To appear in the *Journal of Applied Statistics*
Department of Surgical Outcomes and Analysis, Kaiser Permanente
Vol. 00, No. 00, Month 20XX, 1–6

To appear in the *Journal of Applied Statistics*
Vol. 00, No. 00, Month 20XX, 2–6

Checking Identifiability of Covariance Parameters in Linear Mixed Effects Models Supplementary Material

Wei Wang

(Received 00 Month 20XX; accepted 00 Month 20XX)

1. Histograms of $\hat{\sigma}$ and $\hat{\rho}$ (MLE)

Email: wei.wang@kp.org

Figure 1. Maximum likelihood estimated covariance parameters in model A. Vertical lines indicate the true parameter values: $\sigma = \sqrt{0.6} \approx 0.77$, and $\rho = 0.4$.

Figure 2. Maximum likelihood estimated covariance parameters in model B. Vertical lines indicate the true parameter values: $\sigma = \sqrt{0.6} \approx 0.77$, and $\rho = 0.5$.

2. Histograms of $\hat{\sigma}$ and $\hat{\rho}$ (REML)

Figure 3. REML estimated covariance parameters in model A. Vertical lines indicate the true parameter values: $\sigma = \sqrt{0.6} \approx 0.77$, and $\rho = 0.4$.

Figure 4. REML estimated covariance parameters in model B. Vertical lines indicate the true parameter values: $\sigma = \sqrt{0.6} \approx 0.77$, and $\rho = 0.5$.

3. Coverage plots of ρ

Figure 5. Coverage plot of the parameter ρ in model A. True value is $\rho = 0.4$.

Figure 6. Coverage plot of the parameter ρ in model B. True value is $\rho = 0.5$.

4. Proofs

In our study, a matrix Σ with parameter θ is obtained through the map $\Sigma = \sum_{k=1}^m \theta_k \mathbf{E}_{\mathcal{I}_k}$ with the domain $\Theta = \{\theta : \Sigma \in \mathcal{S}_+^n\}$. We first show that the parameter space Θ is open in $\mathcal{R}^m$. This result helps to establish the necessary conditions of model identifiability

in the consequent results. The openness is immediate to visualize for some structured Σ , e.g. an MI- Σ . For other structures such as UN or Toeplitz, the verification is not straightforward. Although it is well known that $\mathbf{S}_+^n$ is an open subset of $\mathbf{S}^n$ [1, page 55], the set $\mathbf{S}_+^n$ itself is too general for our discussion since each Σ_j is parameterized by θ_j and $\theta = (\theta'_1 \cdots \theta'_J)'$ is the target of the study. In general, we have the following result. By the lemma, each parameter space Θ_j is open in $\mathcal{R}^{m_j}$. Hence, the space $\Theta = \Theta_1 \times \cdots \times \Theta_J$ of θ is open in $\mathcal{R}^{\sum_{j=1}^J m_j}$.

LEMMA 4.1 $\Theta = \{\theta : \Sigma \in \mathcal{S}_+^n\}$ is an open set of $\mathcal{R}^m$.

Proof of Lemma 4.1:

We consider the Euclidean norm $\|\cdot\|$ of θ . Let θ be an arbitrary point of Θ . We define a ball centered at θ with radius r as $B(\theta, r) = \{\mathbf{x} : \|\theta - \mathbf{x}\| < r\}$. We need to show that given θ , there exists an r such that $B(\theta, r) \subset \Theta$.

Since $\theta \in \Theta$, all the leading principal minors of Σ are positive. For a given positive leading principal minor, θ lies in an open set (not necessarily Θ). There is a radius such that a ball with this radius centered at θ is in this open set. Considering all the leading principal minors together, there exists a minimum radius enabling a ball to be in all these open sets. That is, every point of Θ is an interior point. $\square$

Proof of Theorem 3.1:

We prove that the model is not identifiable if and only if elements of $\mathcal{F}$ are linearly dependent. Let $\mathbf{c}_j$ be an arbitrary vector of length m_j with elements c_{jk} , $k = 1, \dots, m_j$. Let $\mathbf{c} = (\mathbf{c}'_1 \cdots \mathbf{c}'_J)' \neq \mathbf{0}$. Suppose $\sum_{j=1}^J \mathbf{Z}_j \left(\sum_{k=1}^{m_j} c_{jk} \mathbf{E}_{\mathcal{I}_k}^j \right) \mathbf{Z}'_j = \mathbf{0}$. Linear dependence of elements of $\mathcal{F}$ is equivalent to $\mathbf{c} \neq \mathbf{0}$. Define $\theta^* = \theta + \mathbf{c}$. It is clear that both θ^* and θ produce the same Σ_y . Let d be a constant and $\theta^* = \theta + d\mathbf{c}$. By Lemma 4.1, $\theta^* \in \Theta$ for small enough $|d|$. $\square$

Proof of Corollary 3.1:

We have $\text{vec} \Sigma_y = \mathbf{V}\theta$. Define $\theta^* = \theta + \mathbf{c}$ where $\mathbf{c}$ is arbitrary. Suppose $\text{vec} \Sigma_y = \mathbf{V}\theta^*$. Then, $\mathbf{V}$ of full column rank is equivalent to $\mathbf{c} = \mathbf{0}$. For small enough $|d|$, $\theta^* = \theta + d\mathbf{c} \in \Theta$ by Lemma 4.1. $\square$

Proof of Theorem 3.2:

Proof of the theorem is very similar to the proof of Corollary 3.1, and is omitted. $\square$

References

- [1] A. Berman. *Cones, Matrices and Mathematical Programming*. Springer-Verlag Berlin, 1973.