Asymptotics of a Gauss hypergeometric function with two large parameters: a new case

J. F. Harper¹

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Abstract

Asymptotic expansions of the Gauss hypergeometric function with large parameters, $F(\alpha + \epsilon_1\tau, \beta + \epsilon_2\tau; \gamma + \epsilon_3\tau; z)$ as $|\tau| \rightarrow \infty$, are known for many special cases, but not for one that the author encountered in recent work on fluid mechanics: $\epsilon_2 = 0$ and $\epsilon_3 = \epsilon_1 z$. This paper gives the leading term for that case if β is not a negative integer and z is not on the branch cut $[1, \infty)$, and it shows how subsequent terms can be found.

1 Introduction

Asymptotic expansions as $|\tau| \rightarrow \infty$ of the hypergeometric function

$$F(\alpha + \epsilon_1\tau, \beta + \epsilon_2\tau; \gamma + \epsilon_3\tau; z), \quad (1)$$

where $\alpha, \beta, \gamma, \epsilon_i$ and z are finite, have been studied for over 100 years but the theory is still not complete [2, 5, 6, 7, 8]. Most recently Paris [6] and then Cvitković et al. [2] considered $\epsilon_2 = 0$ but their methods did not apply if $\epsilon_3 = \epsilon_1 z$. This paper deals with that case by putting $a = \alpha - \gamma/z$, $b = \beta$, $\lambda = \gamma + \epsilon_3\tau$, so that unless $z = 0$, $|\lambda| \rightarrow \infty$ and

$$F(\alpha + \epsilon_1\tau, \beta; \gamma + \epsilon_3\tau; z) = F(a + \lambda/z, b; \lambda; z). \quad (2)$$

We refer to this function simply as F if no ambiguity would arise. If $z = 0$, then (1) gives $F = 1$; its asymptotic expansion is just 1.

Our theory will assume that b is not a negative integer, so that there is a branch cut in the complex z -plane, and that $z \notin [1, \infty)$ in order to avoid

it. If b were a negative integer, F would be a polynomial in z and a rational function of λ , no branch cut would exist, and the asymptotic series would converge if $|\lambda| > |b| - 1$.

The motivation for studying the case $\varepsilon_2 = 0, \varepsilon_3 = \varepsilon_1 z$ was the author's recent theory [3] for a gas bubble rising in a solution of a substance (e.g. common salt in water) that raises the surface tension. Finding the drag on the bubble led to a series $\sum f_n \sigma_n$, in which $|f_n| \rightarrow 0$ as $n \rightarrow \infty$, and the values of σ_n were

$$\sigma_n = \frac{F(3c_n + 2, 1; 2c_n + 2; \frac{2}{3})}{1 + 2c_n} - \frac{F(3c_n + 3, 1; 2c_n + 3; \frac{2}{3})}{1 + c_n}, \quad (3)$$

where $c_n \rightarrow \infty$ as $n \rightarrow \infty$, and every $c_n > 0$. Equation (3) involves two special cases of (2): $a = -\frac{1}{2}d$, $b = 1$, $z = \frac{2}{3}$, $\lambda = 2c_n + d$, where $d = 2$ or 3 . Section 3 will show that $|\sigma_n| = O(c_n^{-1/2})$; the series $\sum f_n \sigma_n$ converges.

2 Integral representation of F

Let $\text{ph} : \mathbb{C} \setminus \{0\} \rightarrow (-\pi, \pi]$ be the single-valued phase function [5, §1.9(i)], let $\theta_\lambda = \text{ph}(\lambda)$, $\theta_z = \text{ph}(z)$, let a , b and z be finite real or complex constants and let $G(\lambda, b)$, $I(\lambda, a, b, z)$, $p(t, z)$, $q(t, a, b, z)$, $r(t, z, \theta_\lambda)$ be the functions

$$G(\lambda, b) = \frac{\Gamma(\lambda)}{\Gamma(b)\Gamma(\lambda - b)}, \quad (4)$$

$$I(\lambda, a, b, z) = \int_0^1 t^{b-1} (1-t)^{\lambda-b-1} (1-zt)^{-(\lambda/z)-a} dt \quad (5)$$

$$= \int_0^1 \exp(-\lambda p(t, z)) q(t, a, b, z) dt, \quad (6)$$

$$p(t, z) = z^{-1} \ln(1-zt) - \ln(1-t), \quad (7)$$

$$q(t, a, b, z) = t^{b-1} (1-t)^{-b-1} (1-zt)^{-a}, \quad (8)$$

$$r(t, z, \theta_\lambda) = \text{Re}(e^{i\theta_\lambda} p(t, z)). \quad (9)$$

If $z \neq 0$ or 1 , $|\text{ph}(1-z)| < \pi$, and $\text{Re}(\lambda) > \text{Re}(b) > 0$, then [5, §15.6]

$$F(a + \lambda/z, b; \lambda; z) = G(\lambda, b) I(\lambda, a, b, z). \quad (10)$$

The asymptotic form of $G(\lambda, b)$ is well known [5, §5.11(iii)]. If $|\lambda| \rightarrow \infty$ and $|\theta_\lambda| \leq \pi - \delta < \pi$, then

$$G(\lambda, b) \sim \frac{\lambda^b}{\Gamma(b)} \sum_{s=0}^{\infty} \binom{b}{s} \frac{B_s^{(b+1)}}{\lambda^s} = \frac{\lambda^b}{\Gamma(b)} \sum_{s=0}^{\infty} \frac{G_s}{\lambda^s}, \text{ say,} \quad (11)$$

where the $B_s^{(b+1)}$ are Bernoulli numbers of order $b+1$ [5, §24.16(i)]:

$$G_0 = 1, \quad (12)$$

$$G_1 = -b(b+1)/2, \quad (13)$$

$$G_2 = b(b-1)(b+1)(3b+2)/24. \quad (14)$$

3 Real $z < 1, z \neq 0$

Because we use both (6) and (10) we assume henceforth that $|\theta_\lambda| < \frac{1}{2}\pi$, $z \neq 0$, and $z \notin [1, \infty)$. We also assume for the time being that $\operatorname{Re}(\lambda) > \operatorname{Re}(b) > 0$ and $z < 1$, but the restriction on $\operatorname{Re}(b)$ will be relaxed in Section 3.2 and the restriction on z will be relaxed in Section 4.

3.1 $\operatorname{Re}(b) > 0$

Let S be the open sector $\{t : 0 < |t| < 1, |\operatorname{ph}(t)| < \frac{1}{2}\pi\}$. Let D be the open disc $\{t : |t| < \min(1, 1/|z|)\}$. Let A be the closed annular sector

$$A = \{\lambda : |\lambda| \geq R_{\min} > \max(|b| + 1, |z| + |a|), |\theta_\lambda| \leq \frac{1}{2}\pi - \delta < \frac{1}{2}\pi\}. \quad (15)$$

We may use Laplace's method [5, §2.4(iii)] if $z \in (-\infty, 0) \cup (0, 1)$ to find the asymptotic series of $I(\lambda, a, b, z)$ by integrating along $P_1 = [0, 1]$ because the following conditions are all satisfied.

- (i) Both $p(t, z)$ and $q(t, a, b, z)$ are analytic for $t \in S$, and the path of integration with t real, from 0 to 1, is in S except for its ends.
- (ii) If $t \in D$ then $p(t, z)$ and $q(t, a, b, z)$ have these convergent series in powers of t , with $p_0 \neq 0$ and $q_0 \neq 0$:

$$p(t, z) = \sum_{n=0}^{\infty} p_n t^{n+2}, \quad p_n = \frac{1 - z^{n+1}}{n+2}, \quad (16)$$

$$q(t, a, b, z) = \sum_{n=0}^{\infty} q_n t^{n+b-1}, \quad q_n = (-1)^n \sum_{m=0}^n \binom{-b-1}{n-m} \binom{-a}{m} z^m. \quad (17)$$

If b is not an integer we use the principal value of t^b in (17).

- (iii) $I(\lambda, a, b, z)$ converges at $t = 1$ absolutely and uniformly with respect to $\lambda \in A$.
- (iv) If $0 < t < 1$ then $r(t, z, \theta_\lambda) > r(0, z, \theta_\lambda) = 0$.
- (v) If $\lambda \in A$, $r(t, z, \theta_\lambda)$ is bounded away from zero uniformly with respect to θ_λ as $t \rightarrow 1$ along P_1 .

Then, as $\lambda \rightarrow \infty$ in S ,

$$I(\lambda, a, b, z) \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+b}{2}\right) \frac{a_s}{\lambda^{(s+b)/2}}, \quad (18)$$

where the coefficient a_s is the residue at $t = 0$ of

$$\frac{q(t, a, b, z)}{2p(t, z)^{(s+b)/2}}, \quad (19)$$

which has a pole of order s there. Expressions for a_s are also known in terms of partial ordinary Bell polynomials [9, 10] and of ordinary potential polynomials [4]. We have

$$a_0 = \frac{q_0}{2p_0^{b/2}} = \frac{2^{b/2-1}}{(1-z)^{b/2}}, \quad (20)$$

$$a_1 = \left[\frac{q_1}{2} - \frac{(b+1)p_1q_0}{4p_0} \right] \frac{1}{p_0^{(b+1)/2}}, \quad (21)$$

$$a_2 = \left[\frac{q_2}{2} - \frac{(b+2)p_1q_1}{4p_0} + \{(b+4)p_1^2 - 4p_0p_2\} \frac{(b+2)q_0}{16p_0^2} \right] \frac{1}{p_0^{(b+2)/2}}. \quad (22)$$

By the duplication formula for gamma functions [5, (5.5.5)], the leading term when z is real is

$$F\left(a + \frac{\lambda}{z}, b; \lambda; z\right) \sim \frac{2^{-b/2} \pi^{1/2} \lambda^{b/2}}{(1-z)^{b/2} \Gamma(\frac{1}{2} + b/2)}. \quad (23)$$

3.2 $\operatorname{Re}(b) \leq 0$

Equation (23) has been proved for $\operatorname{Re}(b) > 0$. It still holds if $b = 0$, because then $F = 1$. It also holds for any $b \in \mathbb{C}$ except a negative integer. That is because the value for $\operatorname{Re}(b) > -1$ follows from those for $b + 1$ and $b + 2$ [1, (15.2.11)]. It agrees with (23), and shows that each term after the first in the asymptotic series is $O(\lambda^{-1/2})$ times the previous one. In the same way we find the same results for $\operatorname{Re}(b) > -2, -3, -4, \dots$. If on the other hand b is a negative integer, the right-hand side of (23) is zero but the left-hand side is not. The asymptotic expansion then begins further along the series, which now converges and contains only integer powers of λ .

4 Complex z

We still require $\theta_\lambda \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, $z \neq 0$, and $z \notin [1, \infty)$. Checking where $r(t, z, \theta_\lambda) > 0$ when $z \in \mathbb{C}$ need not require dealing with $\operatorname{Im}(z) < 0$ because $r(t, z, \theta_\lambda) = r(t, \bar{z}, -\theta_\lambda)$. If the least value of $r(t, z, \theta_\lambda)$ for $t \in P_1 = [0, 1]$ is still at $t = 0$ even when $\operatorname{Im}(z) \neq 0$, then the result of Section 3 still holds.

We now show that a sufficient condition is $\ddot{r}(t, z, \theta_\lambda)|_{t=0} > 0$, where dots indicate t -derivatives. With θ_λ and z fixed, $r(t, z, \theta_\lambda)$ defined in (9) is a real function of t infinitely differentiable on P_1 ; its least value on P_1 is thus either at $t = 0$ or at another point $t = t'$ where $\dot{r}(t, z, \theta_\lambda) = 0$. That gives a quadratic equation for t ; its two roots are 0 and t' . Because $r(t, z, \theta_\lambda) \rightarrow +\infty$ as $t \rightarrow 1$ along P_1 , $t' \in (0, 1)$ would require $r(t', z, \theta_\lambda) < 0$ and $\dot{r}(t', z, \theta_\lambda) = 0$. Now

$$\dot{r}(t, z, \theta_\lambda) = \operatorname{Re} \left(\frac{e^{i\theta_\lambda}}{1-t} - \frac{e^{i\theta_\lambda}}{1-zt} \right), \quad (24)$$

$$\therefore t' = \frac{\ddot{r}(0, z, \theta_\lambda)}{|z| \cos(\theta_\lambda - \theta_z) - |z|^2 \cos \theta_\lambda}, \quad (25)$$

$$\ddot{r}(t, z, \theta_\lambda) = \operatorname{Re} \left(\frac{e^{i\theta_\lambda}(1-z)(1-zt^2)}{(1-t)^2(1-zt)^2} \right). \quad (26)$$

If $\ddot{r}(0, z, \theta_\lambda) > 0$ then $r(t, z, \theta_\lambda) > 0$ when $0 < t \ll 1$, and there would have to be a maximum of r at $t'' \in (0, t')$ before r decreased to its minimum at t' . The quadratic equation would then have three roots, which is impossible. Therefore if $\ddot{r}(0, z, \theta_\lambda) > 0$ then $t' \notin (0, 1)$, and the leading contribution to the integral (6) is from t near 0, with P_1 still the path of integration.

Figure 1: The complex t -plane ($t = x + iy$), in a case where P_2 is useful but P_1 is not: $|z| = 3$, $\theta_z = 30^\circ$, $\theta_\lambda = -36^\circ$. Dotted lines on which $r(t, z, \theta_\lambda) = 0$ separate regions where it is of constant sign, marked $+$ and $-$. The lines spiral in towards the branch points on other Riemann sheets; only the principal sheet is shown. Dashed lines: the straight line P_1 and arc P_2 . Dot-dash-dot-dash lines: the branch cuts for $\ln(1 - zt)/z$ and $\ln(1 - t)$. Black circles: the branch points $t = 1$ and $1/z$.

Experiments with Waterloo Maple computer algebra revealed (to the author's initial surprise) that the results of Section 3 still held even when $\ddot{r}(0, z, \theta_\lambda) < 0$. That is because there are other paths through complex values from $t = 0$ to $t = 1$ along which $r(t, z, \theta_\lambda)$ has its least value at $t = 0$. Figures 1 and 2 illustrate the possibilities. In Figure 1 $|z| = 3$, $\theta_z = 30^\circ$, $\theta_\lambda = -36^\circ$, $\omega \approx 86^\circ$, and in Figure 2 $|z| = 8$, $\theta_z = 3^\circ$, $\theta_\lambda = 83^\circ$, $\omega \approx 47^\circ$, where $\omega = -\frac{1}{2}\{\text{ph}(1 - z) + \theta_\lambda\}$.

Let $\theta_t = \text{ph}(t)$. Then

$$r(t, z, \theta_\lambda) \sim \text{Re} \left(\frac{1}{2}(1 - z)t^2 e^{i\theta_\lambda} \right) \text{ as } t \rightarrow 0 \quad (27)$$

$$= \left| \frac{1}{2}(1 - z)t^2 \right| \cos(2\theta_t - 2\omega). \quad (28)$$

Therefore $r(t, z, \theta_\lambda)$ begins increasing as t leaves 0 if $|\theta_t - \omega| < \frac{1}{4}\pi$, and it does so as fast as possible if $\theta_t = \omega$. The other direction of steepest ascent of r (or steepest descent of $e^{-\lambda p}$) from $t = 0$ is $\theta_t = \omega \pm \pi$, but it will not concern us. Let P_2 be the circular arc passing through $t = 0$ and $t = 1$ with

Figure 2: The complex t -plane in a case where P_3 is useful but P_1 and P_2 are not: $|z| = 8$, $\theta_z = 3^\circ$, $\theta_\lambda = 83^\circ$. Solid line: P_3 . Other symbols as in Figure 1.

its tangent at 0 in the direction $\theta_t = \omega$, so that if $\sin \omega \neq 0$

$$t = \frac{(\sin \phi + \sin \omega) + i(\cos \phi - \cos \omega)}{2 \sin \omega}, \quad \phi \in [-\omega, \omega]. \quad (29)$$

Numerical work with $|z|$ from 0.5 to 10, θ_z from 3° to 177° , and θ_λ from -87° to $+87^\circ$ showed that $r(t, z, \theta_\lambda) > 0$ everywhere except $t = 0$ on at least one of P_1 and P_2 , except for a few cases where $r(t, z, \theta_\lambda) < 0$ at some points on each of P_1 and P_2 . In those cases there is a path P_3 on which $r(t, z, \theta_\lambda) > 0$ everywhere except $t = 0$; we obtain P_3 by modifying P_2 to bring $y = \text{Im}(t)$ nearer to 0 by an amount gradually increasing as ϕ increases:

$$y = \frac{\cos \phi - \cos \omega}{2 \cosh(1 + \phi/\omega) \sin \omega}, \quad \phi \in [-\omega, \omega]. \quad (30)$$

Because $t = 0$ when $\phi = -\omega$ and $t = 1$ when $\phi = \omega$, P_3 leaves $t = 0$ in the same direction as P_2 but arrives at $t = 1$ from a direction much closer to the real axis $y = 0$; see Figure 2. P_3 was needed if $|z| \gg 1$, $|\theta_z| \ll \frac{1}{2}\pi$, so that z was near the positive real axis but far from 1, and θ_λ was near $\pm \frac{1}{2}\pi$. Then $r < 0$ both on the straight line P_1 very near $t = 0$ and on much of the arc P_2 ; hence both paths were unsuitable for Laplace's method. Because $\omega \approx 47^\circ$ in Figure 2, the lower right negative region is between $\theta_t \approx +2^\circ$ and $\theta_t \approx -88^\circ$

near $t = 0$, and the path P_1 is initially inside it. The circular-arc path P_2 is in a negative region if $x > 0.53$, but P_3 is in a positive region for its whole length. Note that P_2 and P_3 are not paths of steepest descent though they do leave $t = 0$ in the steepest-descent direction. As the NIST Handbook says [5, §2.4(iv)], “for the purpose of simply deriving the asymptotic expansions the use of steepest descent paths is not essential”.

The remaining task was to choose the correct branch of $p_0^{b/2}$ in (23). That was the one on which $\text{ph}(p_0) = \text{ph}(1 - z)$ [5, eq. 2.4.13], whether the path of integration was P_1 , P_2 or P_3 .

5 Conclusion

The asymptotic form (23) for $F(a + \lambda/z, b; \lambda; z)$ as $|\lambda| \rightarrow \infty$ if $|\text{ph}\lambda| < \frac{1}{2}\pi$ was found in Section 3.1 for $\text{Re}(b) > 0$, z real, $z \neq 0$, and $z < 1$. Section 3.2 shows that it still holds for any real or complex value of b except a negative integer if $z \in \mathbb{R} \setminus [1, \infty)$, and Section 4 extends that region to $z \in \mathbb{C} \setminus [1, \infty)$. In a recent paper on bubbles rising in a liquid [3], $b = 1$ and $z = \frac{2}{3}$, a special case which revealed that the present investigation was needed. The present results appear not to be in previous work on hypergeometric functions.

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References

- [1] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions*. Dover, 1972 (cit. on p. 5).

- [2] Mislav Cvitković, Ana-Sunčara Smith, and Jayant Pande. “Asymptotic expansions of the hypergeometric function with two large parameters—application to the partition function of a lattice gas in a field of traps”. In: *J. Phys. A: Math. Theor.* 50 (2017). (24pp), p. 265206 (cit. on p. 1).
- [3] J. F. Harper. “Effect of a negatively surface-active solute on a bubble rising in a liquid”. In: *Q. J. Mech. Appl. Maths* 71 (2018), pp. 427–439. DOI: doi.org/10.1093/qjmam/hby012 (cit. on pp. 2, 8).
- [4] Gergő Nemes. “An Explicit Formula for the Coefficients in Laplace’s Method”. In: *Constr. Approx.* 38 (2013), pp. 471–487. DOI: [10.1007/s00365-013-9202-6](https://doi.org/10.1007/s00365-013-9202-6) (cit. on p. 4).
- [5] Frank W. J. Olver, ed. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010 (cit. on pp. 1–4, 8).
- [6] R. B. Paris. “Asymptotics of the Gauss hypergeometric function with large parameters, I”. In: *J. Classical Anal.* 2 (2013), pp. 183–203 (cit. on p. 1).
- [7] R. B. Paris. “Asymptotics of the Gauss hypergeometric function with large parameters, II”. In: *J. Classical Anal.* 3 (2013), pp. 1–15 (cit. on p. 1).
- [8] G. N. Watson. “Asymptotic expansions of hypergeometric functions”. In: *Trans. Cambridge Philos. Soc.* 22 (1918), pp. 277–308 (cit. on p. 1).
- [9] John Wojdylo. “On the coefficients that arise from Laplace’s method”. In: *J. Comput. Appl. Math.* 196 (2006), pp. 241–266 (cit. on p. 4).
- [10] John Wojdylo. “Computing the Coefficients in Laplace’s Method”. In: *SIAM Rev.* 48 (2006), pp. 76–96 (cit. on p. 4).

Author address

- (i) **J. F. Harper**, School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, NEW ZEALAND.
<mailto:john.harper@vuw.ac.nz>