Online Supplement to “Attractor Tempi for Metrical Structures”

Mark Gotham, University of Cambridge, Cambridge, UK, mrhgotham@gmail.com

1. Finding the local maxima of combined peaks using derivatives

Recall that

$$S = \exp\left(-\frac{(\log(x/\mu))^2}{2\sigma^2}\right).$$

Maxima occur when $\frac{dS}{dx} = 0$; (minima do not occur for this exponential function). The derivative of this equation is found using a composite form of the chain rule which states that

$$\frac{dS}{dx} = \frac{dS}{du} * \frac{du}{dv} * \frac{dv}{dw} * \frac{dw}{dx}.$$

In this case, $S(x)$ decomposes to

$$\begin{aligned} S &= f(u) = e^u, \\ u &= g(v) = -v^2/2\sigma^2, \\ v &= h(w) = \log(w), \\ w &= i(x) = x/\mu; \end{aligned}$$

and so

$$\begin{aligned} \frac{dS}{du} &= e^u = \exp\left(-\frac{(\log(x/\mu))^2}{2\sigma^2}\right), \\ \frac{du}{dv} &= -2v/2\sigma^2 = -v/\sigma^2 = \frac{-\log(x/\mu)}{\sigma^2}, \\ \frac{dv}{dw} &= \frac{1}{w\ln(10)} = \frac{\mu}{x\ln(10)}, \\ \frac{dw}{dx} &= \frac{1}{\mu}. \end{aligned}$$

Combining these terms (and noting that the μ -values of the last two terms cancel) gives

$$\frac{dS}{dx} = \exp\left(-\frac{(\log(x/\mu))^2}{2\sigma^2}\right) * \frac{-\log(x/\mu)}{\sigma^2} * \frac{1}{x\ln(10)}. \quad (1)$$

For the related curve, $S(nx)$, the cancellation of the final two terms includes n as well as μ :

$$\begin{aligned} \frac{dS}{du} &= \exp\left(-\frac{(\log(nx/\mu))^2}{2\sigma^2}\right), \\ \frac{du}{dv} &= \frac{-\log(nx/\mu)}{\sigma^2}, \\ \frac{dv}{dw} &= \frac{\mu}{nx\ln(10)}, \\ \frac{dw}{dx} &= \frac{n}{\mu}, \end{aligned}$$

and so

$$\frac{dS(nx)}{dx} = \exp\left(-\frac{(\log(nx/\mu))^2}{2\sigma^2}\right) * \frac{-\log(nx/\mu)}{\sigma^2} * \frac{1}{x\ln(10)}. \quad (2)$$

Combining curves (such as $S(x)$ and $S(nx)$) does not complicate matters: the respective derivatives can simply be deduced separately and summed. When combining two consecutive levels of a binary metre for instance,

$$M = \exp\left(-\frac{(\log(x/\mu))^2}{2\sigma^2}\right) + \exp\left(-\frac{(\log(2x/\mu))^2}{2\sigma^2}\right),$$

and

$$\frac{dM}{dx} = \exp\left(-\frac{(\log(x/\mu))^2}{2\sigma^2}\right) * \frac{-\log(x/\mu)}{\sigma^2} * \frac{1}{x\ln(10)} + \exp\left(-\frac{(\log(2x/\mu))^2}{2\sigma^2}\right) * \frac{-\log(2x/\mu)}{\sigma^2} * \frac{1}{x\ln(10)}.$$

This can be generalised for any combination of metrical levels by

$$\frac{dM}{dx} = \sum_{n=1,p,q,r\dots} \frac{dS}{dx} = \sum_{n=1,p,q,r\dots} \exp\left(-\frac{(\log(nx/\mu))^2}{2\sigma^2}\right) * \frac{-\log(nx/\mu)}{\sigma^2} * \frac{1}{nx\ln(10)}. \quad (3)$$

M is maximised by the x -value which gives $\frac{dM}{dx} = 0$.

2. Quantified values for some example cases

The table below sets out the quantified salience values which are generated by the model advanced in this paper for some of the main tempo-metrical structures discussed: two binary metres with pulse streams of 0.6 and $0.6/\sqrt{2}$ seconds, and two ternary metres each with a single three-unit grouping of the 0.2 and $(0.6/\sqrt{3})$ second pulse streams respectively (the left-most column identifies the rows on which these referential values fall). This illustrates some properties and discussions in the main article; the first of these example metres corresponds to the example of Table 1 in the main article, for instance.

As set out in the article, the pulse IOI is given by x , $S(x)$ denotes that pulse’s individual salience value, and $M(x)$ gives the combined “metrical salience” values for some combination of metrical levels. The S and M values are relative, set in proportion to the referential value of 1.0 for $S(0.6)$, (which is $S(\mu)$). This table includes data for every possible number of metrical levels. For instance, in the “two levels” column, the values refer to the sum of the S -values for the metrical level of the x -value on that row, and the previous level only. The first entry in this column (0.737934) refers to the sum of $S(0.15) + S(0.3)$. In this case the value is identical in this case to the entry in the previous M column, which gives the sum of all $S(x)$ values up to the present level. By contrast, the following value in the ‘2 level’ column refers to the sum of $S(0.3) + S(0.6) = 1.604448$, which is less than the corresponding value in the previous column which also includes the $x = 0.15$ stream ($S(0.15) + S(0.3) + S(0.6) = 1.737934$). All values are rounded to 6 decimal places.

	x	$S(x)$ (1 level)	$M(x)$ (all up to given)	2 levels	3 levels	4 levels	5 levels	6 levels
Binary 0.6	0.15	0.133486	0.133486					
	0.3	0.604448	0.737934	0.737934				
	0.6	1	1.737934	1.604448	1.737934			
	1.2	0.604448	2.342383	1.604448	2.208897	2.342383		
	2.4	0.133486	2.475869	0.737934	1.737934	2.342383	2.475869	
	4.8	0.010770	2.486639	0.144257	0.748705	1.748705	2.353153	2.486639
Binary $0.6/\sqrt{2}$	0.10607	0.043003	0.043003					
	0.21213	0.322150	0.365153	0.365153				
	0.42426	0.881738	1.246891	1.203888	1.246891			
	0.84853	0.8821738	2.128629	1.763477	2.085627	2.128629		
	1.697056	0.322150	2.450779	1.203888	2.085627	2.407777	2.450779	
	3.394113	0.043003	2.493782	0.365153	1.246891	2.128629	2.450779	2.493782
Ternary 0.2	0.1	0.034597	0.034597					
	0.2	0.282326	0.316923	0.316923				
	0.6	1	1.316923	1.282326	1.316923			
	1.2	0.604448	1.921371	1.604448	1.886774	1.921371		
	2.4	0.133486	2.054857	0.737934	1.737934	2.020261	2.054857	
	4.8	0.010770	2.065628	0.144257	0.748705	1.748705	2.031031	2.065628
Ternary $0.6/\sqrt{3}$	0.173205	0.198385	0.198385					
	0.346410	0.728933	0.927318	0.927318				
	1.039230	0.728933	1.656252	1.457866	1.656252			
	2.078461	0.198385	1.854637	0.927318	1.656252	1.854637		
	4.156922	0.019726	1.874363	0.218112	0.947045	1.675978	1.874363	