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Assessing the Likelihood of Panic-Based Bank Runs

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Abstract

Conditional on the considered equilibrium, the probability of a bank run in the demand-deposit contract models of Bryant (1980) and of Diamond and Dybvig (1983) is either one or zero. In contrast, we establish the existence of an interval - being a strict subset of the unit-interval - of possible bank run probabilities for a two-player demand-deposit contract model where players receive independent signals about their liquidity desire from a continuous type space. As our main result we demonstrate that this interval reduces to a unique probability of a panic-based bank strictly smaller than one if and only if there exist types for which not running on the bank is a dominant action. In addition to existing models of bank runs such as, e.g., Goldstein and Pauzner (2005), our approach also provides some assessment of the likelihood of a bank run if there are no types for which not running on the bank is a dominant action. As a consequence, we can investigate the comparative statics of the likelihood of bank runs with respect to a larger range of payoff parameters than considered in previous models. Furthermore, we derive a technical result by which the findings of Morris and Shin (2005) on the dominance-solvability of binary action games with strategic complements also apply to nice games in the sense of Moulin (1984) if players' best response functions are increasing.

KEYWORDS: demand-deposit contract, bank run, dominance solution, supermodular game, nice game, binary action game, heterogeneity, uniqueness

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1 Introduction

1.1 Motivation

Ever since the seminal contributions of Bryant (1980) and Diamond and Dybvig (1983) strategic models of bank runs have been the subject of intensive study in the literature. Diamond and Dybvig (1983) consider the strategic implications of a demand deposit contract under the assumption that every agent receives a private signal determining whether he has a high or a low desire for liquidity. Diamond and Dybvig observe that there exists - besides a *good* equilibrium implementing the optimal allocation - a *bad* equilibrium in which all agents prematurely withdraw their funds due to strategic considerations. Thus, according to the Diamond–Dybvig model, the probability of a bank run is - conditional on the considered equilibrium - either one out of two extreme cases. Namely, there is either a bank run with certainty or there is no bank run at all.

In this paper we assess the likelihood of a panic-based bank run under the assumption that the agents receive independent private signals about their liquidity desire from a continuous type space. In contrast to the extreme case of the Diamond–Dybvig model, a non-trivial interval of possible bank run probabilities exists as a strict subset of the unit-interval for all payoff parameters. Furthermore, we specify the range of payoff parameters for which there exists a unique bank run probability.

While the early models of Bryant and of Diamond and Dybvig describe the occurrence of bank runs as a coordination problem arising from the existence of multiple equilibria, more recent models try to deduce a unique bank run probability from the existence of a unique Bayesian Nash equilibrium (Postlewaite and Vives, 1987; Rochet and Vives, 2004; Goldstein and Pauzner, 2005). For a two-player demand-deposit contract game, Postlewaite and Vives (1987) modify the intertemporal preferences such that each agent has a unique dominant strategy. Thus, the model of Postlewaite and Vives does not capture the *strategic dimension* associated with panic-based bank runs but it rather describes a situation of bank runs based on *fundamentals*.

Alternative models of panic-based bank runs with a unique Bayesian Nash equilibrium are presented in Rochet and Vives (2004) and in Goldstein and Pauzner (2005) who adopt the *global game approach* (cf. Carlsson and van Damme, 1993; Morris and Shin, 1998). In a global game, agents infer from their own signals relevant information about the signals of the other agents. Such an information structure may apply to situations where the signals refer to the long-term project’s profitability as assumed in Rochet and Vives (2004) and in Goldstein and Pauzner (2005). But in situations where the signals refer to, e.g., a player’s time-discount factor or survival probability our assumption of independent signals appears as more plausible.

The uniqueness result in Goldstein and Pauzner (2005) is derived under the assumption that the so-called *upper dominance regions* are non-empty, i.e., that there exists for every agent a subset of types with measure strictly greater zero for which not running on the bank is a dominant action. While this assumption is technically very convenient,

it excludes plausible situations where - regardless of his own information - an agent will run on the bank if he learned that the other agents are going to withdraw their funds. As one main insight of our approach we demonstrate that there exists a unique probability of a bank run that is strictly smaller than one if and only if the upper dominance regions of all agents are non-empty. Even more interestingly we also characterize the interval of possible bank-run probabilities for the case that the agents' upper dominance regions are empty. Thus, in contrast to, e.g., the models of Postlewaite and Vives (1987), Rochet and Vives (2004), and Goldstein and Pauzner (2005), our model provides some assessment of the likelihood of a bank run even for payoff parameters that do not ensure the existence of a unique bank run probability. Key to our approach is the assumption that the agents may possibly choose any strategy that survives the iterated elimination of strictly dominated strategies, i.e., that belongs to the model's dominance solution. This assumption is plausible for one-shot strategic situations where players lack the opportunity to learn how to coordinate their strategy choices in a Nash equilibrium.

1.2 Relation to the Game-theoretic Literature on Uniqueness from Heterogeneity

As a prerequisite for deriving our main result we prove a technical lemma stating that strictly quasiconcave utility functions combined with increasing best response functions entail a dominance solution that is a non-empty closed interval whose minimal and maximal points are Nash equilibria. Milgrom and Roberts (1990) establish an analogous result for so-called *supermodular games* (see, e.g., Topkis, 1978; Milgrom and Roberts, 1990; Vives, 1990, 2005; Amir, 2004). Supermodular games are characterized by players whose strategies are *strategic complements* (Bulow, Geneakoplos, and Klemperer, 1985) so that a player's incentive to choose a larger strategy increases when his opponents also choose larger strategies. Since our model violates this complementarity assumption, the proof of our lemma differs from the approach of Milgrom and Roberts. In particular, our proof partly builds on a result of Moulin (1984) who studies so-called *nice games* where the players have strictly quasiconcave utility functions.

The model of panic-based bank runs presented in this paper belongs to the class of so-called *binary action games*. Under the assumption of strategic complements, Morris and Shin (2005) investigate how heterogeneity in payoff-types may induce dominance-solvability of binary action games. For a benchmark binary action game, Morris and Shin establish that this game is dominance-solvable if and only if the function $\text{prob}(\theta_j < \bar{\theta}_i \mid \bar{\theta}_i)$ has a unique fixed point in Θ_i , where θ_j denotes the payoff-type of player j and $\bar{\theta}_i \in \Theta_i$ stands for the *cutoff-point* of player i , (that is, all types $\theta_i \leq \bar{\theta}_i$ choose one particular action whereas all types $\theta_i > \bar{\theta}_i$ choose the other action). Morris and Shin demonstrate that $\text{prob}(\theta_j < \bar{\theta}_i \mid \bar{\theta}_i)$ has a unique fixed point in Θ_i if and only if the first order derivative of $\text{prob}(\theta_j < \bar{\theta}_i \mid \bar{\theta}_i)$ is smaller one; which is in particular the case if $\text{prob}(\theta_j < \bar{\theta}_i \mid \bar{\theta}_i)$ is close to a constant function. Moreover, under the assumption of a joint normal distribution on the players' type space, $\Theta_i \times \Theta_j$,

Morris and Shin show that $\text{prob}(\theta_j < \bar{\theta}_i \mid \bar{\theta}_i)$ approaches a constant function either if (i) the variance of players's types is sufficiently great or (ii) if players' types are almost (!) perfectly correlated. This is a remarkable insight since it helps to explain why rather contrary assumptions on the information structure - on the one hand models that assume independence of types (see, e.g., McKelvey and Palfrey, 1995; Herrendorf, Valentinyi and Waldmann, 2000; Baliga and Sjöström, 2004) on the other hand the global game approach requiring strong correlation - may establish uniqueness of the strategic solution from heterogeneity of payoff-types.

Our technical lemma now implies that the conclusions in Morris and Shin (2005) remain valid for binary action games if we replace the assumption of strategic complements with the assumptions of (i) strictly quasiconcave utility functions and (ii) increasing best response functions. Binary action games with strictly quasiconcave utility functions satisfy Moulin's (1984) definition of nice games since each player's relevant strategy set is the unit-interval on the real line¹. Combined with increasing best response functions the resulting game has all the nice properties of a supermodular game - as exploited in the proofs in Morris and Shin (2005) - in the sense of Milgrom and Roberts (1990). Besides the results on the likelihood of bank runs this technical finding on uniqueness from heterogeneity is the second main contribution of the present paper.

The remainder of the paper is organized as follows. In section 2 we introduce our model. The solution to our model is presented and discussed in section 3. In section 4 we illustrate our main results by investigating the comparative statics of the likelihood of bank runs for two examples. Finally, section 5 concludes. All formal proofs are relegated to the appendix.

2 Model

According to a demand deposit contract all agents of the economy deposit in period 0 one monetary unit in a mutual bank that invests in a long-term investment project. This project produces profits in period 2 while in period 1 it only pays back one monetary unit per one unit invested. In period 1 each agent receives a private signal from a continuous type space that effects his desire for liquidity in period 1. By liquidating the required fraction of the invested money the bank pays out - subject to a sequential service constraint - one monetary unit plus a contracted interest rate to each agent who withdraws in period 1. The agents who did not withdraw money in period 1 then mutually share in period 2 the proceedings of the money that remained invested in the long-term project. Bryant (1980) and Diamond and Dybvig (1983) demonstrate that such a demand deposit contract may improve the welfare of individuals who are ex ante - in period 0 - uncertain about their future liquidity desire. In particular, for the case of only two different privately observable types, Diamond and Dybvig show that a

¹ As demonstrated in Zimper (2005), Moulin's argument does not apply to individual strategy sets that are subsets of $\mathbb{R}^n$ with $n \geq 2$.

demand deposit contract may implement the socially optimal allocation subject to the corresponding incentive compatibility constraints.

In our stylized model we restrict attention to an economy consisting of only two agents $i \in \{A, B\}$. If both agents simultaneously WITHDRAW IN PERIOD 1, then the “first” agent receives - in accordance with a sequential service constraint - the contracted withdrawal amount r such that $1 < r < 2$. The “second” agent has to settle for the remaining amount $(2 - r)$. We assume that both agents assess the likelihood of being “first” respectively “second” with equal probability of 0.5. The long-term project gives a profit R , with $R > r$, per monetary unit remaining invested after any withdrawals in period 1. Thus, if only one agent WITHDRAWS IN PERIOD 1 the remaining agent receives in period 2 the amount $(2 - r) R$. If both agents do NOT WITHDRAW IN PERIOD 1 they mutually share the total profits, $2R$, of the project in period 2. Conditional on the behavior of $j \in \{A, B\}$, the demand-deposit contract therefore induces the following prospects of monetary payoffs for the agent $i \in \{A, B\}$, with $i \neq j$, when he withdraws in period 1 or period 2, respectively:

	j WITHDRAWS IN PERIOD 1	j WITHDRAWS IN PERIOD 2
i WITHDRAWS IN 1	$\begin{cases} r \text{ with probability } \frac{1}{2} \\ 2 - r \text{ with probability } \frac{1}{2} \end{cases}$	r
i WITHDRAWS IN 2	$(2 - r) R$	R

We introduce now our central assumption, namely, that agent i 's evaluation, $i \in \{A, B\}$, of his period 2 monetary payoffs is weighted by some private signal θ_i which is independently drawn from a uniform distribution over $\Theta_i = [0, 1]$. Under the assumption of risk-neutral expected utility maximizers, these signals allow for several different interpretations such as, e.g., the agent's time-discount factor related to his liquidity needs, the agent's survival probability, the probability of the long-term project's success, or the actual profits of the long-term project. The agent receives the signal before he either decides to WITHDRAW or NOT to WITHDRAW in period 1.

Conditional on the agents' strategic decisions to WITHDRAW in period 1 or NOT to WITHDRAW in period 1, the demand-deposit contract therefore induces the following utility matrix for agent i :

		j	
		WITHDRAW	NOT WITHDRAW
i	WITHDRAW	1	r
	NOT WITHDRAW	$(2 - r) R \theta_i$	$R \theta_i$

We say that agent i 's *upper dominance region* is non-empty if there exists a subset of types with measure greater zero for which NOT to WITHDRAW is a strictly dominant action. Suppose there exists some $\bar{\theta}_i \in \Theta_i$ such that $(2 - r) R \bar{\theta}_i = 1$. By assumption $1 < r < R$, so that $\bar{\theta}_i$ is indifferent between WITHDRAW and NOT to WITHDRAW if j withdraws whereas $\bar{\theta}_i$ strictly prefers NOT to WITHDRAW to WITHDRAW if j does not withdraw. As a consequence, NOT to WITHDRAW is a strictly dominant action for a type θ_i if and only if $\theta_i > \bar{\theta}_i$ so that we obtain

Observation: *There exists a non-empty upper dominance region of agent $i \in \{A, B\}$, $(\bar{\theta}_i, 1] \subset \Theta_i$, if and only if*

$$R > \frac{1}{(2 - r)}.$$

For $i \in \{A, B\}$, define a *switching strategy* s_i as a map

$$s_i : \Theta_i \rightarrow \{\text{WITHDRAW}, \text{NOT WITHDRAW}\}$$

such that for a corresponding *cutoff-point* $\sigma_i \in [0, 1]$

$$s_i(\theta_i) = \begin{cases} \text{WITHDRAW} & \text{if } \theta_i \leq \sigma_i \\ \text{NOT WITHDRAW} & \text{if } \theta_i > \sigma_i \end{cases}$$

Observe that every switching strategy s_i is completely characterized by its cutoff-point $\sigma_i \in S_i := [0, 1]$. Our assumption of a uniform distribution entails that if player A expects B to choose cutoff-point σ_B , then A expects B to WITHDRAW with probability σ_B . For a risk-neutral expected utility maximizer $i \in \{A, B\}$ we thus obtain the following type-dependent utility function $U_i : \Theta_i \times S_i \times S_{-i} \rightarrow \mathbb{R}_+$:

$$U_i(\theta_i, \sigma_i, \sigma_{-i}) = \begin{cases} r(1 - \sigma_{-i}) + \sigma_{-i} & \text{if } \theta_i \leq \sigma_i \\ R \theta_i (1 - \sigma_{-i}) + (2 - r) R \theta_i \sigma_{-i} & \text{if } \theta_i > \sigma_i \end{cases} \quad (1)$$

The ex-ante expected utility $U_i(\sigma_i, \sigma_{-i})$ - before learning his type - of agent $i \in \{A, B\}$ is therefore given by

$$\begin{aligned}
& \int_0^1 U_i(\theta_i, s_i, s_{-i}) d\theta_i \\
&= \int_0^{\sigma_i} r(1 - \sigma_{-i}) + \sigma_{-i} d\theta_i + \int_{\sigma_i}^1 R\theta_i(1 - \sigma_{-i}) + (2 - r)R\theta_i\sigma_{-i} d\theta_i \\
&= (r(1 - \sigma_{-i}) + \sigma_{-i})\sigma_i + \left(\frac{R}{2}(1 - \sigma_{-i}) + (2 - r)\frac{R}{2}\sigma_{-i} \right) (1 - \sigma_i^2). \quad (2)
\end{aligned}$$

Thus, our model of panic-based bank runs is formally described as a game in strategic form, $G := (S_i, U_i)_{i \in \{A, B\}}$, such that the strategy sets S_i consist of cutoff-points and the expected utility functions U_i are given by (2) whereby the payoff parameters r and R must satisfy $1 < r < 2$ and $r < R$.

3 Main Results

Before we present our main result let us first recall the formal definition of the *dominance solution* of a game, which serves as our strategic solution concept. For a finite set of players $I := \{1, \dots, N\}$, let $G := (S_i, U_i)_{i \in I}$ denote a strategic game in normal form where S_i denotes the non-empty individual strategy set of player $i \in I$ and where $U_i : S_i \times S_{-i} \rightarrow \mathbb{R}_0$ represents player i 's preferences over strategies in $S := \times_{i \in I} S_i$. Let $\vartheta^0 := S$ and denote by ϑ^k , $k \geq 1$, the set of all strategies that are undominated in S with respect to ϑ^{k-1} , i.e., for all $i \in I$, $s_i \in \vartheta_i^k$ if and only if there does not exist some $t_i \in S_i$ such that, for all $s_{-i} \in \vartheta_{-i}^{k-1}$, $U_i(t_i, s_{-i}) > U_i(s_i, s_{-i})$.

Definition: The dominance solution of game $G = (S_i, U_i)_{i \in I}$ is defined as the set of strategy-profiles that survives iterated elimination of strictly dominated strategies, i.e.,

$$D(G) := \bigcap_{k=0}^{\infty} \vartheta^k.$$

A game is called dominance-solvable if and only if the dominance solution contains a unique strategy-profile such that this strategy-profile is a Nash equilibrium of the game. If a game is dominance-solvable it has the desirable property that all strategic solution concepts - equilibrium solution concepts (e.g., Nash, 1950a,b; Selten, 1975; Myerson, 1978; for an overview see van Damme, 1991) as well as iterative solution concepts (Bernheim, 1984; Moulin, 1984; Pearce, 1984; Börgers, 1993; Ghirardato and Le Breton, 1997 and 2000) - coincide.

The proof of our main result is based on an application of the following lemma.

Lemma: Consider a strategic game $G = (S_i, U_i)_{i \in I}$ with $I := \{1, \dots, N\}$ such that S_i is a non-empty compact and convex subset of $\mathbb{R}$ and U_i is strictly quasiconcave w.r.t.

s_i for all $i \in I$. If all players have increasing and continuous best response functions, then the dominance-solution of G is given as the non-empty closed interval

$$D(G) = [\min D(G), \max D(G)] \subset [0, 1]^N$$

such that $\min D(G)$ is the smallest and $\max D(G)$ is the largest Nash equilibrium of G with respect to the natural order $\leq$.

Clearly, any game that satisfies the assumptions of the lemma is dominance-solvable if and only if it has a unique Nash equilibrium.

Proposition: *The dominance solution of our model of panic-based bank runs is the non-empty closed interval*

$$D(G) = [\min D(G), \max D(G)] \subset [0, 1]^2$$

such that the corresponding smallest, respectively largest, strategy of each agent $i \in \{A, B\}$ is determined by the payoff parameters r and R as follows:

a. If $R \leq \frac{1}{(2-r)}$, then

$$\max D_i(G) = 1,$$

$$\min D_i(G) = \min \left\{ 1, \frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)} \right\}$$

b. If $R > \frac{1}{(2-r)}$, then

$$\begin{aligned} \min D_i(G) &= \max D_i(G) \\ &= \frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)}. \end{aligned}$$

If both agents simultaneously WITHDRAW in period 1 we speak of a bank run. For chosen cutoff-points σ_A and σ_B the probability of a bank run in our model is - due to the independence assumption - given by the product $\sigma_A \sigma_B$. Since a strategically rational player in our sense may choose any strategy that survives the iterated elimination of strictly dominated strategies, the set of possible bank run probabilities is therefore given by all probabilities $\sigma_A \sigma_B$ such that σ_A and σ_B are cutoff-points belonging to the model's dominance solution. The following corollary - combining our observation from section 2 with the above proposition - states our model's main insight.

Corollary:

a. If each agent's upper dominance region is empty, i.e., $R \leq \frac{1}{(2-r)}$, then the possible probabilities by which a bank run realizes lie in the interval

$$\left[\left(\frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)} \right)^2, 1 \right] \subset (0, 1],$$

when this interval is well-defined. In case this interval is not well-defined, then a bank run realizes with the unique probability 1.

- b. If each agent's upper dominance region is non-empty, i.e., $R > \frac{1}{(2-r)}$, then a bank run realizes with the unique probability

$$\left(\frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)} \right)^2 \in (0, 1).$$

Remark 1. By a well-known result of Milgrom and Roberts (1990) the largest as well as the smallest strategy profile of the dominance-solution of a supermodular game are Nash equilibria. A supermodular game in the sense Milgrom and Roberts is basically characterized by *increasing utility differences* with respect to an imposed lattice order. By imposing the natural order $\leq$ on $\mathbb{R}$ for both agents, our model would exhibit increasing utility differences for agent $i \in \{A, B\}$ if and only if we have, for all cutoff points $\tau_i, \sigma_i \in S_i$ such that $\tau_i < \sigma_i$,

$$\frac{\partial [U_i(\sigma_i, \sigma_{-i}) - U_i(\tau_i, \sigma_{-i})]}{\partial \sigma_{-i}} \geq 0. \quad (3)$$

To see that our model is not a supermodular game observe that the utility differences are given by

$$\begin{aligned} & U_i(\sigma_i, \sigma_{-i}) - U_i(\tau_i, \sigma_{-i}) \\ &= (r(1 - \sigma_{-i}) + \sigma_{-i})(\sigma_i - \tau_i) + \left(\frac{R}{2}(1 - \sigma_{-i}) + (2 - r)\frac{R}{2}\sigma_{-i} \right) (\tau_i^2 - \sigma_i^2). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial [U_i(\sigma_i, \sigma_{-i}) - U_i(\tau_i, \sigma_{-i})]}{\partial \sigma_{-i}} &\geq 0 \Leftrightarrow \\ \frac{R}{2}(\sigma_i + \tau_i) &\geq 1 \end{aligned}$$

which only holds for rather large cutoff-points σ_i, τ_i whereas it is violated for rather small cutoff-points σ_i, τ_i .²

Remark 2. Rochet and Vives (2004) introduce financial intermediaries in their model of bank runs in order to obtain a convenient supermodular payoff structure. As a consequence, Rochet and Vives can apply the formal arguments in Morris and Shin (2002) for establishing dominance-solvability of their model. But, as Goldstein and Pauzner (2005, p. 1295) observe, such a supermodularity condition is typically not satisfied for models with standard demand-deposit contracts. While this observation of Goldstein and Pauzner also applies to our demand-deposit contract model, our technical

² Similarly, we cannot adopt the approach of Vives (1990) who analyzes a two-player game with decreasing utility differences as a supermodular game by reversing the order of one player's strategy set.

lemma entails that we could proceed in the proof of the proposition as if our non-supermodular model was a supermodular game in the sense of Milgrom and Roberts.

4 Illustrative Examples: Comparative Statics for the Likelihood of Bank Runs

We illustrate the impact of the value of the long-term project's profits, R , on the likelihood of a bank run by two examples. In the first example we consider a contracted withdrawal amount, r , that is rather low whereas it is high in the second example. As our main insight we conjecture that a high value of the contracted withdrawal amount does not only increase the likelihood of a bank run but also tends to increase the ambiguity of the model's predictions.

1. The probability of a bank run for $r=1.1$.

Example 1. For $r = 1.1$, Figure 1 displays the unique probability of a bank run depending on the values of the long-term project's proceeds $R \in [\frac{10}{9}, 4]$. Notice that this probability equals 1 if $R \leq \frac{10}{9}$. Not surprisingly, the probability of a bank run strictly declines with greater profits of the long-term project, R .

Example 2. For $r = 1.9$ table 1 displays the probabilities, respectively intervals of probabilities, of bank runs depending on the values of the long-term project's profits R . While there exists a unique probability of bank runs for very small and very large values of R , i.e., for $R \in \{2, 3, 4\}$ and for $R \in \{11, 12\}$, for intermediate values of R this probability lies in some non-degenerate interval that increases in R . Thus, in contrast to example 1, an increase in R does not necessarily reduce the likelihood of a bank run but it rather increases - at least on some range of values - the ambiguity of the

model's predictions.

Value of R	Poss. Prob. of a Bank Run
2	1
3	1
4	1
$\simeq 4.873805429511963$	$[0.433155, 1]$
5	$[0.323505, 1]$
6	$[0.161247, 1]$
7	$[0.105273, 1]$
8	$[0.0752917, 1]$
9	$[0.0568504, 1]$
10	$[0.0445679, 1]$
11	0.0359317
12	0.0296105

Table 1: Possible probabilities of a bank run for $r=1.9$

Observe that $r = 1.9$ implies that the strategy profile $\sigma^* = (1, 1)$ is a Nash equilibrium iff $R \leq 10$. For the case $R = 4$, Figure 2 illustrates that the best response function f_i intersects with the 45° line only at $\sigma_{-1} = 1$ so that a bank run occurs with certainty.

Figure 2: $r = 1.9$ and $R = 4$

This situation is similar for $R < 4.8738\dots$, at $R = 4.8738\dots$, however, the dominance solution “explodes” since there now exists another - minimal - fixed point. As a consequence, the possible bank run probabilities lie in the rather large interval $[0.433155, 1]$. Figure 3 demonstrates for $R = 6$ the existence of three fixed points of the best response function whereby all strategies between the smallest and the largest

equilibrium strategy belong to the dominance solution.

Figure 3: $r = 1.9$ and $R = 6$

For $R = 10$, as illustrated in Figure 4, $\sigma^* = (1, 1)$ is still a Nash equilibrium and the corresponding interval of possible bank run probabilities, $[0.0445679, 1]$, is extremely ambiguous.

Figure 4: $r = 1.9$ and $R = 10$

For $R > 10$ the dominance-solution reduces to a unique strategy profile, which induces a unique probability of a bank run that is rather low (< 0.0445679). Thus, around $R = 10$ there is a sharp drop in the maximal probability by which a bank run may occur. Figure 5 illustrates the existence of a unique fixed point for the case $R = 12$.

Figure 5: $r = 1.9$ and $R = 12$

The example 2 demonstrates that small changes of the payoff parameters may have a strong impact on the size of the interval containing the possible bank run probabilities. While in example 1 an increase in the long-term project's profitability R unambiguously decreases the probability of a bank run, the same is true in example 2 only for sufficiently large values of R . For comparably moderate values of R an increase in the project's profits only decreases the minimal probability by which a bank run may realize whereas it increases the ambiguity resulting from the multiplicity of possible bank run probabilities. We conclude that the possible probabilities of a bank run may be highly sensible with respect to the considered payoff-parameters. Thus, even our comparably simple model of panic-based bank runs does not necessarily generate robust predictions.

5 Conclusion

For a two-player demand-deposit contract game we establish the existence of a non-trivial interval of possible probabilities of a panic-based bank run. More specifically, we show that (i) the existence of non-empty upper dominance regions implies a unique probability of a bank run strictly smaller than one, whereas (ii) empty upper dominance regions give either rise to a bank run with certainty or a non-degenerate interval of possible bank run probabilities that are bounded away from zero. Investigating the comparative statics of the likelihood of bank runs we further demonstrate that very small changes in the long-term project's profitability may have a strong impact on the size of the interval of possible bank run probabilities. In our opinion, the fact that such a sensibility with respect to payoff parameters already appears in a highly stylized two-player model casts some doubt on the robustness of model predictions where more complex strategic situations of bank runs are considered.

Our formal argument combines and extends Moulin's (1984) approach for nice

games and Milgrom and Roberts' (1990) approach for supermodular games. In particular, we have derived a technical lemma that states conditions under which the dominance solution of a game with real-valued strategy sets is a closed convex set whose smallest and largest strategy profiles are Nash equilibria even if this game is not supermodular in the sense of Milgrom and Roberts (1990).

Appendix: Proofs

Proof of the lemma

Before we prove the lemma let us introduce the following formal definitions. Let $f_i : S_{-i} \rightarrow S_i$ denote player i 's individual best response function, i.e., for all $s_{-i} \in S_{-i}$,

$$f_i(s_{-i}) := \arg \max_{s_i \in S_i} U_i(s_i, s_{-i}).$$

Further, define $f : S \rightarrow S$ such that $f(s) := \times_{i \in I} f_i(s_{-i})$ for all $s \in S$. Let $\lambda^0 := S$ and define, for all $k \geq 1$,

$$\lambda^k := \bigcup_{s \in \lambda^{k-1}} f(s).$$

At first, we show that under the assumptions of the Lemma $\lambda^k = \vartheta^k$ for all $k \geq 0$. By continuity of f_i and compactness of S_i , every set λ_i^k is compact so that there exist strategies $\max \lambda_i^k$ and $\min \lambda_i^k$ for all $i \in I$ and $k \geq 0$ with respect to the natural order $\leq$. It is easy to show (cf. Moulin, 1984) that strictly quasiconcave utility functions imply

$$\begin{aligned} \max \lambda_i^k &= \max \vartheta_i^k \\ \min \lambda_i^k &= \min \vartheta_i^k \end{aligned}$$

for all $i \in I$ and $k \geq 0$, so that

$$\max \lambda^k = \max \vartheta^k \tag{4}$$

$$\min \lambda^k = \min \vartheta^k \tag{5}$$

for all $k \geq 0$.

By the intermediate value theorem, the values of the continuous function f_i take on every value between the points $\min \lambda_i^k$ and $\max \lambda_i^k$ so that the sets λ_i^k are convex for all $i \in I$ and $k \geq 0$. Thus, the sequence $\{\lambda^k\}_{k \geq 0}$ consists of nested sets that are compact, non-empty and convex. As a consequence, the infinite intersection

$$D(G) = \bigcap_{k=0}^{\infty} \lambda^k$$

is a compact non-empty convex set as well so that

$$D(G) = [\min D(G), \max D(G)]$$

for two strategy-profiles $\max D(G)$, $\min D(G) \in S$.

Since the sequence $\{\max \lambda_i^k\}_{k \geq 0}$ is monotonically decreasing, it converges to its lower limit $\hat{s}_i$, i.e.,

$$\lim_{k \rightarrow \infty} \max \lambda^k = \hat{s}. \quad (6)$$

Accordingly, since $\{\min \lambda_i^k\}_{k \geq 0}$ is monotonically increasing, it converges to its upper limit $\check{s}_i$, i.e.,

$$\lim_{k \rightarrow \infty} \min \lambda^k = \check{s}. \quad (7)$$

By (4) and (5),

$$D(G) \subset [\min \lambda^k, \max \lambda^k]$$

for all $k \geq 0$, so that we obtain, by (6) and (7),

$$D(G) \subset [\check{s}, \hat{s}].$$

Increasing best response functions imply

$$\begin{aligned} \max \lambda^{k+1} &= f(\max \lambda^k) \\ \min \lambda^{k+1} &= f(\min \lambda^k), \end{aligned}$$

for all $k \geq 0$, so that, by (6) and (7),

$$\begin{aligned} \lim_{k \rightarrow \infty} f(\max \lambda^k) &= \hat{s} \\ \lim_{k \rightarrow \infty} f(\min \lambda^k) &= \check{s}. \end{aligned}$$

By continuity of f , we also obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} f(\max \lambda^k) &= f(\hat{s}) \\ \lim_{k \rightarrow \infty} f(\min \lambda^k) &= f(\check{s}), \end{aligned}$$

which establishes that the strategy profiles $\check{s}$ and $\hat{s}$ must be Nash equilibria. As a consequence, $\check{s}, \hat{s} \in D(G)$ so that

$$D(G) = [\min D(G), \max D(G)] = [\check{s}, \hat{s}]$$

where $\check{s}$ is the smallest and $\hat{s}$ is the largest Nash equilibrium of G . $\square$

Proof of the proposition

Step 1. Clearly, the individual strategy sets of our model are non-empty convex compact subsets of $\mathbb{R}$. So, let us proceed by verifying that our model satisfies the remaining assumptions of the lemma.

i.) *Strictly quasiconcave utility functions.* Notice that

$$\begin{aligned} \frac{\partial U_A(\sigma_i, \sigma_{-i})}{\partial \sigma_i} &\leq 0 \Leftrightarrow \\ r(1 - \sigma_{-i}) + \sigma_{-i} - 2 \left(\frac{R}{2}(1 - \sigma_{-i}) + (2 - r) \frac{R}{2} \sigma_{-i} \right) \sigma_i &\leq 0 \Leftrightarrow \\ \frac{r - (r - 1) \sigma_{-i}}{R(1 - (r - 1) \sigma_{-i})} &\leq \sigma_i. \end{aligned}$$

That is, for any fixed $\sigma_{-i} \in [0, 1]$, U_i strictly increases in S_i until cutoff-point

$$\sigma_i^* = \min \left\{ 1, \frac{r - (r - 1) \sigma_{-i}}{R(1 - (r - 1) \sigma_{-i})} \right\} \quad (8)$$

whereas it strictly decreases afterwards. $\square$

ii.) *Continuous and increasing best response functions.* By equation (8), agent i 's best response function with respect to cutoff-points is given by

$$f_i(\sigma_{-i}) = \min \left\{ 1, \frac{r - (r - 1) \sigma_{-i}}{R(1 - (r - 1) \sigma_{-i})} \right\}, \quad (9)$$

which is continuous. Furthermore, we obtain as first order derivative of agent i 's best response function

$$\begin{aligned} \frac{df_i(\sigma_{-i})}{d\sigma_{-i}} &\geq 0 \Leftrightarrow \\ -(r - 1) \cdot R(1 - (r - 1) \sigma_{-i}) + R(r - 1) \cdot (r - (r - 1) \sigma_{-i}) &\geq 0 \text{ if } r > 1 \Leftrightarrow \\ -1 + r &\geq 0. \end{aligned}$$

Since, by assumption, $r > 1$, the best response function of each agent is strictly increasing. $\square$

Step 2. By the results of step 1 and the lemma, the dominance solution of our demand-deposit contract model is given as the interval

$$D(G) = [\min D(G), \max D(G)]$$

where $\min D(G)$ denotes the smallest and $\max D(G)$ denotes the largest Nash equilibrium of the model with respect to the natural order $\leq$.

At first observe that only symmetric Nash equilibria may qualify as the smallest respectively largest Nash equilibrium in our symmetric model. Suppose to the contrary that we have for a non-symmetric strategy profile $(\sigma_A, \tau_B) = \min D(G)$. But then the symmetric payoff-structure of our model implies that the strategy profile (τ_B, σ_A) is also a Nash equilibrium. However, since $(\sigma_A, \tau_B) \not\leq (\tau_B, \sigma_A)$, we obtain a contradiction. (The analogous argument applies to the case where $(\sigma_A, \tau_B) = \max D(G)$.)

Since we are exclusively interested in symmetric Nash equilibria, it suffices to iden-

tify the smallest and the largest fixed points x^* of the function $g(x) : [0, 1] \rightarrow [0, 1]$ such that

$$g(x) := \min \left\{ 1, \frac{r - (r-1)x}{R(1 - (r-1)x)} \right\}.$$

Because of

$$\frac{r - (r-1)x}{R(1 - (r-1)x)} > 0,$$

the possible candidates of fixed points of $g(x)$ are the corner solution $x^* = 1$ as well as the two possible solutions to the equation

$$x^* = \frac{r - (r-1)x^*}{R(1 - (r-1)x^*)}, \quad (10)$$

which are given by

$$\begin{aligned} x_1^* &= \frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r-1)} \\ x_2^* &= \frac{R + r - 1 + \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r-1)} \end{aligned}$$

In what follows we investigate the relevant cases.

Case i.) *The corner-solution $(\sigma_A, \sigma_B) = (1, 1)$ is a Nash equilibrium.* By increasing best response functions, this is the case if and only if

$$\frac{r - (r-1)\sigma_{-i}}{R(1 - (r-1)\sigma_{-i})} \geq 1$$

at $\sigma_{-i} = 1$, which is equivalent to the condition

$$\frac{1}{(2-r)} \geq R. \quad (11)$$

Thus, whenever condition (11) is satisfied, we have $\max D(G) = (1, 1)$. Moreover, $(1, 1)$ is also the unique Nash equilibrium if and only if there are no solutions to the equation (10) which lie in $[0, 1]$. Since $x_1^* < x_2^*$ and $0 < x_1^*$, we have

$$\min D(G) = (\sigma_A^*, \sigma_B^*)$$

with

$$\sigma_A^* = \sigma_B^* = \frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r-1)}$$

whenever $x_1^* \leq 1$. This proves the first part of the proposition.

Case ii.) *The corner-solution $(\sigma_A, \sigma_B) = (1, 1)$ is not a Nash equilibrium, i.e., we have*

$$\frac{1}{(2-r)} < R. \quad (12)$$

In this case,

$$\min D(G) = (\sigma_A^*, \sigma_B^*)$$

with

$$\sigma_A^* = \sigma_B^* = \frac{R + r - 1 - \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)}$$

since this strategy profile is the only candidate for the smallest Nash equilibrium. Moreover, we either have

$$\max D(G) = \min D(G)$$

or

$$\max D(G) = (\sigma_A^{**}, \sigma_B^{**})$$

with

$$\sigma_A^{**} = \sigma_B^{**} = \frac{R + r - 1 + \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)}. \quad (13)$$

However, mathematical rearrangement shows that

$$\frac{R + r - 1 + \sqrt{1 - 2r + r^2 - 2R + 6rR - 4r^2R + R^2}}{2R(r - 1)} \leq 1 \Leftrightarrow R \leq \frac{1}{(2 - r)}.$$

That is, $(\sigma_A^{**}, \sigma_B^{**})$ is only a Nash equilibrium if $(\sigma_A, \sigma_B) = (1, 1)$ is also a Nash equilibrium so that

$$\max D(G) \neq (\sigma_A^{**}, \sigma_B^{**}).$$

This establishes the existence of a unique Nash equilibrium (σ_A^*, σ_B^*) whenever the inequality (12) is satisfied. This proves the second part of the proposition. $\square\square$

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