the literature.

Based on the results of the numerical examples, the following key conclusions of general significance can be drawn:

- For the case of SDOF oscillators, excellent agreement has been found between the proposed closed-form expressions and the spectral moments calculated numerically. Only minor differences have been observed for relatively large natural periods of vibration (i.e. $T > 2$ s) and large viscous damping ratios (i.e. $\zeta > 0.10$), which are consistent with the assumptions underpinning the formulation itself.
- Using the analytical spectral moments to reproduce the target response spectrum returns excellent results. The difference between reproduced and target 4-branch response spectra (RS) are below the 10% threshold prescribed by Eurocode 8 for acceptance. This has been verified also for RS proposed by other international building codes, including those with only 2 and 3 branches, always providing excellent results.
- For SDOF oscillators, very good agreement has also been found between the probability distributions of the peak displacements determined by MCS and those obtained with the proposed formulation. Similar to the case of the spectral moments, minor discrepancies have only been observed for structural periods over two seconds.
- The case study performed on a base-isolated multi-story building (with isolation period $T_{iso} = 2.086$ s) has shown that, in comparison with MCS, the proposed approach entails negligible computational effort while achieving a level of accuracy that is either comparable or higher than the RSM (response spectrum method).

- The computational efficiency of the proposed analysis method, relying on closed-form expressions, particularly shines when utilized for processes involving a large number of iterations, such as preliminary parametric analyses or seismic design optimization through, for example, the application of heuristic methods.

References

- [1] Clough R, Penzien J. Dynamics of Structures. Civil Engineering Series; McGraw-Hill; 1993. ISBN 9780071132411.
- [2] Lin Y, Cai G. Probabilistic Structural Dynamics: Advanced Theory and Applications. Engineering Reference Series; McGraw-Hill; 2004. ISBN 9780071438001.
- [3] Ahmadi G. Generation of artificial time-histories compatible with given response spectra—a review. *Solid Mechanics Archives* 1979;4:207–39.
- [4] Kanai K. Semi-empirical formula for the seismic characteristics of the ground. *Bulletin of the Earthquake Research Institute* 1957;.
- [5] Tajimi H. A statistical method of determining the maximum response of a building structure during an earthquake. In: 2nd World Conference of Earthquake Engineering. Tokyo and Kyoto; 1960..
- [6] Falsone GaNF. Stochastic modelling of earthquake excitation following the EC8: power spectrum and filtering equations. *European Earthquake Engineering* 2000;39(1):3–12.
- [7] CEN . Eurocode 8: Design of structures for earthquake resistance - Part 1: General rules, seismic actions and rules for buildings, EN 1998-1. Brussels: Comité Européen de Normalisation; 2004.
- [8] Pfaffinger DD. Calculation of Power Spectra from Response Spectra. *Journal of Engineering Mechanics* 1983;109(1):357–72. doi:10.1061/(ASCE)0733-9399(1983)109:1(357).
- [9] Preumont A. The generation of spectrum compatible accelerograms for the design of nuclear power plants. *Earthquake Engineering & Structural Dynamics* 1984;12(4):481–97. doi:10.1002/eqe.4290120405.
- [10] Preumont A. A method for the generation of artificial earthquake accelerograms. *Nuclear Engineering and Design* 1980;59(2):357–68. doi:10.1016/0029-5493(80)90205-8.
- [11] Kaul MK. Stochastic characterization of earthquakes through their response spectrum. *Earthquake Engineering & Structural Dynamics* 1978;6(5):497–509. doi:10.1002/eqe.4290060506.
- [12] Sundararajan C. An iterative method for the generation of seismic power spectral density functions. *Nuclear Engineering and Design* 1980;61(1):13–23. doi:10.1016/0029-5493(80)90074-6.
- [13] Unruh JF, Kana DD. An iterative procedure for the generation of consistent power/response spectrum. *Nuclear Engineering and Design* 1981;66(3):427–35. doi:10.1016/0029-5493(81)90172-2.
- [14] Cacciola P, Colajanni P, Muscolino G. Combination of Modal Responses Consistent with Seismic Input Representation. *Journal of Structural Engineering* 2004;130(1):47–55. doi:10.1061/(ASCE)0733-9445(2004)130:1(47).
- [15] Crespi PG, Floris C, Paganini p. A Probabilistic Method for Generating Spectrum Compatible Earthquake Time Histories. *European Earthquake Engineering* 2002;16(3):3–17.
- [16] Cacciola P. A stochastic approach for generating spectrum compatible fully nonstationary earthquakes. *Computers and Structures* 2010;88(15-16):889–901. doi:10.1016/j.compstruc.2010.04.009.
- [17] Cacciola P, Zentner I. Generation of response-spectrum-compatible artificial earthquake accelerograms with random joint time-frequency distributions. *Probabilistic Engineering Mechanics* 2012;28:52–8. doi:10.1016/j.probengmech.2011.08.004.
- [18] Giaralis A, Spanos P. Wavelet-based response spectrum compatible synthesis of accelerograms—Eurocode application (EC8). *Soil Dynamics and Earthquake Engineering* 2009;29(1):219–35. doi:10.1016/j.soildyn.2007.12.002.
- [19] Giaralis A, Spanos PD. Derivation of response spectrum compatible non-stationary stochastic processes relying on Monte Carlo-based peak factor estimation. *Earthquakes and Structures* 2012;3(3.4):581–609.
- [20] Conte JP, Peng BF. Fully Nonstationary Analytical Earthquake Ground-Motion Model. *Journal of Engineering Mechanics* 1997;123(1):15–24. doi:10.1061/(ASCE)0733-9399(1997)123:1(15).
- [21] Gupta I, Trifunac M. Defining equivalent stationary PSDF to account for nonstationarity of earthquake ground motion. *Soil Dynamics and Earthquake Engineering* 1998;17(2):89–99. doi:10.1016/S0267-7261(97)00031-6.
- [22] Spanos P, Giaralis A, Politis N. Time–frequency representation of earthquake accelerograms and inelastic structural response records using the adaptive chirplet decomposition and empirical mode decomposition. *Soil Dynamics and Earthquake Engineering* 2007;27(7):675–89. doi:10.1016/j.soildyn.2006.11.007.
- [23] Cecini D, Palmeri A. Spectrum-compatible accelerograms with harmonic wavelets. *Computers & Structures* 2015;147:26–35. doi:10.1016/j.compstruc.2014.10.013.
- [24] Spanos PD, Giaralis A, Jie L. Synthesis of accelerograms compatible with the Chinese GB 50011-2001 design spectrum via harmonic wavelets: Artificial and historic records. *Earthquake Engineering and Engineering Vibration* 2009;8(2):189–206. doi:10.1007/s11803-009-9017-4.
- [25] Rezaeian S, Der Kiureghian A. A stochastic ground motion model with separable temporal and spectral nonstationarities. *Earthquake Engineering & Structural Dynamics* 2008;37(13):1565–84. doi:10.1002/eqe.831.
- [26] Priestley MB. Evolutionary spectra and non-stationary processes. *Journal of the Royal Statistical Society Series B (Methodological)* 1965;27(2):204–37.
- [27] Spanos PD. Non-stationary random vibration of a linear structure. *International Journal of Solids and Structures* 1978;14(10):861–7. doi:10.1016/0020-7683(78)90076-8.
- [28] Spanos PD, Vargas Loli LM. A statistical approach to generation of design spectrum compatible earthquake time histories. *International Journal of Soil Dynamics and Earthquake Engineering* 1985;4(1):2–8. doi:10.1016/0261-7277(85)90029-4.
- [29] Di Paola M, La Mendola L, Navarra G. Stochastic Seismic Analysis of Structures with Nonlinear Viscous Dampers. *Journal of structural engineering* 2007;133(10):1475–8. doi:10.1002/stc.254.

- [30] Di Paola M, Navarra G. Stochastic seismic analysis of MDOF structures with nonlinear viscous dampers. *Structural Control and Health Monitoring* 2009;16(3):303–18. doi:10.1002/stc.254.
- [31] Navarra G, Barone G, Lo Iacono F. Stochastic seismic analysis by using an analytical model of PSD consistent with Response Spectra. *Vienna Recent Advance in Earthquake Engineering and Structural Dynamics (VEESD2013)* 2013;2013(387):1–10.
- [32] Barone G, Lo Iacono F, Navarra G, Palmeri A. A novel analytic model of power spectral density function coherent with earthquake response spectra. In: M. Papadrakakis, V. Papadopoulos GS, editor. *Proceedings of the 1st International Conference on Uncertainty Quantification in Computational Sciences and Engineering (UNCECOMP 2015)*. Athens: Institute of Structural Analysis and Antiseismic Research School of Civil Engineering National Technical University of Athens (NTUA) Greece. ISBN 978-960-99994-9-6; 2015, p. 394–406. doi:10.7712/120215.4280.805.
- [33] Vanmarcke EH. Properties of spectral moments with applications to random vibration. *Journal of the Engineering Mechanics Division* 1972;98(2):425–46.
- [34] Vanmarcke EH. On the Distribution of the First-Passage Time for Normal Stationary Random Processes. *Journal of Applied Mechanics* 1975;42(1):215. doi:10.1115/1.3423521.
- [35] Vanmarcke EH. Chapter 8 - structural response to earthquakes. In: Lomnitz C, Rosenblueth E, editors. *Seismic Risk and Engineering Decisions*; vol. 15 of *Developments in Geotechnical Engineering*. Elsevier; 1976, p. 287 –337. doi:https://doi.org/10.1016/B978-0-444-41494-6.50011-4.
- [36] Miller KS, Rochwarger MM. On estimating spectral moments in the presence of colored noise. *IEEE Transactions on Information Theory* 1970;16(3):303–9. doi:10.1109/TIT.1970.1054460.
- [37] Der Kiureghian A. Structural response to stationary excitation. *Journal of the Engineering Mechanics Division* 1970;106(6):1195–213.
- [38] Di Paola M, Muscolino G. Analytic evaluation of spectral moments. *Journal of Sound and Vibration* 1988;124(3):479–88. doi:10.1016/S0022-460X(88)81389-0.
- [39] Giaralis A, Spanos PD. Effective linear damping and stiffness coefficients of nonlinear systems for design spectrum based analysis. *Soil Dynamics and Earthquake Engineering* 2010;30(9):798–810. doi:10.1016/j.soildyn.2010.01.012.
- [40] Spanos PD, Giaralis A. Third-order statistical linearization-based approach to derive equivalent linear properties of bilinear hysteretic systems for seismic response spectrum analysis. *Structural Safety* 2013;44:59–69. doi:10.1016/j.strusafe.2012.12.001.
- [41] Mitseas IP, Kougioumtzoglou IA, Giaralis A, Beer M. A novel stochastic linearization framework for seismic demand estimation of hysteretic MDOF systems subject to linear response spectra. *Structural Safety* 2018;72:84–98. doi:10.1016/j.strusafe.2017.12.008.
- [42] Muscolino G, Palmeri A, Versaci C. Damping-adjusted combination rule for the response spectrum analysis of base-isolated buildings. *Earthquake Engineering & Structural Dynamics* 2013;42(2):163–82. doi:10.1002/eqe.2200.
- [43] Chopra A. *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 5th edition. Always learning; Pearson; 2017. ISBN 9780134555126.
- [44] Rosenblueth E. A basis for aseismic design of structures. Ph.d. thesis; University of Illinois at Urbana-Champaign; 1951.
- [45] Wilson EL, Der Kiureghian A, Bayo EP. A replacement for the SRSS method in seismic analysis. *Earthquake Engineering & Structural Dynamics* 1981;9(2):187–92. doi:10.1002/eqe.4290090207.
- [46] International Code Council . *International Building Code – IBC2012*. USA; 2015.
- [47] Ministerio de Vivienda . E.030 – Reglamento Nacional de Edificaciones. Diseño Sismo-Resistente (Technical Standard of Building, Earthquake-Resistant Design Chapter. Peru: Ministerio de Vivienda; 2006.
- [48] Muscolino G, Palmeri A. Maximum response statistics of MDof linear structures excited by non-stationary random processes. *Computer Methods in Applied Mechanics and Engineering* 2005;194(12-16):1711–37. doi:10.1016/j.cma.2004.01.044.
- [49] Muscolino G, Palmeri A. Peak response of non-linear oscillators under stationary white noise. *Computers & Structures* 2007;85(5-6):255–63. doi:10.1016/j.compstruc.2006.10.005.
- [50] Muscolino G. *Dinamica delle strutture con fondamenti ed applicazioni di ingegneria sismica e dinamica aleatoria*. Bologna, Italy: Pitagora Editrice; 2012. ISBN 88-371-1858-9.
- [51] Ministero delle Infrastrutture e dei trasporti . D.M. 17.01.2018 - Aggiornamento delle “Norme tecniche per le Costruzioni”. Italy; 2018.
- [52] Jennings P, Housner G, Tsai C. Simulated earthquake motions for design purpose. In: *4th World Conference on Earthquake Engineering*; vol. A-1. Santiago (Chile); 1969, p. 145–60.

Appendix A. Analytical derivation of PSD parameters

This section reports the analytical derivation of the PSD parameters defined in Eqs. (6) to (10). The procedure is initially presented in detail for the exponent e_1 , and then a similar approach is applied in turn to derive the remaining parameters. In all cases, the starting point is Eq. (3), here reported for sake of clarity:

$$G_{\ddot{U}_g}(\omega) = \frac{\gamma}{\omega} \left[\left(\frac{S_{pa}(\omega, \xi)}{\eta_U(\omega, \xi)} \right)^2 - \int_0^\omega G_{\ddot{U}_g}(\hat{\omega}) d\hat{\omega} \right], \quad (A.1)$$

Appendix A.1. Determination of exponent e_1

To determine the first parameter, the strategy is to particularize Eq. (A.1) considering two points belonging to the same branch of the RS-consistent PSD ($0 \leq \omega \leq \omega_1$), and then algebraically manipulating the resulting mathematical

relationships. It is worth reporting here, as a reminder, the mathematical expressions of both the RS and its consistent PSD functions (see Eq. (5)), both written in terms of the circular frequency $\omega = 2\pi/T$:

$$S_{\text{pa}}(\omega) = \begin{cases} \alpha S_0 \left(\frac{\omega_1}{\omega_2}\right)^{k_1} \left(\frac{\omega}{\omega_1}\right)^{k_2}, & 0 \leq \omega < \omega_1; \\ \alpha S_0 \left(\frac{\omega}{\omega_2}\right)^{k_1}, & \omega_1 \leq \omega < \omega_2; \\ \alpha S_0, & \omega_2 \leq \omega \leq \omega_3; \\ S_0 \left[1 + (\alpha - 1) \frac{\omega_3}{\omega}\right], & \omega > \omega_3; \end{cases} \quad (\text{A.2a})$$

$$G_{\ddot{U}_g}(\omega) = \begin{cases} G_0 \left(\frac{\omega_1}{\omega_2}\right)^{e_2} \left(\frac{\omega}{\omega_1}\right)^{e_1}, & 0 \leq \omega < \omega_1; \\ G_0 \left(\frac{\omega}{\omega_2}\right)^{e_2}, & \omega_1 \leq \omega < \omega_2; \\ G_0 \left(\frac{\omega}{\omega_2}\right)^{e_3}, & \omega_2 \leq \omega \leq \omega_3; \\ G_0 \left(\frac{\omega_3}{\omega_2}\right)^{e_3} \left(\frac{\omega}{\omega_3}\right)^{e_4}, & \omega > \omega_3, \end{cases} \quad (\text{A.2b})$$

Setting $\omega = \omega_1$ into Eq. (A.1) leads to:

$$G_0 \left(\frac{\omega_1}{\omega_2}\right)^{e_2} = \frac{\gamma}{\omega_1} \left[\frac{\left(\alpha S_0 \left(\frac{\omega_1}{\omega_2}\right)^{k_1}\right)^2}{\eta_U^2(\omega_1, \zeta)} - \int_0^{\omega_1} G_{\ddot{U}_g}(\omega) d\omega \right]. \quad (\text{A.3})$$

Since the PSD function is given in analytical form, the integral term to the right-hand side can be solved in closed form, leading to:

$$G_0 \omega_2 \left(\frac{\omega_1}{\omega_2}\right)^{e_2+1} \frac{\gamma + e_1 + 1}{\gamma(e_1 + 1)} = \left(\frac{\alpha S_0}{\eta_U(\omega_1, \zeta)} \left(\frac{\omega_1}{\omega_2}\right)^{k_1} \right)^2. \quad (\text{A.4})$$

Following the same two steps, but considering now the frequency $\omega = \omega_1/\rho$ (with $\rho > 1$), the following expression is obtained:

$$G_0 \omega_2 \left(\frac{\omega_1}{\omega_2}\right)^{e_2+1} \frac{\gamma + e_1 + 1}{\gamma(e_1 + 1)} = \left(\frac{\alpha S_0}{\eta_U(\omega_1/\rho, \zeta)} \left(\frac{\omega_1}{\omega_2}\right)^{k_1} \right)^2 \rho^{e_1+1-2k_2}. \quad (\text{A.5})$$

Comparison of equations (A.4) and (A.5) leads to:

$$e_1 = \frac{\ln \left(\frac{\eta_U^2(\omega_1/\rho, \zeta)}{\eta_U^2(\omega_1, \zeta)} \right)}{\ln(\rho)} + 2k_2 - 1. \quad (\text{A.6})$$

In order to simplify the notation for the coming algebra, two new functions $A(\omega)$ and $B(\omega)$ are introduced as follows:

$$A(\omega) = 2\eta_U(\omega); \quad (\text{A.7a})$$

$$B(\omega) = \exp \left[-q_U^{1,2}(\omega) \sqrt{\pi \ln(A(\omega))} \right], \quad (\text{A.7b})$$

in which the explicit dependency on the viscous damping ratio ζ has been dropped to simplify the notation.
The Vanmarcke's peak factor $\eta_U(\omega)$ can now be rewritten in the form:

$$\eta_U(\omega) = \sqrt{2 \ln \{A(\omega) [1 - B(\omega)]\}} . \quad (\text{A.8})$$

Now, taking the limit of Eq. (A.6) with $\rho \rightarrow 1$, the following equation is obtained:

$$e_1 = 2 k_2 - 1 - L(\omega) , \quad (\text{A.9})$$

where:

$$L(\omega) = 2 \frac{P(\omega)}{\eta_U^2(\omega)} ; \quad (\text{A.10a})$$

$$P(\omega) = 1 + \frac{B(\omega) \sqrt{\pi} q_U^{1,2}(\omega)}{2 (1 - B(\omega)) \sqrt{\ln(A(\omega))}} . \quad (\text{A.10b})$$

On the other hand, taking advantage of the positions in Eqs. (A.7), the first derivative of the peak factor $\eta_U(\omega)$ can be written as:

$$\frac{d}{d\omega} \eta_U(\omega) = \frac{1}{\omega} \frac{P(\omega)}{\eta_U(\omega)} . \quad (\text{A.11})$$

Comparison of Eqs. (A.10) and (A.11) gives then:

$$L(\omega) = 2 \omega \frac{d}{d\omega} \ln(\eta_U(\omega, \zeta)) , \quad (\text{A.12})$$

confirming the validity of both eqs.(6) and (11).

Appendix A.2. Determination of exponent e_2 and parameter G_0

The closed-form expression for e_2 and G_0 are derived using a similar mathematical process, but considering in this case a new point positioned in the second branch of the PSD function, $\omega_1 \leq \omega \leq \omega_2$. First, based on the analytical PSD model, the following integral can be solved in closed form:

$$\int_0^{\omega_2} G_{\ddot{U}_g}(\omega) d\omega = G_0 \omega_2 \left[\frac{1}{1 + e_1} \left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} + \frac{1}{1 + e_2} \left(1 - \left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} \right) \right] . \quad (\text{A.13})$$

Then, rewriting Eq. (A.1) by setting $\omega = \omega_2$ and reordering all resulting terms, the following expression is obtained:

$$\frac{G_0 \omega_2}{\gamma} \left[\left(1 - \left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} \right) \frac{\gamma + e_2 + 1}{e_2 + 1} + \left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} \frac{\gamma + e_1 + 1}{e_1 + 1} \right] = \left(\frac{\alpha S_0}{\eta_U(\omega_2)} \right)^2 . \quad (\text{A.14})$$

The latter returns G_0 as a function of the exponents e_1 and e_2 , that can be rewritten in the form used for Eq. (10):

$$G_0 = \frac{\gamma}{\beta_2 \omega_2} \left(\frac{\alpha S_0}{\eta_U(\omega_2)} \right)^2 , \quad (\text{A.15})$$

where the coefficient β_2 has been given in Eq. (12).

To derive the exponent e_2 in closed-form, Eq. (A.1) is rewritten for $\omega = \omega_2/\rho$ (assuming $1 < \rho < \omega_2/\omega_1$):

$$G_0 \omega_2 \left[\left(\frac{\omega_1}{\omega_2} \right)^{e_2+1} - \frac{1}{\rho^{e_2+1}} \right] \frac{\gamma + e_2 + 1}{\gamma (e_2 + 1)} = \left[\frac{\alpha S_0}{\eta_U(\omega_1)} \left(\frac{\omega_1}{\omega_2} \right)^{k_1} \right]^2 - \left[\frac{\alpha S_0}{\eta_U(\omega_2/\rho)} \right]^2 \frac{1}{\rho^{2k_1}} . \quad (\text{A.16})$$

By using Eq. (A.14) and Eq. (10), and simplifying opportunely, the following relationship is obtained:

$$e_2 + 1 = -\frac{1}{\ln(\rho)} \ln \left\{ 1 + \beta_2 \frac{e_2 + 1}{\gamma + e_2 + 1} \left[\left(\frac{\eta_U(\omega_2)}{\rho^{k_1} \eta_U(\omega_2/\rho)} \right)^2 - 1 \right] \right\} . \quad (\text{A.17})$$

536 The limit for $\rho \rightarrow 1$ of Eq. (A.17) leads to:

$$e_2 + 1 = -\frac{e_2 + 1}{\gamma + e_2 + 1} \beta_2 (L(\omega_2) - 2k_1). \quad (\text{A.18})$$

537 Finally, by approximating that:

$$\beta_2 \cong \frac{\gamma + e_2 + 1}{e_2 + 1}, \quad (\text{A.19})$$

538 the closed-form expression for e_2 reported in Eq. (7) is obtained.

539 *Appendix A.3. Determination of the exponent e_3*

540 For the evaluation of the exponent e_3 , a point of the third branch is considered. Eq. (A.1) is then rewritten at the
541 frequency $\omega = \rho \omega_2$ (again, with $\rho > 1$):

$$G_0 \omega_2 (\rho^{e_3+1} - 1) \frac{\gamma + e_3 + 1}{\gamma (e_3 + 1)} = \left(\frac{\alpha S_0}{\eta_U(\rho \omega_2)} \right)^2 - \left(\frac{\alpha S_0}{\eta_U(\omega_2)} \right)^2. \quad (\text{A.20})$$

542 Substitution of Eq. (10) into Eq. (A.20) leads to:

$$e_3 + 1 = \frac{1}{\ln(\rho)} \ln \left\{ 1 + \beta_2 \frac{e_3 + 1}{\gamma + e_3 + 1} \left[\left(\frac{\eta_U(\omega_2)}{\eta_U(\rho \omega_2)} \right)^2 - 1 \right] \right\}. \quad (\text{A.21})$$

543 Performing the limit for $\rho \rightarrow 1$, as done in the previous subsection, the closed-form expression for the exponent e_3 is
544 obtained, as reported in Eq. (8).

545 *Appendix A.4. Determination of the exponent e_4*

546 The determination of the closed-form expression for the last exponent follows the same reasoning presented for the
547 previous cases, considering Eq. (A.1) for two frequencies $\omega = \omega_3$ and $\omega = \rho \omega_3$ (again, with $\rho > 1$), that is positioned
548 on the last branch of the PSD function. The algebra, slightly more cumbersome than in the previous cases, leading to
549 the expression:

$$e_4 + 1 = \frac{1}{\ln(\rho)} \ln \left\{ 1 + \beta_3 \frac{e_4 + 1}{\gamma + e_4 + 1} \left[\left(\frac{\rho + \alpha + 1}{\rho \alpha} \right)^2 \left(\frac{\eta_U(\omega_3)}{\eta_U(\rho \omega_3)} \right)^2 - 1 \right] \right\}, \quad (\text{A.22})$$

550 where β_3 has the expression reported in Eq. (13). Finally, by considering the limit $\rho \rightarrow 1$ and simplifying opportunely,
551 the closed-form expression for the exponent e_4 , reported in Eq. (9) is obtained.