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A dialog in the footsteps of the book: "A Journey in Mathematics Education Research – Insights from the Work of Paul Cobb" (2011)

Edited by Erna Yackel, Koeno Gravemeijer, & Anna Sfard.

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A Prologue

When we were asked to write a review on Paul Cobb's new book, we willingly and happily accepted the invitation and challenge, and sat down to read the book. We had found a book in which the *main author*, Paul Cobb, the three editors, Erna Yackel, Koeno Gravemeijer and Anna Sfard, and the other contributors in the book seemed to interweave their efforts and wisdom to bring into realization a very ambitious goal—discussing and drawing a meaningful developmental trajectory in mathematics education research, in the last decades, through the insightful work of Cobb and his colleagues. As Greeno expressed it in his forward to the book: “Cobb's trajectory is a prototype of the field's progress” (2011, p. vii).

The book is structured in an interesting and systematic way, intended to emphasize the above goal. It has six parts, which present *main stations* in Cobb's and colleagues' journey within mathematics education research. In each part there are two chapters: In each case the second chapter consists of an already published or an expansion of a paper from the past (The interval among the different six papers is 1983-2007). Each of the six papers was written by Cobb, or by Cobb and colleague/s and was chosen to represent in the best way a *station* in the journey. The six titles are: Radical Constructivism, Social Constructivism, Symbolizing and Instructional Design, Developing instructional Sequences to Support Students' Mathematical Learning, Classroom Mathematical Practices, Diversity and Equity, and The Institutional Setting of Mathematics Teaching and Learning.

The first chapter in each part is a thoughtful introduction, written especially for this part of the book by Cobb alone, or by Cobb with colleague/s (often with co-author/s of the reprinted paper in the same part). These introductory chapters interweave the different parts together to form a whole book which indeed demonstrates an insightful *Journey in Mathematics Education Research* of Paul Cobb and colleagues' work. The introductory chapters describe the advancements with time of the journey: development and changes in approaches, goals, methodologies and the constructed theories. These chapters explain and also portray the needs of the researchers, which evolved through their investigations, and the various external resources which had an effect on their work and gave a solution to their needs. In this way the introductory chapter and the reprinted paper in each part complement each other to form a part of the Journey, a vivid, meaningful and paradigmatic

section of the journey. For example, in Part II, Social Constructivism, the introductory chapter by Paul Cobb with Erna Yackel (pp. 33-39), examines with scientific integrity the influence of work outside mathematics education as compared with that of mathematics education researchers Bauersfeld (1980) and Voigt (1985) (p. 36). This illuminates the next step of collaboration between the American team (Paul Cobb, Erna Yackel and Terry Wood) and the German researchers (Heinrich Bauersfeld, Jorg Voigt and Gotz Krummheuer), with the aim of "integrating social and cognitive perspectives on mathematical learning", an important and well understood advancement" (p. 37).

At the beginning of the book, there is also a Foreword by Greeno (pp. vii-xii) and an Introduction by Yackel and Gravemeijer (pp. 1-5), which complement one another, as a review for the book. The book ends with Anna Sfard's Epilogue: "On the Importance of Looking Back" (p. 231).

Reading these three chapters we felt that there was no point in writing a classic review of the book, and therefore decided (a) to try illuminating the perspectives of the many mathematics education researchers who were influenced by Paul Cobb's work, and who often found an insight or answer to their research need in a station or stations of Cobb's Journey to date; (b) to try to have some authentic examples which demonstrate the above influential relationships; and (c) to have our common writing work as a dialogue between both of us in a free and flexible way, knowing that what we write are only some first *seeds* of a discussion. These seeds might grow and have life in our dialogue, or they might be completely neglected. The following is the outcome of our collaboration.

Rina and Barbara

Dear Rina

As we agreed, I am starting our discussion of the book we have just read relating to the work of Paul Cobb. It has been very interesting for me to read these chapters and recall stages in my own work to which they relate and ways in which Paul's writings were of influence. I have read the book in sequence, although the editors make clear that it is possible to dip in and read at almost any point. I found the metaphor of *journey* especially vivid in characterizing this chronological sequence.

Shifts in perspectives and theory

Paul's early work, with von Glasersfeld and Steffe, was rooted in radical constructivist theory and focused on the researcher as teacher and model builder, trying to discern children's mathematical constructions through a clinical interview process. A 1983 paper (Cobb & Steffe, 1983) took me back into those early days of radical constructivism and the constructs promoted

by Ernst von Glasersfeld and others. A desire to gain access to children's mathematical thinking and associated mental constructions motivated this research and the theory behind it, drawing considerably on the work of Piaget. In the book, such published papers are interspersed with new writing from Paul Cobb and colleagues, reflecting on the (historical) pieces and setting ideas. One important theme for me was development of theory and concomitant methodology. Paul and other colleagues extended the research with Steffe and von Glasersfeld from the clinical interview model with individual students, with the researcher acting as teacher, to the first of a series of classroom experiments, based originally in cognitive constructivism. In the book, these are addressed globally from within a paradigm of design research, whereas in work published at the time (Wood, Cobb, Yackel, & Dillon, 1993) the theoretical focus is described as taking initially a "psychological constructivist position" (1993, p. 1), considering that "mathematical learning is a constructive, problem-solving process in which experience, activity, and communication are essential" (1993, p. 3). As it became clear that a psychological perspective alone would not account for children's learning, the research team "widened our purview to include a sociological perspective" (1993, p. 1). In Chapter 2 of the book, the introduction to Part I, Paul writes about a shift in his epistemological stance over the years from radical constructivism to a pragmatic realist perspective, based on Dewey, Bernstein, Rorty, and Putnam (p. 14). Pragmatic realism "leads us to acknowledge that we make choices when we adopt a particular theoretical perspective"; it "underscores that such choices need to be justified and that the justification should be pragmatic in nature" (p. 16).

It therefore seems appropriate in the book that the first two parts are headed respectively Radical Constructivism (Part I in the book) and Social Constructivism (Part II in the book); the former dealing with the clinical interview research and the latter with the classroom experiments. Here we gain insight to the development of ideas about the widely cited socio-mathematical norms in classroom practice.

Paul writes that originally the research team differentiated between social aspects of the classroom and the cognitive aspects that included students' mathematical reasoning. They came to a view that any aspect of the classroom can be analyzed from either a social or a cognitive perspective. The coordination of these perspectives became central to a so-called emergent perspective, emphasizing that "the classroom social content or micro-culture emerges from (and is continually regenerated by) the teacher's and students' coordinated actions" (p. 38).

The paper which is reproduced in Part II as chapter 5 (Cobb, Yackel, & Wood, 1989), was originally a chapter in a book on Affect and Mathematical Problem Solving (McLeod &

Adams, 1989). It focuses on “Children’s emotional acts while engaged in mathematical problem solving.” In the chapter, Cobb, Yackel and Wood draw on their year-long teaching experiment in a second grade classroom, in which the teacher used a problem-solving approach engaging children in solving problems together and providing reasons for their solutions: “The general instructional goals included the promotion of intellectual and moral autonomy and task involvement as a form of motivation” (p. 41). The authors state that they subscribe to the cognitive and constructivist approach to emotions (p. 42) and go on to address classroom *social norms*, writing that their analysis of emotional acts must be placed in the context of the social norms created in the classroom: “These acts must be placed in the social context within which they were performed and within which they take on meaning” (p. 43). In introducing this chapter, the authors refer to the classroom experiment as “what we would now call the classroom design experiment” (p. 34).

Part III of the book takes us overtly into considerations of instructional design. So, we see in this progression of research both a chronological development of ideas about discerning children’s mathematical construction and providing opportunity for such construction (the design of activity) and a parallel theoretical development in which the cognitive shades into the social and the two become merged in classroom complexity. The pragmatic epistemology underpinning theory seems to imply that theory has to fit with practice and explain realistically what is emerging from practice.

I’ll leave it there for the moment Rina and look forward to your first thoughts from your reading.

Regards

Barbara

Dear Barbara (2. 8. 2011)

In your message you wrote: I found the metaphor of journey especially vivid in characterizing this chronological sequence (of Cobb and his colleagues' book chapters). I would like to

continue this point regarding Cobb's and his colleagues' journey, and to raise some additional points of view.

In his foreword, Greeno, uses a similar metaphor and says that:

Cobb's trajectory is a prototype of the field's progress (p.vii).

And later on he explains: "Cobb and his colleagues have travelled along a conceptual and methodological pathway, and the opportunity this book gives us to travel their route provides valuable insights into important features, structures and resources of the domain of mathematics education research" (p. viii).

I completely agree with the above paragraph and would like to further expand this point through a related example. I would like to demonstrate through the example the value of providing these "important insightful features, structures and resources" to us—people in the mathematics education research community, who needed it, in order to analyze, express, and present (theoretically and methodologically) what we have found, while focusing on our own data, evolving out from our own research situations and goals. In other words, I would like to focus, at least partially, on "the second purpose of the book", as it is described by Yackel and Gravemeijer in their comprehensive introduction for the book as a whole (Chapter 1), and for each part in particular: "... to demonstrate with this particular body of work how exemplary educational research is conducted, how it evolves, and why it useful" (p. 2).

I will do that mainly through the lenses of *the user*, who found in the theories and methodologies evolved along the years, collected and illuminated in this book, support and resource for modification upon need. For authentic reasons I will borrow an example from the work of my colleagues and myself.

The inspiration and support of Cobb and his colleagues' work to others—an example

Not all mathematics educators and researchers in the field were lucky or wise enough to have a similar *wide angle journey* as the one illustrated in the book's chapters. For example, because of the pressure and the necessity which practitioners in many countries have experienced to create and implement a curriculum in schools, and to educate teachers quite quickly and on large scale, their own journey is different from Cobb's and his colleagues' journey. Even if some of the *milestones* (stations as they are called by Yackel & Gravemeijer, p. 2), along the different journeys were quite similar, the chronological order of the milestones was different and occasionally even in an opposite direction. Yet, in many cases, Cobb and his theoretical and methodological models were there for these educators, and provided them with

the most relevant frames, by which they were, and still are, able to think, theorize, and interpret their own data, and more importantly, to modify these models to their own research situation.

Through the following example I try to illuminate a reality in which, the insightful features of Cobb and his colleagues' work inspired and practically helped in our search for frames and methodologies for presenting and interpreting our own data.

Example: Socio-mathematical norms in a computerized learning environment.

Discussions concerning norms and socio mathematical norms appear mainly at Parts II, III and IV of Cobb's book, and express the way these notions were developed and became part of the scientific language in our field. For example, more than 15 years ago Voigt (1995), in Cobb's and Bauersfeld's book, writes:

"The *terms mathematical norm or socio-mathematical norms* are used here to describe a criteria of values with regard to mathematical activities. Socio-mathematical norms are not obligations that students have to fulfil; they facilitate the students' attempts to direct their activities in an environment providing relative freedom for interpreting and solving mathematical problems." (p. 196)

Social norms and *socio-mathematical norms* are notions within the process in which the emergent perspective framework evolved (Yackel & Cobb, 1996). As you said above, Barbara, the emergent perspective framework is a powerful framework for describing socio-cognitive development within classroom and was established upon the need to better understand and interpret what was observed in the mathematics classroom. The following is the current description of the above process in Cobb with Yackel's (2011) words (Chapter 3, Part I):

"... we differentiated between what we termed the social aspects of the classroom, which included classroom social norms, and the cognitive aspects that included students' mathematical reasoning. We subsequently revised this view while developing the notions of socio-mathematical norms and classroom mathematical practices, and in doing so questioned our assumption that some aspects of the classroom are inherently social and other aspects are inherently cognitive. We instead came to the view that any aspect of the classroom can be analyzed from either a social or a cognitive perspective. This revised view is central to the resulting emergent perspective as it involved the explicit coordination of a social perspective on classroom events with a cognitive perspective on the teacher's and students' individual interpretations as they participate in those events." (p. 37)

In 1993, we (Hershkowitz et al, 2002) had started the CompuMath curriculum development, implementation and research project for the three years of junior high school level. One of the

main characteristics of the project was integrating of interactive computerized learning environments into the inquiry type mathematics classroom. We were familiar theoretically, with the potential of computerized learning environments and thought that we could hypothesize what might happen in such environments, and we had the goal to create an existence example of what can happen, through which we will investigate how it happens. The team thus has to deal with many faces of theory, research and practice of development and implementation, where practices are fed by theory and research, and vice versa. Vivid cycles of design—classroom research—redesign became the day to day life of the project activity.

From quite an early stage, we were carried away by the exciting and surprising learning processes we observed, and by the extent to which they differed from what we had observed during the previous two decades of our development and research work. What started as naïve observation and documentation by taking field notes, was soon transformed into coherent research with videotape documentation and detailed analysis and interpretation. The need to describe, understand, explain and analyze what was going on in these classrooms naturally brought us closer to the concerns of socio-cultural psychology. Like many others (c.f. Perret-Clermont, 1993; Yackel & Cobb, 1996), we felt the shortcomings of cognitive theories, methodologies and tools we had at our disposal to describe and interpret learning and teaching processes in the classroom. Research became a crucial component in the curriculum development activity.

At this stage we had found that the emergent perspective (Yackel & Cobb, 1996) fits very well, enriches and triggers our thinking and our theoretical and practical approaches. The socio-mathematical norms notion was especially useful for us in describing parts from the innovative changes that were observed in the classroom, both from cognitive and social points of view, (Hershkowitz & Schwarz, 1999). The context differed from the context of Cobb and his colleagues (Cobb & Bauersfeld, 1995)—junior-high-mathematics rather than Grade two mathematics, computerised learning environment rather than paper and pencil learning environment. So, obviously we had to reconsider many contextual issues such as the role of tools in the classroom's activity, for example, what is the direct potential of multi-representational software in learning functions, or, the role of non-verbal interactions between students and tools in classroom activities (Hershkowitz & Schwarz, 1999, p. 151). An additional very important question to investigate was the role of such a rich learning environment, consisting of some holistic computerized tools and their potential power to assist students' mathematization processes.

By analyzing and interpreting our data within the frame of socio-mathematical norms, we were able to put our finger on socio-mathematical norms which are typical in such computerized environments of mathematics learning and to describe and explain how these norms were created in classrooms (Hershkowitz & Schwarz, 1999).

This example demonstrates the wide meaning of socio-mathematical norms and their flexible usefulness: The socio-mathematical norms in the mathematics classroom, which were framed in Cobb's and his colleagues' work, can be modified for many different mathematics learning contexts and learning environments, and in each of them the socio-mathematical norms might have different features which describe what is going on in that classroom from socio-cognitive points of view.

With my best wishes

Rina

Dear Rina (26.9.11)

It is interesting to see how the metaphor of journey, and the various stations of the journey, emerges from our reading of this book as something central to which we relate in our own work. Your reflections on your CompuMath project resonate strongly with experiences from my work with colleagues in Norway on our projects Learning Communities in Mathematics (LCM) and Teaching Better Mathematics (TBM), in which we were didacticians in a university working with teachers in schools. You write "From quite an early stage, we were carried away by the exciting and surprising processes we observed, and by the extent to which they differed from what we had observed during the previous two decades of our development and research." This was also true in our projects. It is at least partially due to the ambitious aims of these projects, although the degree of ambition only became clear as the projects progressed. A major factor in our projects was the aim to work with teachers as partners, with each group bringing important knowledge to the relationship and each group having responsibilities related to their particular goals and associated tasks.

The first project, LCM, was designed by the didacticians who invited schools and teachers to join. This meant that ownership' rested originally with didacticians despite a rhetoric of partnership. An outcome was that partnership was an emergent quality in the project and had considerable associated learning for both groups (Jaworski, 2008). How this learning played out in the progress of the LCM project was central to the development we observed and an

important factor in conceptualizing the second project, TBM, together with teachers and school leaders to enable joint ownership.

I see these factors relating strongly to what I read in the book. Three factors (at least) are relevant:

- The nature of the roles of the participants' (teachers, university academics, researchers) in research activity and development of practice.
- The emergent nature of the developmental process with context as a central but elusive factor.
- The development of theory in and with practice.

Cobb and Steffe, (1983) write:

"We believe that nothing could be more useful to teachers than the type of knowledge represented by a model. However, just as children construct mathematical knowledge, so teachers construct their own understanding of children's mathematical realities. We can no more give teachers our counting-type model than we can give children our knowledge that subtraction is the inverse of addition. The question of how to help teachers as they strive to understand children's mathematical realities is of critical importance. "(p. 29)

In 1983, at the heart of constructivist epistemology in mathematics education, this was a very significant observation. The LCM project was conceptualized in 2003, 20 years later. What did we learn in those 20 years and how has this played out in our research into mathematics teaching? The journey charted by Cobb and colleagues is illuminating on the question articulated here, namely, how to help teachers as they strive to understand children's mathematical realities?

The question is presented as part of the station of radical constructivism (Part I in the book). The next stations, following this first one, are social constructivism and symbolizing and instructional design. In the journey portrayed we see radical constructivism developing into social constructivism, and the methodology of classroom experiment evolving into a process of instructional design with a methodology rooted in a design research. However, the design research process was not clearly articulated until well into the new millennium (see, e.g., Cobb et al., 2003). Thus, during these 20 years we see important theoretical and methodological advances, with the generation of new knowledge which has been widely influential.

I believe that “how to help teachers to ...” is a fundamental question for teacher educators in the education of practicing teachers at all levels of education. My expectation is that all mathematics teacher educators have addressed this question during their practice. As editor of *JMTE (Journal of Mathematics Teacher Education)* for some years, I read many paper submissions in which educators reported on their research in programmes in which they sought to promote some models or forms of practice with teachers and the teachers’ concomitant learning in these programmes. It surprised me that rarely did I find the teacher educators themselves reporting on their own learning and development in such programs (cf. Chapman, 2008). It seemed to be taken for granted that the educator knew what was needed and the teachers needed to learn this. The success of the programme was evaluated in terms of the degree to which teachers learned and were able to implement their new learning.

One of the most significant outcomes of the LCM project was the learning of the didacticians as they worked with teachers. We had a lot to learn about teachers’ epistemologies, the social context of schools, and teachers’ working practices, all interrelated. Correspondingly, teachers had to learn about didacticians’ epistemologies, theoretical perspectives, and practical insights, all interrelated. We see here centrally both social and contextual factors in the interactions between these groups of people, their roles in developmental activity and the ways in which their differing epistemologies played out in relation to each other.

Cobb with Yackel (2011, p. 38) write about emergent perspectives. They refer to stages of their journey in which classroom social norms were contrasted with cognitive aspects of students’ mathematical reasoning, referring to their earlier writing (Cobb, Yackel and Wood, 1989, reprinted as Chapter 5 of the book) that “some aspects of the classroom are inherently social and other aspects are inherently cognitive”. They report going on to question this assumption:

"We instead came to the view that any aspect of the classroom can be analyzed from either a social or a cognitive perspective. This revised view is central to the emergent perspective as it involved the explicit coordination of a social perspective on classroom events with a cognitive perspective on the teachers’ and students’ individual interpretations as they participate in those events." (p. 38)

We see here a deep analysis of the researchers’ growth of understanding about classroom processes and an associated theoretical development. The classroom research projects on which they report show paradigmatic examples of students’ learning of mathematics and teachers’ implementation of classroom processes associated with developing socio-mathematical norms;

they show also how the university-educator-researcher-theorists themselves developed knowledge and understanding. While many mathematics educators around the world have drawn on Cobb et al.'s theory of classroom mathematical norms and its importance for teachers and classrooms, I suggest that not so many have recognized the importance of the emergent process of synthesis and theorizing which typify the journey articulated in this book.

Just one more insight at this stage, Rina, before I pass back to you again, and this concerns the design experiment methodology which emerged from classroom experiments. Cobb, together with Gravemeijer and Yackel (Chapter 6, Part III, pp. 75-82) reflect on their accommodation of heuristics for instructional design (the Realistic Mathematics Education, RME model) developed at the Freudenthal Institute in the Netherlands with the theoretical development mentioned above. The account given of the merging of these two highly significant areas of theory is worth reading for its own specific detail. However, what seems fundamental for me is what Cobb et al. report as being central to the learning they experienced:

"As the US researchers came to realize, it is one thing to attempt to understand a sophisticated form of practice such as RME by reading about it and quite another to become competent in developing instructional designs for supporting students' mathematical learning ... we had to co-participate in the process of formulating, testing and revising designs before we could begin to contribute to the development of adequate designs." (p. 82)

These words reflect the important relations between developing practice and developing theory. These authors can only write about such sophisticated forms of theory-related practice with the depths of insight we see here by having been immersed in both development of practice and development of theory in integrated ways. These insights seem to me to have consequences for how we work with teachers in developmental projects to improve teaching and learning. In Chapter 7, Cobb speaks of their design research projects as involving the teacher such that "the collaborating teacher is a full member of the research and development team" (p. 87). I am curious to know what it meant for the teacher to be a "full member" of the team, how this worked out in practice, and what learning ensued for both the teacher and other members of the team. In LCM, it was our experience that such learning reflected importantly on the design of activity and that the re-design of activity, consequent on this learning, led to new learning for both teachers and didacticians in our project. The impact showed in the developing practice of both groups of practitioners and also in the theoretical development on which we have reported (e.g. Jaworski, 2006, 2008; Jaworski & Goodchild, 2006).

Over to you again Rina.

best regards

Barbara

Dear Barbara, (1. 1. 2012)

The way you described the relevancy of Cobb and his colleagues' research question: "how to help teachers as they strive to understand children's mathematical realities," is very interesting and triggers my thoughts. Especially its influence on you and your colleagues' work in the LCM project, which was conceptualized in 2003, about twenty years later than Cobb's work on this question.

Your example is quite typical of the kind of influence of Cobb and his colleagues' research - it is amazing how Paul and his colleagues' pioneering work on a certain issue is so often a core seed of so many important now-a-days works on the same issue, and also on its continuation. We may learn so much on trends of theory, research and methodology in our area by examining and learning Cobb's and his colleagues' work itself, on one side, and, in parallel, the changing forms this work gathered while it is being adopted by many other researchers. We may also see and become aware of the main streams of our field's concerns and needs. The theoretical and research activity of Cobb and colleagues, which was done and expressed with such an awareness, maturity and explicitness, enables many other mathematics educators to continue and take it into various directions and contexts.

Some of these issues were, and still are, the big questions in education as a whole and in mathematics education in particular, big questions that, in spite of being extensively investigated, will never have full answers. I see the question you raised Barbara: "how to help teachers, as they strive to understand children's mathematical realities," as one of these never ending issues.

The issue I would like to focus on, in this round, is the one which is engaged in understanding the meaning of classroom's learning; or, in other words; understanding the meaning of constructing the classroom's mathematical knowledge. The way it is expressed in the classroom: within the whole class community, groups and individuals of the working classroom community, including the processes of *shifts* of a constructed knowledge between and within the above social settings in the class. A few scientists consider the understanding of the learning

process in classroom community as very important. Saxe et al. (2009) opened their chapter: A methodological framework and empirical techniques for studying the travel of ideas in classroom communities, with the following sentence: “Powerful methods for analyzing the travel of ideas are essential for understanding the process of learning in classroom communities” (p. 203). Like many others, I consider these issues as a big challenge, a secret (at least partially), and even a never ending issue. I consider various parts in Cobb's new book as an important contribution to these issues.

Cobb seemed to be attracted by the fascinating processes of human constructing of mathematical knowledge and the role of interaction within these processes from the very beginning of his research journey. In his first station, Radical Constructivism, which is represented in the paper of Cobb and Steffe (1983), reprinted as Chapter 3 of the book, the authors claim: “Our emphasis on the researcher as teacher stems from our view that children's construction of mathematical knowledge is greatly influenced by the experience they gain through their interaction with their teacher” (p.19). Later in the same paper they say: “... we believe researchers must act as teachers is that the experiences children gain through interactions with adults greatly influence their construction of mathematical knowledge” (p. 20).

In a sense this Vygotskian approach might be considered as an *early bird* which hints the travel into the second station in the journey of Cobb and colleagues, Social Constructivism.

It seems that both of us, Barbara, are very interested in Cobb's travelling into the second station, which is told in a very interesting and efficient way in Chapter 4 of the book (The introduction to Part II, Social Constructivism, written by Cobb with Yackel). The authors explain that their “theoretical perspective on mathematical learning in that time acknowledged social interactions as a source of learning opportunities for individual children” (p.35). The investigation of the individual's mathematical knowledge construction is still a leading goal, but the perspective concerning an interaction with an adult—the researcher-teacher in the radical constructivism's station—broadened to a more general perspective—a social interaction, with all the ingredients of a classroom context: the teacher, the students as individuals in different social setting in the classroom, the designed activities, and so forth. Typical features of such classroom research are its richness, on one side, and the overloading of the data which emerged from the diverse resources and relationships among them, on the other side.

For controlling such a research situation and the needed research activities, like collecting, analyzing and interpreting data, a reliable theoretical-methodological framework is necessary,

as well as a research team's work. The way in which Cobb and his colleagues verified their theoretical-methodological framework, which is flexible enough to incorporate changes which necessarily emerged from the successive experiments' findings and research work they did (Parts II, III and IV in the book), and/or from the use of common corpus of data to elaborate different perspectives (Cobb & Bauersfeld, (Eds.), 1995), were and still are insightful for me.

Cobb and Bauersfeld's (1995) book demonstrates meaningful research concerning mathematics learning in classrooms via the psychological and the socio-cultural lenses in parallel. The book represents a collective effort to create a theoretical-methodological frame for classroom research in mathematics. The authors use a common corpus of data from mathematics classrooms to elaborate their different perspectives. While the Cobb et al., 2011 book has a *linear flavour* and we all relate to it as to a documentation of an academic journey, the 1995 book, edited by Cobb and Bauersfeld, has a *star structure*, where the star's centre is the emergence perspective (also Parts II and III in Cobb's 2011 book), and the *star's edges* are the many views given by the authors of the various chapters in the 1995 book. In this sense, the 1995 book illuminated deeply some perspectives in the second and third stations of Cobb's and his colleagues' journey, described in the 2011 book.

In any case, the issue of the meaning of mathematical knowledge construction in the classroom is emphasized and intensively investigated and interpreted in Cobb's and his colleagues' work, and, by doing so, their work turned to be a core seed of the trend in our field of mathematical classroom research.

On the whole, the overall goal of this trend in mathematics classroom research is to study the development of mathematical knowledge and reasoning in the classroom, as it is expressed in the emergence of collective activity of the classroom community, and in cognitive and socio-interactive processes of knowledge construction by individuals working in the classroom. While Cobb and his colleagues made a seminal core work of classroom research that put the interactive knowledge construction in the classroom at the centre of the stage, they also presented basic questions which typically emerged from this kind of work. These questions are used as links to other researchers' active work and thus accumulated more investigation and new or more refined findings, by different methodologies and theories, and in different contextual areas, not only in mathematics (Schwarz, Dreyfus, & Hershkowitz, 2009). Again, Cobb's and his colleagues' work seemed to provide the first bricks in this wall. In the following I draw specific examples of what I discussed above in a general way.

In their efforts to analyze the collective learning of a mathematics classroom community, Cobb and colleagues (Cobb, Stephan, McClain, & Gravmeijer, 2001, Part IV, Chapter 9 in the book) focus on the evolution of mathematical practices. For this purpose, they combined “a social perspective on communal practices with a psychological perspective on individual students’ diverse ways of reasoning as they participate in those practices” (p. 117). They discussed the notion of *taken-as-shared* activities of the students in the same classroom and explained their approach:

We speak of normative activities being taken as shared rather than shared, to leave room for the diversity in individual students’ ways of participating in these activities. The assertion that a particular activity is taken as shared makes no deterministic claims about the reasoning of the participating students, least of all that their reasoning is identical. (p. 122)

The above quotations by Cobb et al., raise two issues that relate one to each other: (a) practices as communal activities, carried by the reasoning of the different students participating in these practices; where (b) these practices are taken as shared rather than shared.

It is well understood why the authors are taking care not to use the term *shared* that might raise the assumption that the students “participating in the practice” are considered as having identical reasoning. But, unfortunately this does not clarify the meaning of practices taken as shared. In a sense this might leave the notion of mathematical practices as largely intuitive (Cobb, with Stephan & Bowers, 2011, Part IV, Chapter 8 in the book, p. 109). Might it be that this vague meaning concerning taken as shared pushes more researchers to invest their efforts in making the above two notions more explicit. Here are some examples.

The first example is of the inspiration of the seminal pioneer work of Cobb and his colleagues, and the direction in which it is being continued by researchers, who are taking a more explicit turn.

The notion of the mathematical practices as mathematical activity of a classroom community was defined and investigated in an explicit way by Stephan and Rasmussen (2002) and Rasmussen and Stephan (2008). In their chapter from 2008, they write: “One issue of theoretical and pragmatic concern that has emerged from design research is documentation of the normative or collective ways of reasoning that develop as learners engage in mathematical or scientific activity. The purpose of this chapter is to explicate a methodology for documenting learners’ collective activity.” (p.195). They define collective activity as: “the normative ways of reasoning of a classroom community” (p. 195), and claim that: “rigorous qualitative

methodologies are needed to document the collective activity of a class over extended periods of time” (p. 196). And, indeed, they use a methodological approach, in which two criteria based on Toulmin's model (1958) of argumentation are used for documenting collective activity in the classroom and for analyzing when ideas begin "to function as if shared". They explain: “Our use of the phrase ‘function as-if-shared’ is intended to emphasize that normative ways of reasoning in a community function as if everyone shares this way of reasoning” (p.196).

The second example is presented by Hershkowitz, Hadas, Dreyfus, and Schwarz (2008), in which they apply their model (RBC - a theoretical - methodological model, according to which the emergence of a new knowledge construct is described and analyzed by means of three observable epistemic actions: recognizing (R), building-with (B) and constructing (C).), for analyzing students' abstraction processes of a group in a working classroom. They observe and analyze the discourse of:

... "groups of individual students [who] construct shared knowledge and consolidate it in the mathematical classroom. The data emphasize the interactive flow of knowledge from one student to the others in the group, until they reach a shared knowledge — a common basis of knowledge which allows them to continue the construction of further knowledge in the same topic together." (p. 41)

This example describes an additional research effort to apply a notion of shared knowledge which follows in the footsteps of Cobb's and his colleagues' path: considering both the reasoning of the individual and how it feeds, and is also fed by, the interactions in a small group, in a working mathematics classroom.

Dear Barbara, up to here I have tried to write about and even demonstrate a main point I wanted to raise concerning what we called "Paul Cobb's book" (Yackel, Gravemeijer, & Sfard, 2011). That is to say, that the value of the book is not only in the journey of Cobb and colleagues itself, as *the* prototypical example for the development of mathematics education research in the last three decades, but also in the insightful ability to verify, as a result of their wise and intensive empirical team's work, the big questions, which are relevant for so many mathematics education researchers in their own work, where answering these questions is still a main research goal for so many.

With my best greetings

Rina

Dear Rina (25 .1.12)

It is time to end our dialogue, which has spanned a much greater timescale than I think we originally intended. It is clear from our writing above how much we have both valued not only our reading of this book but also the body of work it reflects. There is no doubt of the significance of this work for mathematics education as a whole. Having moved myself from my early work in constructivist frames, to current work with socio-cultural theories, I am aware that we still have the big challenge of theorizing the individual within the social, and relating this to our interactions in practical settings. While some energy over the years has gone into debating the theoretical pros and cons, what emerges from our considerations here is that research brings theory closer to practice, and practice informs theory in deep and reforming ways. Currently, as a teacher studying my own teaching, I come up against, every day, ways in which the practical setting challenges my theoretical knowledge. It is an ongoing story!

All the best

Barbara

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