Supplementary Appendix to Autoregressive Moving Average Infinite Hidden Markov-Switching Models

Abstract

This Appendix contains additional empirical results with respect to the published article. In Section 1, the posterior results for the HDP parameters of the IHMS-ARMA models are presented for the U.S. GDP growth rate and inflation series. In Section 2, we report additional in-sample and forecasting results for the same series. In Section 3, some results for a different truncation choice of the number of regimes in the approximate model are reported.

1 Posterior results for the HDP parameters

For the the U.S. GDP growth rate and inflation series, Table 1 provides the prior means (see also Table 3 of the paper) and the prior standard deviations, as well as the posterior counterparts of the HDP parameters of the ARMA(1,1) models, for both types of prior (CP and MS). The posterior results for the other parameters of these models are presented in Section 6 of the paper.

As expected, the data are informative about the HDP parameters when some breaks are detected. In particular, we observe that the posterior moments for the MS prior always differ from the corresponding prior moments more than for the CP prior. In addition, Table 1 provides the autocorrelation times (ACT), computed by batch means (see Geyer (1992)), for each parameter, indicating that the MCMC mixing is quite good.

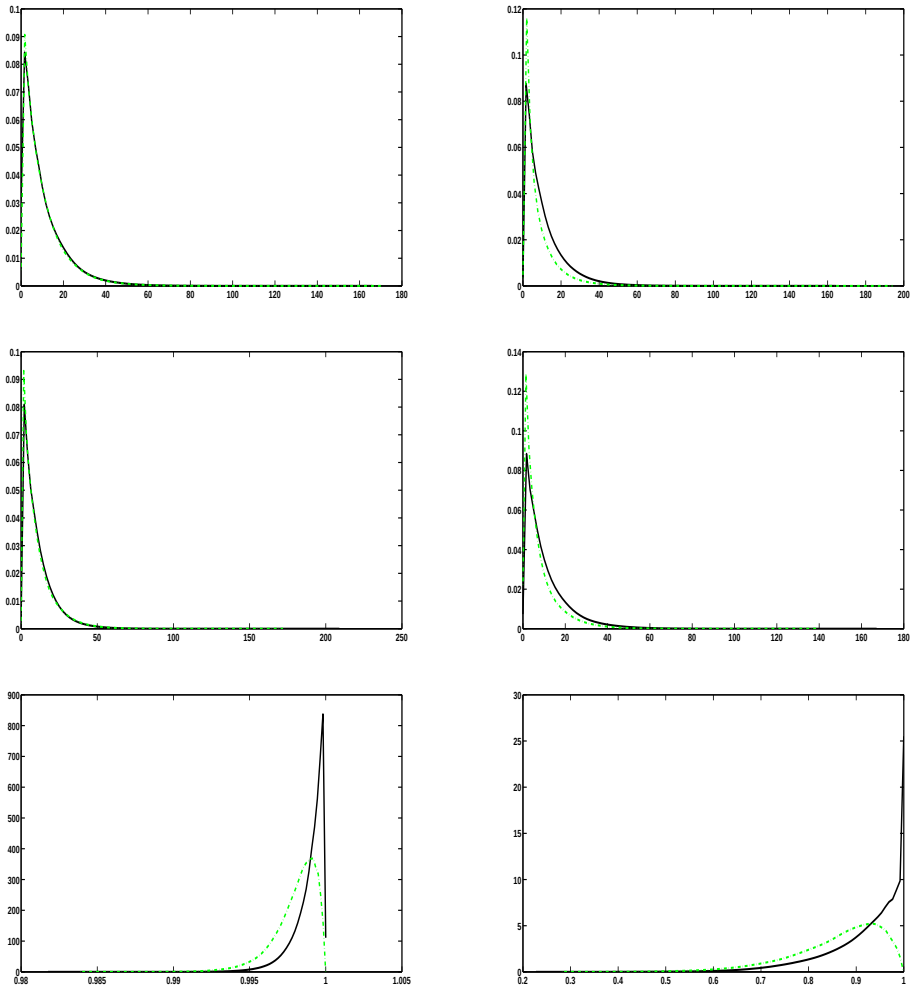
Table 1: U.S. GDP and Inflation: Prior and Posterior Results for HDP parameters

	CP prior						MS prior					
Parameters	η_ψ	$\alpha_\psi+\kappa_\psi$	ρ_ψ	η_σ	$\alpha_\sigma+\kappa_\sigma$	ρ_σ	η_ψ	$\alpha_\psi+\kappa_\psi$	ρ_ψ	η_σ	$\alpha_\sigma+\kappa_\sigma$	ρ_σ
Prior mean	10	10	1.00	10	10	1.00	10	10	0.91	10	10	0.91
Prior std	10	10	0.00	10	10	0.00	10	10	0.08	10	10	0.08
					U.S. GDP growth rate							
Post mean	9.80	9.79	1.00	9.04	20.52	1.00	6.28	7.10	0.86	4.11	13.67	0.85
Post std	9.97	10.29	0.00	8.97	13.63	0.00	8.10	8.14	0.09	3.97	10.05	0.07
ACT	2.76	59.87	5.50	16.12	73.59	7.88	21.08	125.55	38.12	23.40	114.43	34.81
					U.S. Inflation							
Post mean	11.24	18.96	1.00	5.64	18.48	1.00	5.80	11.58	0.89	2.30	14.93	0.83
Post std	10.11	13.27	0.00	6.17	11.79	0.00	5.92	11.41	0.07	1.83	10.28	0.07
ACT	9.24	77.93	47.73	3.06	84.97	24.45	104.21	268.83	135.94	19.72	92.62	43.98

CP and MS refer to the IHMS hyper-parameters: CP (MS) imply high (weak) regime persistence. ACT: autocorrelation times.

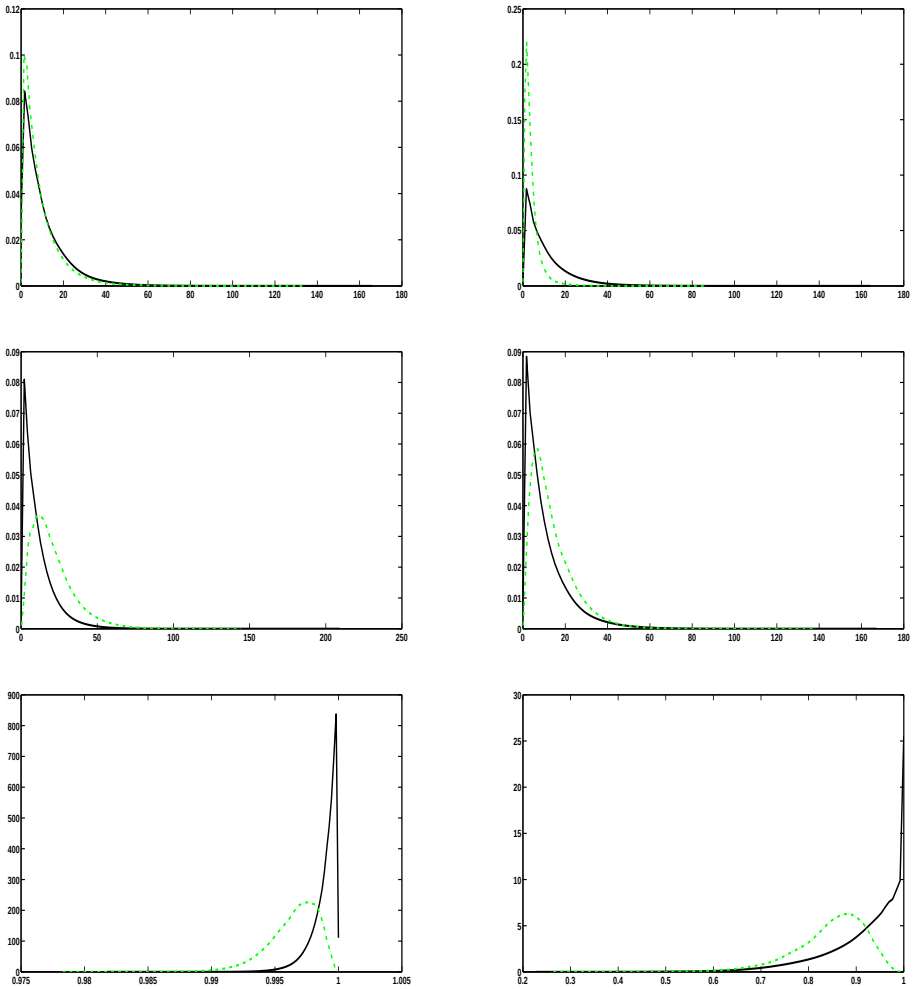
To get a more detailed view, Figures 1-4 display all the prior and posterior densities of these parameters. In addition to the comments above, we also observe that the posterior densities of the parameter ρ remarkably deviate from the prior densitites whatever the prior setup.

Figure 1: U.S. GDP: posterior densities of the hyperparameters for the mean state vector



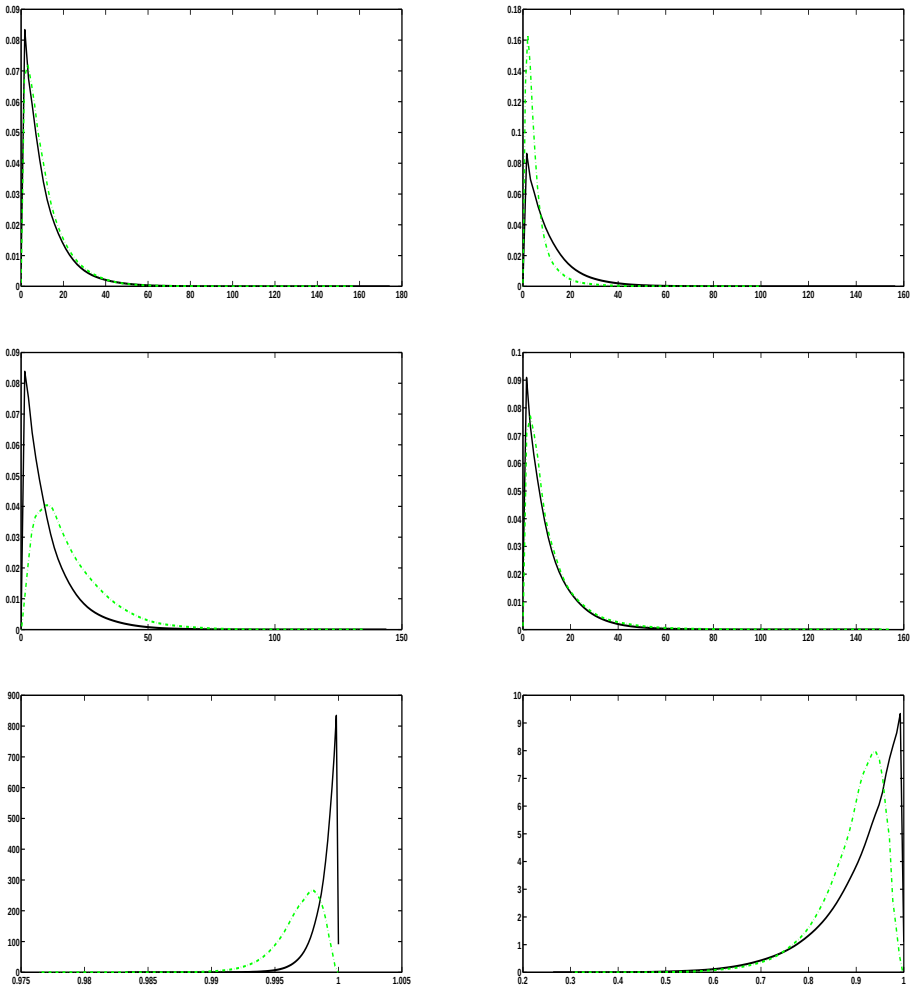
The left column corresponds to the CP-prior, the right one to the MS-prior. Black (continuous) lines correspond prior densities and dashed (green) to posterior densities. First row: parameter η_ψ. Second row: parameter $\alpha_\psi + \kappa_\psi$. Last row: persistence parameter ρ_ψ.

Figure 2: U.S. GDP: posterior densities of the hyperparameters for the variance state vector



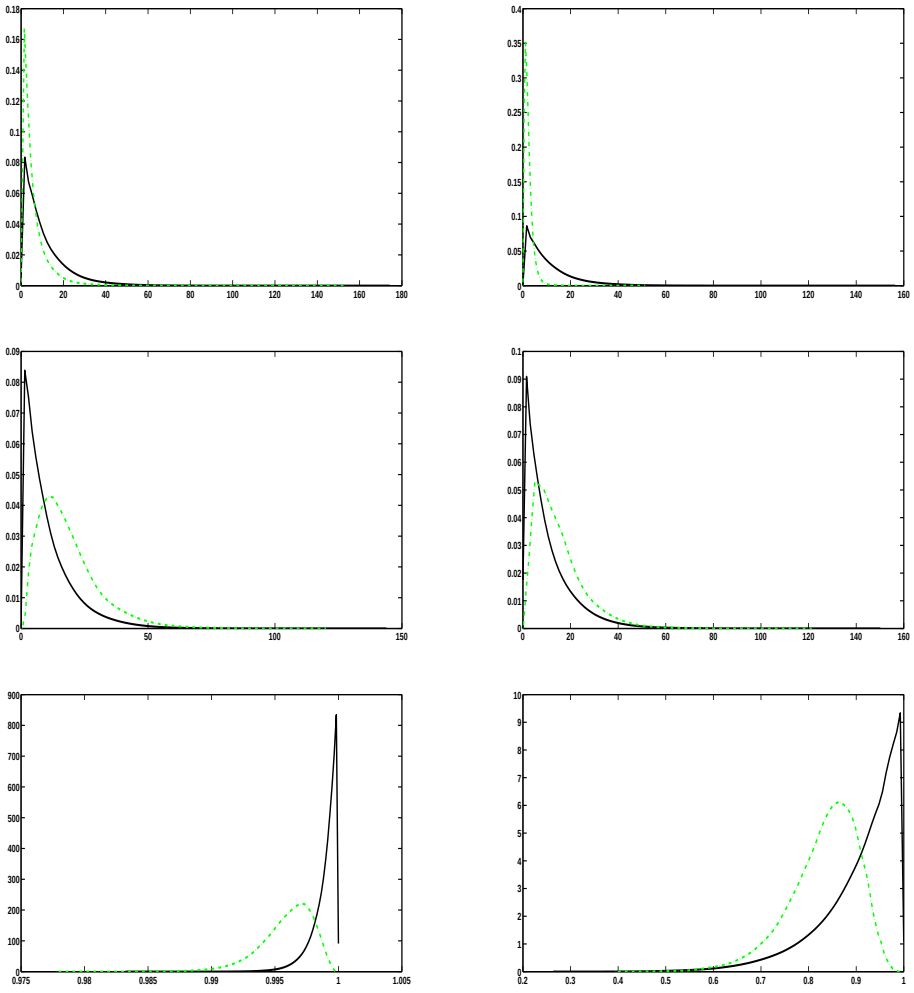
The left column corresponds to the CP-prior, the right one to the MS-prior. Black (continuous) lines correspond prior densities and dashed (green) to posterior densities. First row: parameter η_σ. Second row: parameter $\alpha_\sigma + \kappa_\sigma$. Last row: persistence parameter ρ_σ.

Figure 3: U.S. Inflation: posterior densities of the hyperparameters for the mean state vector



The left column corresponds to the CP-prior, the right one to the MS-prior. Black (continuous) lines correspond prior densities and dashed (green) to posterior densities. First row: parameter η_ψ. Second row: parameter $\alpha_\psi + \kappa_\psi$. Last row: persistence parameter ρ_ψ.

Figure 4: U.S. Inflation: posterior densities of the hyperparameters for the variance state vector



The left column corresponds to the CP-prior, the right one to the MS-prior. Black (continuous) lines correspond prior densities and dashed (green) to posterior densities. First row: parameter η_σ. Second row: parameter $\alpha_\sigma + \kappa_\sigma$. Last row: persistence parameter ρ_σ.

2 Additional Empirical Results

We consider several additional models: the restricted IHMS-ARMA(1,1) models, restricted in the sense that the mean parameters jointly move with the variance one, the Auto-Regressive model with a number of lags chosen by the marginal likelihood criterion (MLL), and the random walk (RW) model (only used for the inflation series).

2.1 U.S. GDP Growth Rate

As in Table 4 of the paper, we first document the MLL in Table 2. According to the Akaike information criterion (AIC), the best AR model has sixteen lags. We observe a vast improvement in the MLL of models using the IHMS structure compared to fixed parameter models. Focusing on the processes with time-varying parameters, we observe that the best models remain (as in the paper) the unrestricted ARMA models which disentangle the break in the mean parameters with respect to the variance one. The second best are the IHMS-AR models, closely followed by the restricted versions of the IHMS-ARMA models.

Table 2: U.S. GDP: marginal log-likelihood values

AR(16)	ARMA(1,1)	IHMS-AR(1)		Rest. IHMS-ARMA(1,1)		IHMS-ARMA(1,1)	
		CP	MS	CP	MS	CP	MS
-396.74	-378.57	-355.07	-353.54	-355.75	-355.62	-353.13	-351.90

CP and MS refer to the IHMS hyper-parameters: CP (MS) imply high (weak) regime persistence. Rest. stands for the restricted model in which the break dynamic in common for the mean and the variance parameters.

Table 3 (extending Table 5 of the paper) provides the posterior probabilities of having a specific number of regimes. The restricted model exhibits probabilities similar to the most flexible IHMS-ARMA process. Obviously, from an interpretation point of view, one cannot capture that only the variance varies over time with the restricted model.

Table 4 is similar to Table 6 of the paper, except that the results of the significance tests are given for comparing the IHMS models to the AR(16) models (instead of the ARMA model). Hence, the reported values are identical but the stars may differ.

Table 3: U.S. GDP: posterior probabilities of having a specific number of regimes

# Regimes	1	2	3	4	5	6	7	8	9
Restricted IHMS-ARMA with MS prior									
$\mu, \beta, \phi, \sigma^2$	0	0.06	**0.29**	0.28	0.16	0.10	0.07	0.05	0
IHMS-ARMA with MS prior									
μ, β, ϕ	**0.62**	0.20	0.06	0.07	0.03	0.01	0	0	0
σ^2	0	0.06	0.19	**0.26**	0.23	0.15	0.07	0.03	0.01
Restricted IHMS-ARMA with CP prior									
$\mu, \beta, \phi, \sigma^2$	0	**0.54**	0.37	0.09	0.01	0.00	0	0	0
IHMS-ARMA with CP prior									
μ, β, ϕ	**0.99**	0.01	0.00	0	0	0	0	0	0
σ^2	0	**0.80**	0.16	0.03	0.01	0	0	0	0

Table 4: U.S. GDP: APD, MSFE and CRPS

Forecast Horizons	One quarter	Two quarters	One year	Two years	Three years	Four years
			APD			
AR(16)	0.36	0.33	0.32	0.31	0.31	0.31
ARMA	0.33	0.30	0.28	0.27	0.27	0.27
Rest. IHMS-ARMA (CP)	0.46**	0.44**	0.40**	0.39*	0.38	**0.39**
Rest. IHMS-ARMA (MS)	0.45**	0.42**	0.38**	0.35	0.34	0.34
IHMS-AR (CP)	0.46**	0.43**	0.39**	0.38*	0.37	0.38
IHMS-AR (MS)	0.43**	0.39**	0.36**	0.34	0.33	0.33
IHMS-ARMA (CP)	**0.47****	**0.44****	**0.40****	**0.39***	**0.38**	0.38
IHMS-ARMA (MS)	0.44**	0.41**	0.37**	0.34	0.33	0.33
			MSFE			
AR(16)	0.43	0.52	0.58	0.60	0.61	0.58
ARMA	0.40	0.53	0.68	0.78	0.82	0.82
Rest. IHMS-ARMA (CP)	**0.36**	**0.43***	**0.52**	**0.57**	**0.61**	**0.57**
Rest. IHMS-ARMA (MS)	0.37	0.45	0.55	0.64	0.69	0.69
IHMS-AR (CP)	0.39	0.49	0.60	0.67	0.70	0.68
IHMS-AR (MS)	0.40	0.52	0.64	0.70	0.73	0.73
IHMS-ARMA (CP)	0.37	0.45*	0.56	0.65	0.69	0.68
IHMS-ARMA (MS)	0.38	0.47	0.60	0.69	0.73	0.73
			CRPS			
AR(16)	0.37	0.40	0.42	0.43	0.44	0.43
ARMA	0.37	0.42	0.47	0.50	0.51	0.51
Rest. IHMS-ARMA (CP)	0.33**	0.36**	**0.39****	**0.42**	**0.42**	**0.42**
Rest. IHMS-ARMA (MS)	0.33**	0.37**	0.41	0.44	0.45	0.45
IHMS-AR (CP)	0.34*	0.38*	0.42	0.44	0.45	0.44
IHMS-AR (MS)	0.35	0.39	0.43	0.46	0.47	0.47
IHMS-ARMA (CP)	**0.33****	**0.36****	0.41	0.44	0.45	0.45
IHMS-ARMA (MS)	0.34**	0.37**	0.42	0.46	0.47	0.47

APD: average of predictive densities. MSFE: mean squared forecast error. CRPS: continuous ranked probability score. CP and MS refer to the prior IHMS hyper-parameters. Forecasts are from 1987Q1 to 2014Q1. Bold numbers identify the best performing model. A star indicates that there exists a significant statistical difference with respect to the AR(16) model at the 10% level (two-sided test). A double star denotes significance at 5%.

2.2 U.S. Inflation

Judging by the AIC, the best AR model for the U.S. inflation exhibits twelve lags. Table 5 (extending Table 7 of the paper) documents the MLL of all the considered models. Looking at the MLL, considering models with fixed parameters is too restrictive as the models with a Markov-switching structure strongly improve the MLL values. Furthermore, disentangling the break dynamics of the mean and the variance parameters is beneficial. Indeed, the log Bayes factor amounts at minimum to 8.10 in favor of the flexible break structure. The added models do not improve the fit compared to those included in Table 7 of the paper.

Table 5: U.S. Inflation: marginal log-likelihood values

RW		AR(12)		ARMA(1,1)	
-1465.12		-1406.57		-1398.11	
IHMS-AR(1)		Rest. IHMS-ARMA(1,1)		IHMS-ARMA(1,1)	
CP	MS	CP	MS	CP	MS
-1335.81	-1332.85	-1342.18	-1343.39	-1334.08	-1331.84

CP and MS refer to the IHMS hyper-parameters: CP (MS) imply high (weak) regime persistence. Rest. stands for the restricted model in which the break dynamic in common for the mean and the variance parameters.

In Table 6 (extending Table 8 of the paper), we provide the posterior probabilities of having a specific number of regimes for the restricted IHMS-ARMA model. As expected, the restricted model leads to a higher number of regimes and ultimately to an over-parametrized specification.

Table 7 is similar to Table 9 of the paper, except that the results of the significance tests are given for comparing the IHMS models to the AR(12) models (instead of the ARMA model). Hence, the reported values are identical but the stars may differ.

Table 6: U.S. Inflation: posterior probabilities of having a specific number of regimes

# Regimes	1	2	3	4	5	6	7	8	9
Restricted IHMS-ARMA with MS prior									
$\mu, \beta, \phi, \sigma^2$	0	0	0.02	0.05	0.17	**0.47**	0.23	0.04	0.01
IHMS-ARMA with MS prior									
μ, β, ϕ	0	0.08	**0.28**	0.27	0.20	0.13	0.03	0.01	0.00
σ^2	0	0.11	0.22	**0.26**	0.20	0.13	0.06	0.02	0.00
Restricted IHMS-ARMA with CP prior									
$\mu, \beta, \phi, \sigma^2$	0	0	0	0.24	**0.36**	0.18	0.13	0.09	0.00
IHMS-ARMA with CP prior									
μ, β, ϕ	0	**0.70**	0.29	0.01	0	0	0	0	0
σ^2	0	**0.87**	0.12	0.01	0.00	0.00	0	0	0

Table 7: U.S. Inflation: APD, MSFE and CRPS

Forecast Horizons	One month	Two months	Four months	Eight months	One year	Sixteen months
			APD			
RW	0.13	0.10	0.08	0.06	0.05	0.04
AR(12)	0.14	0.14	0.14	0.13	0.12	0.12
ARMA	0.15**	0.14	0.14	0.13	0.12	0.12
Rest. IHMS-ARMA (CP)	0.16**	0.15**	0.15**	0.14	0.13	0.13
Rest. IHMS-ARMA (MS)	0.16**	0.15**	0.15**	0.14*	0.14	0.13
IHMS-AR (CP)	0.16**	0.15**	0.15**	**0.14****	0.14**	**0.14****
IHMS-AR (MS)	0.16**	0.15**	0.15**	0.14**	**0.14****	0.14**
IHMS-ARMA (CP)	0.16**	0.15**	0.15**	0.14*	0.13	0.13*
IHMS-ARMA (MS)	**0.16****	**0.16****	**0.15****	0.14**	0.13*	0.13**
			MSFE			
RW	6.65	10.26	10.33	12.10	12.99	9.94
AR(12)	6.58	6.73	6.95	6.87	7.01	6.62
ARMA	5.52**	6.60	6.52	6.93	7.24	6.53
Rest. IHMS-ARMA (CP)	5.25**	6.19	**5.79**	6.29	6.94	6.56
Rest. IHMS-ARMA (MS)	5.28**	6.32	5.98	6.53	7.20	6.82
IHMS-AR (CP)	5.00**	6.09	5.84	**6.15**	6.55	6.26
IHMS-AR (MS)	**4.94****	**6.04**	5.85	6.17	**6.51**	6.33
IHMS-ARMA (CP)	5.19**	6.33	6.08	6.56	7.13	6.62
IHMS-ARMA (MS)	5.26**	6.32	5.98	6.35	6.78	**6.12**
			CRPS			
RW	1.37	1.66	1.74	2.11	2.35	2.46
AR(12)	1.29	1.25	1.28	1.31	1.29	1.30
ARMA	1.19**	1.28	1.24	1.34	1.37	1.33
Rest. IHMS-ARMA (CP)	1.16**	1.22	**1.18**	1.25	1.32	1.30
Rest. IHMS-ARMA (MS)	1.16**	1.24	1.20	1.28	1.34	1.33
IHMS-AR (CP)	1.13**	1.22	1.19	1.25	1.29	**1.27**
IHMS-AR (MS)	**1.12****	**1.21**	1.18	**1.24**	**1.28**	1.28
IHMS-ARMA (CP)	1.15**	1.24	1.20	1.28	1.33	1.30
IHMS-ARMA (MS)	1.14**	1.22	1.18	1.27	1.31	1.27

APD: average of predictive densities. MSFE: mean squared forecast error. CRPS: continuous ranked probability score. CP and MS refer to the prior IHMS hyper-parameters. Forecasts are from April, 1991 to November 2012. Bold numbers identify the best performing model. A star indicates that there exists a significant statistical difference with respect to the AR(12) model at the 10% level (two-sided test). A double star denotes significance at 5%.

3 Impact of the Truncation on the Regime Number

Tables 8 and 9 document the posterior probabilities of having a specific number of regimes for the U.S. GDP growth rate and the inflation series when we fix the truncation to 20 regimes. To compare the probabilities, the results of the paper (for 10 regimes) are also reported. Whatever the series, the probabilities do not change for the CP prior settings. Regarding the MS prior, although the probabilities are very similar, the posterior mode of the number of regimes differs by one unit for the variance of the U.S. GDP growth rate and by two for the mean parameters of the U.S. inflation.

Table 8: U.S. GDP: posterior probabilities of having a specific number of regimes

# Regimes	1	2	3	4	5	6	7	8	9	≥ 10
IHMS-ARMA with MS prior ($L_{max} = 10$)										
μ, β, ϕ	**0.62**	0.20	0.06	0.07	0.03	0.01	0	0	0	0
σ^2	0	0.06	0.19	**0.26**	0.23	0.15	0.07	0.03	0.01	0
IHMS-ARMA with MS prior ($L_{max} = 20$)										
μ, β, ϕ	**0.50**	0.25	0.17	0.05	0.02	0.01	0.00	0	0	0
σ^2	0	0.05	0.11	0.16	**0.18**	0.16	0.13	0.09	0.06	0.06
IHMS-ARMA with CP prior ($L_{max} = 10$)										
μ, β, ϕ	**0.99**	0.01	0.00	0	0	0	0	0	0	0
σ^2	0	**0.80**	0.16	0.03	0.01	0	0	0	0	0
IHMS-ARMA with CP prior ($L_{max} = 20$)										
μ, β, ϕ	**0.99**	0.01	0.00	0	0	0	0	0	0	0
σ^2	0.00	**0.80**	0.16	0.03	0.01	0.00	0	0	0	0

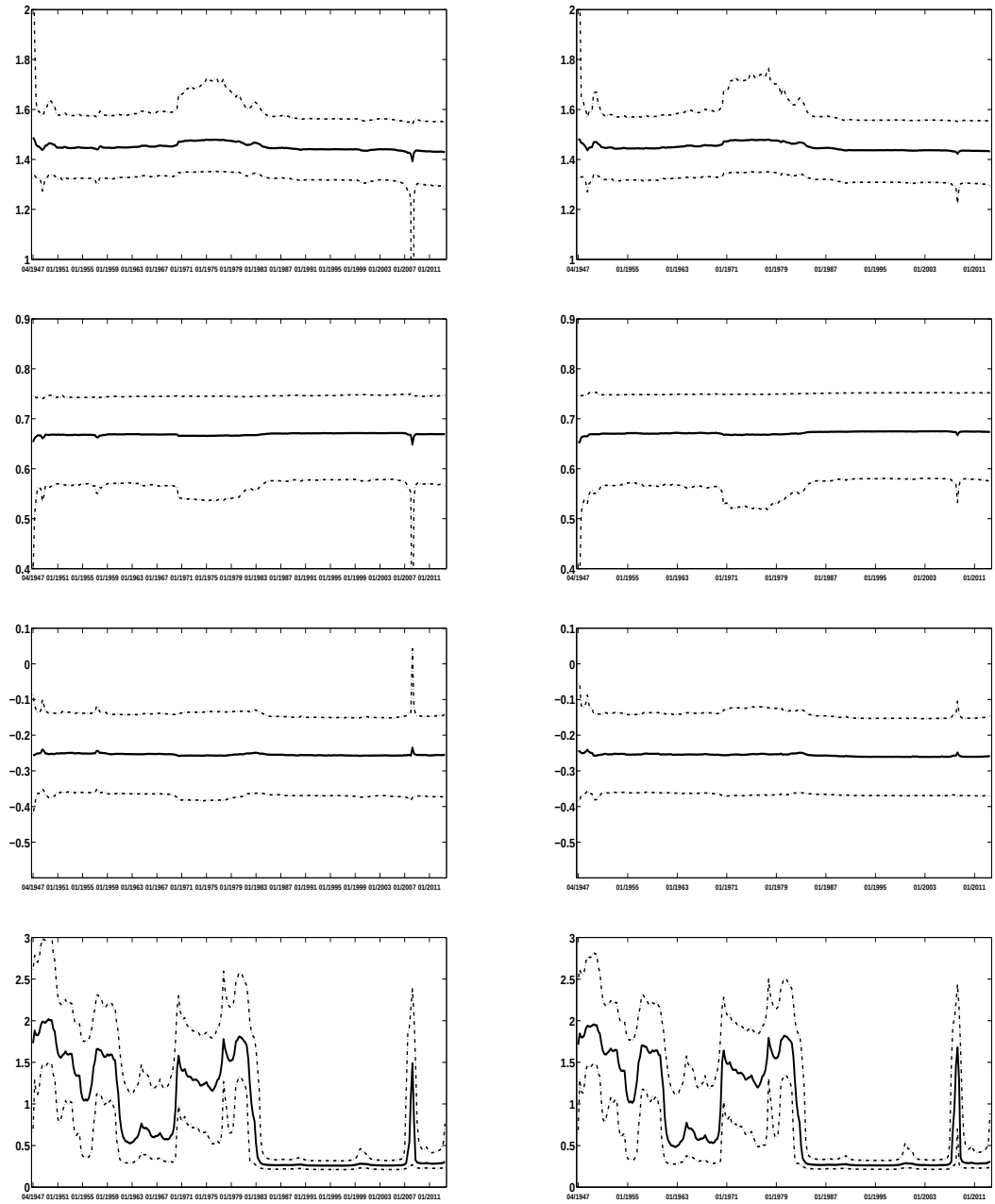
To go into more details, Figures 5 and 6 document the posterior medians as well as the 70% confidence intervals of the model parameters with MS priors for the two series. For ease of comparison, the results of the paper ($L_{Max} = 10$) are also provided. The pattern of the medians over time are identical for the U.S. GDP growth rate and very similar for the U.S. inflation. One can conclude that the upper bound of ten regimes is a reasonable

Table 9: U.S. inflation: posterior probabilities of having a specific number of regimes

# Regimes	1	2	3	4	5	6	7	8	9	≥ 10
IHMS-ARMA with MS prior ($L_{max} = 10$)										
μ, β, ϕ	0	0.08	**0.28**	0.27	0.20	0.13	0.03	0.01	0.00	0
σ^2	0	0.11	0.22	**0.26**	0.20	0.13	0.06	0.02	0.00	0
IHMS-ARMA with MS prior ($L_{max} = 20$)										
μ, β, ϕ	0	0	0.03	0.20	**0.33**	0.22	0.13	0.06	0.02	0.00
σ^2	0	0.08	0.16	**0.21**	0.19	0.14	0.10	0.06	0.04	0.03
IHMS-ARMA with CP prior ($L_{max} = 10$)										
μ, β, ϕ	0	**0.70**	0.29	0.01	0	0	0	0	0	0
σ^2	0	**0.87**	0.12	0.01	0.00	0.00	0	0	0	0
IHMS-ARMA with CP prior ($L_{max} = 20$)										
μ, β, ϕ	0	**0.97**	0.03	0.00	0.00	0	0	0	0	0
σ^2	0	**0.84**	0.14	0.02	0.00	0.00	0	0	0	0

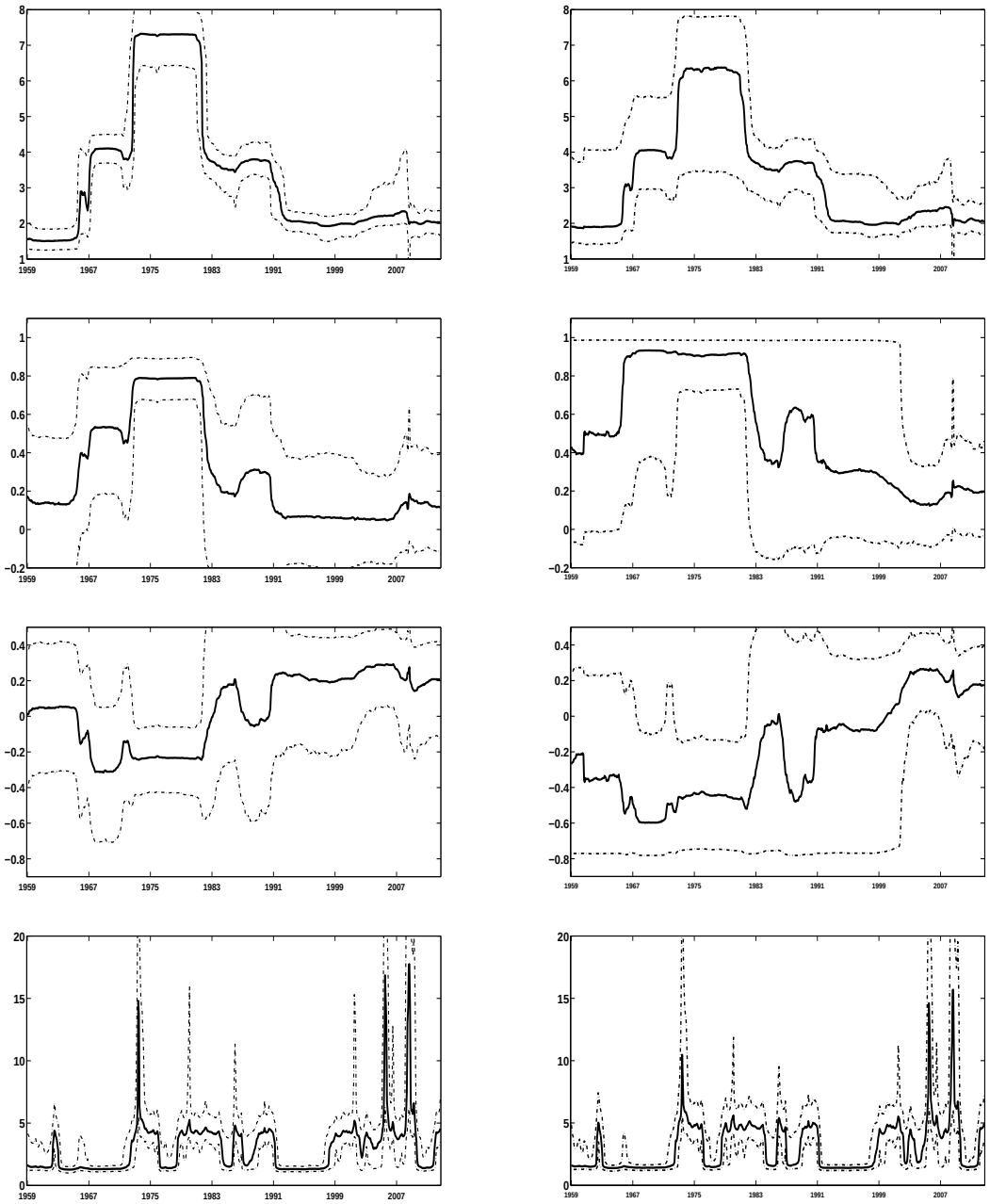
choice of truncation.

Figure 5: U.S. GDP: posterior medians and 70% credible intervals of the MS prior



Left column: $L_{Max} = 20$. Right: $L_{Max} = 10$. Thick horizontal line: posterior median of the ARMA model with fixed parameters. Thin continuous and dotted lines: IHMS-ARMA posterior median and the limits of the 70% posterior credible interval. Top row: long term mean $\mu_t/(1-\beta_t)$. Second row: AR coefficient. Third row: MA coefficient. Last row: variance.

Figure 6: U.S. inflation: posterior medians and 70% credible intervals of the MS prior



Left column: $L_{Max} = 20$. Right: $L_{Max} = 10$. Thick horizontal line: posterior median of the ARMA model with fixed parameters. Thin continuous and dotted lines: IHMS-ARMA posterior median and the limits of the 70% posterior credible interval. Top row: long term mean $\mu_t/(1-\beta_t)$. Second row: AR coefficient. Third row: MA coefficient. Last row: variance.

References

Geyer, C. J. (1992), 'Practical markov chain monte carlo', *Statistical Science* **7(4)**, 473–511.