Online Supplement for “Approximations to Optimal Sequences in Single-Gripper and Dual-Gripper Robotic Cells with Circular Layouts” by

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Online Appendix A: Definitions

The cycle time is the summation of the following robot actions: load/unload times at machines, move times between machines, gripper rotation times (in a dual-gripper robotic cell), and waiting times at machines. There are two ways that the robot may wait at a machine. If the robot loads machine M_i and waits at M_i throughout its entire processing (p_i), then the robot experiences *full waiting* at machine M_i . This is represented by the activity sequence (M_i^l, M_i^u) in a dual-gripper robotic cell and (M_i^l, M_{i+1}^l) in a single-gripper robotic cell. Alternatively, the robot may load a part onto M_i , move elsewhere to perform some other activities, return to M_i , wait for it to complete processing, and unload the part. This is called *partial waiting* at M_i . If M_i has completed processing before the robot returns, then the partial waiting time at M_i is $w_i = 0$.

A *cyclic schedule* specifies the times within a cycle that each robot action begins. We consider only *active* cyclic schedules, in which, upon completion of one action, the robot executes the next action as soon as possible. In 1-unit cycles for a dual-gripper cell, Dawande et al. (2010) explain that a robot can load and unload a machine four distinct ways, referring to these as *usages*.

- $\mathcal{U}_1$: the robot loads a part X_r onto a machine M_i , waits at the machine during the entire processing of X_r , and then unloads X_r . During this usage, the robot’s second gripper is either empty or occupied by a part.
- $\mathcal{U}_2$: the robot arrives with a part X_r and waits at the machine for the previously loaded part X_{r+1} to be completed, unloads X_{r+1} from the machine, rotates the grippers, loads X_r onto the machine, and moves onto the next machine.
- $\mathcal{U}_3$: the robot loads a part X_r onto a machine M_i , leaves M_i to perform other activities at other machines, and then returns to M_i to unload X_r . During this usage the robot’s second gripper is either empty or occupied by a part.
- $\mathcal{U}_4$: the robot arrives with a part X_r at M_i (to be processed on M_i) and waits at M_i for the previously loaded part X_{r+1} to be completed, unloads X_{r+1} from M_i , moves to M_{i+1} , loads X_{r+1} onto M_{i+1} , may or may not perform other activities, and returns to M_i to load the part X_r onto it.

Note that $\mathcal{U}_1$ and $\mathcal{U}_3$ are the only valid usages for single-gripper cycles. These and the following definitions for 1-unit single-grippers cycle (due to Rajapakshe et al. 2011) will be used in the proof of Claim 3 in Online Appendix B.1:

- *Primary part-transfer activities*: in a 1-unit cycle, there are $m + 1$ part-transfer activities between successive machines: from machine M_i to machine M_{i+1} , $i = 0, 1, \dots, m$. The total time required for these part-transfer activities is $(m + 1)\delta$. In addition, a minimum travel time of δ is required

for the robot to return to M_0 after loading M_{m+1} to complete the cycle. Thus, the total primary time in a 1-unit cycle is $(m+2)\delta$.

- *Secondary activities:* as mentioned earlier, the robot may load a part onto a machine and travel empty (i.e., without a part) to another machine(s) while the part is being processed. Secondary travel refers to this empty travel during the cycle, except for the robot's pass from M_{m+1} to M_0 that completes the cycle. The total time required for the secondary travel is referred to as the robot's secondary time.

We also require the following definitions for k -unit cycles in dual-gripper cells:

- *Residence time at a machine:* this includes load and unload times, gripper switch time, partial waiting time, and full waiting time at a machine.
- *Movement time:* this includes all robot travel times.

In order to derive results in Section 6 and in Online Appendix B.1, we also need the lower bound on the cycle time established by Geismar et al. (2006), who note that a k -unit cycle (for k even) can be represented by $k/2$ sequences of two consecutive loadings and unloadings of machine M_i of the form $\sigma(i, r) = (M_{i,2r-1}^l, \sigma_1, M_{i,2r-1}^u, \sigma_2, M_{i,2r}^l, \sigma_3, M_{i,2r}^u, \sigma_4)$, where $i = 0, 1, \dots, m+1$, $r = 1, \dots, \frac{k}{2}$. $M_{i,2r}^l$ (resp., $M_{i,2r}^u$) denotes $2r^{th}$ loading (resp., unloading) of machine M_i . In $\sigma(i, r)$ at least one of $A(\sigma_1)$, $A(\sigma_2)$, $A(\sigma_3)$, and $A(\sigma_4)$ is not empty, where $A(\sigma_h)$ is the set of activities in σ_h , $h = 1, 2, 3, 4$.

We now establish a lower bound on the aggregate residence time of the robot at all machines in a k -unit cycle by considering each sequence of two consecutive loadings and unloadings of machines. The following possible cases are considered: Cases 0 and 1 deal with sequences for $i = 0$ and $i = m+1$, respectively, whereas Cases 2-10 deal with all other i , $1 \leq i \leq m$.

Case 0: $M_i = M_0$ (the “machine” is I). Since we do not have loading onto M_0 , the sequence is simply $(M_{0,2r-1}^u, \sigma_1, M_{0,2r}^u, \sigma_2)$, where at least one of $A(\sigma_1)$ and $A(\sigma_2)$ is not empty. If $A(\sigma_1) \neq \emptyset$ and $A(\sigma_2) \neq \emptyset$, then the robot is occupied at M_0 for 2ϵ time units. If $A(\sigma_1) = \emptyset$ (or $A(\sigma_2) = \emptyset$), then the robot is occupied at M_0 for $2\epsilon + \theta$ time units.

Case 1: $M_i = M_{m+1}$ (the “machine” is O). Since we do not have unloading from M_{m+1} , the sequence is simply $(M_{m+1,2r-1}^l, \sigma_1, M_{m+1,2r}^l, \sigma_2)$, where at least one of $A(\sigma_1)$ and $A(\sigma_2)$ is not empty. If $A(\sigma_1) \neq \emptyset$ and $A(\sigma_2) \neq \emptyset$, then the robot is occupied at M_{m+1} for 2ϵ time units. If $A(\sigma_1) = \emptyset$ (or $A(\sigma_2) = \emptyset$), then the robot is occupied at M_{m+1} for $2\epsilon + \theta$ time units.

Case 2: $A(\sigma_1) \neq \emptyset, A(\sigma_2) \neq \emptyset, A(\sigma_3) \neq \emptyset, A(\sigma_4) \neq \emptyset$. The robot is occupied at M_i for at least 4ϵ time units.

Case 3: $A(\sigma_1) = \emptyset$ (or $A(\sigma_3) = \emptyset$). First consider $A(\sigma_1) = \emptyset$. The robot is occupied at M_i for at least $4\epsilon + p_i$ time units. As $A(\sigma_3) = \emptyset$ has same schedule structure as $A(\sigma_1) = \emptyset$, it will have the same residence time.

Case 4: $A(\sigma_2) = \emptyset$ (or $A(\sigma_4) = \emptyset$). The robot is occupied at M_i for at least $4\epsilon + \theta$ time units.

- Case 5:** $A(\sigma_1) = A(\sigma_2) = \emptyset$ (or $A(\sigma_3) = A(\sigma_4) = \emptyset$). The robot is occupied at M_i for at least $4\epsilon + \theta + p_i$ time units.
- Case 6:** $A(\sigma_1) = A(\sigma_3) = \emptyset$. The robot is occupied at M_i for at least $4\epsilon + 2p_i$ time units.
- Case 7:** $A(\sigma_1) = A(\sigma_4) = \emptyset$ (or $A(\sigma_2) = A(\sigma_3) = \emptyset$). The robot is occupied at M_i for at least $4\epsilon + \theta + p_i$ time units.
- Case 8:** $A(\sigma_2) = A(\sigma_4) = \emptyset$. The robot is occupied at M_i for at least $4\epsilon + 2\theta$ time units.
- Case 9:** $A(\sigma_1) = A(\sigma_2) = A(\sigma_3) = \emptyset$ (or $A(\sigma_1) = A(\sigma_3) = A(\sigma_4) = \emptyset$). The robot is occupied at M_i for at least $4\epsilon + \theta + 2p_i$ time units.
- Case 10:** $A(\sigma_1) = A(\sigma_2) = A(\sigma_4) = \emptyset$ (or $A(\sigma_2) = A(\sigma_3) = A(\sigma_4) = \emptyset$). This schedule structure is infeasible.

Lower bounds on the residence time of the robot at machines for each case are shown in Table 8. Similarly, a lower bound on the time of robot movements is established by analyzing the robot movement time in each case, as outlined next.

Cases	Sets	Residence times	Movement times	Characteristics	Robot Actions
Case 0	U_0	$2\epsilon + \theta$	δ	$A(\sigma_1) = \emptyset$ (or $A(\sigma_2) = \emptyset$)	M_0^u, M_0^u
Case 0	U_0	2ϵ	2δ	$A(\sigma_1) \neq \emptyset$ and $A(\sigma_2) \neq \emptyset$	$M_0^u - M_0^u$
Case 1	U_1	$2\epsilon + \theta$	δ	If $A(\sigma_1) = \emptyset$ (or $A(\sigma_2) = \emptyset$)	M_{m+1}^l, M_{m+1}^l
Case 1	U_1	2ϵ	2δ	If $A(\sigma_1) \neq \emptyset$ and $A(\sigma_2) \neq \emptyset$	$M_{m+1}^l - M_{m+1}^l$
Case 2	U_2	4ϵ	4δ	$A(\sigma_1) \neq \emptyset, A(\sigma_2) \neq \emptyset, A(\sigma_3) \neq \emptyset, A(\sigma_4) \neq \emptyset$	$M_i^l - M_i^u - M_i^l - M_i^u$
Case 3	U_3	$4\epsilon + p_i$	3δ	$A(\sigma_1) = \emptyset$ (or $A(\sigma_3) = \emptyset$)	$M_i^l, M_i^u - M_i^l - M_i^u$
Case 4	U_4	$4\epsilon + \theta$	3δ	$A(\sigma_2) = \emptyset$ (or $A(\sigma_4) = \emptyset$)	$M_i^u, M_i^l - M_i^u - M_i^l$
Case 5	U_5	$4\epsilon + \theta + p_i$	2δ	$A(\sigma_1) = A(\sigma_2) = \emptyset$ (or $A(\sigma_3) = A(\sigma_4) = \emptyset$)	$M_i^l, M_i^u, M_i^l - M_i^u$
Case 6	U_6	$4\epsilon + 2p_i$	2δ	$A(\sigma_1) = A(\sigma_3) = \emptyset$	$M_i^l, M_i^u - M_i^l - M_i^u$
Case 7	U_7	$4\epsilon + \theta + p_i$	2δ	$A(\sigma_1) = A(\sigma_4) = \emptyset$ (or $A(\sigma_2) = A(\sigma_3) = \emptyset$)	$M_i^u, M_i^l, M_i^u - M_i^l$
Case 8	U_8	$4\epsilon + 2\theta$	2δ	$A(\sigma_2) = A(\sigma_4) = \emptyset$	$M_i^u, M_i^l - M_i^u, M_i^l$
Case 9	U_9	$4\epsilon + \theta + 2p_i$	δ	$A(\sigma_1) = A(\sigma_2) = A(\sigma_3) = \emptyset$ (or $A(\sigma_1) = A(\sigma_3) = A(\sigma_4) = \emptyset$)	$M_i^l, M_i^u, M_i^l, M_i^u$

Table 8: Robot residence times and movement times in $\sigma(i, r)$ for each case (set). A dash between operations in the last column indicates robot departure and return.

If a machine M_i is included in Case 2, then there are in each sequence at least four movements to and four movements away from M_i , so M_i will be incident to at least eight movements. If a machine M_i is included in Case 3 or 4, there are in each sequence at least three movements to and three movements away from M_i , so M_i will be incident to at least six movements. If a machine M_i is included in Case 5, 6, 7, or 8, there are in each sequence at least two movements to and two movements away from M_i , so M_i will be incident to at least four movements. If a machine M_i is included in Case 9, there are in each sequence at least one movement to and one movement away from M_i , so M_i will be incident to at least two movements. The total robot movements to all machines can then be easily calculated. When the robot moves from one machine, its destination is another machine. Thus, a lower bound on the aggregate robot movement time is found by summing the number of movements to which each machine is incident, dividing the total by two, and multiplying by δ .

Online Appendix B.1: Proof of Property A

We first state and prove Property A because it is used to prove Lemmas 3 and 4 in Online Appendix B.2.

Let $B_1 = \{j | p_j \geq 2\delta\}$, $B_2 = \{j | \delta \leq p_j < 2\delta\}$, and $B_3 = \{j | p_j < \delta\}$.

Property A: If $\theta > 2\delta$ and $[\frac{4}{3}|B_1|\delta + |B_2|\delta + \sum_{j \in B_3} p_j] < (m+2)\delta$, then the cycle-time $\frac{T(S_r'')}{k}$ of a k -unit cycle S_r'' satisfies $\frac{T(S_r'')}{k} \geq (m+2)\delta + 2(m+1)\epsilon + \frac{4}{3}|B_1|\delta + |B_2|\delta + \sum_{j \in B_3} p_j$.

Proof: Let $\mathcal{I}$ be an arbitrary instance of our problem in a dual-gripper cell. In order to simplify the proof, we modify $\mathcal{I}$ by truncating its processing times to obtain a transformed instance $\mathcal{I}'$ with processing times p' defined as follows:

- If $j \in B_1$, then $p'_j = 2\delta, j = 1, 2, \dots, m$.
- If $j \in B_2$, then $p'_j = \delta, j = 1, 2, \dots, m$.
- If $j \in B_3$, then $p'_j = p_j, j = 1, 2, \dots, m$.

Let $B'_1 = \{j \in M : p'_j = 2\delta\}$, $B'_2 = \{j \in M : p'_j = \delta\}$, and $B'_3 = \{j \in M : p'_j = p_j\}$. Note that $|B'_1| = |B_1|$, $|B'_2| = |B_2|$ and $|B'_3| = |B_3|$. Let cycle $S''_{\mathcal{I}}$ (resp., $S''_{\mathcal{I}'}$) be an optimal k -unit (resp., k' -unit) cycle for the instance $\mathcal{I}$ (resp., $\mathcal{I}'$) with corresponding per unit cycle time $\frac{T^{\mathcal{I}}(S''_{\mathcal{I}})}{k}$ (resp., $\frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'}$).

(For generality, we allow the two cycles $S''_{\mathcal{I}}$ and $S''_{\mathcal{I}'}$ to produce different numbers of units.) Let $\frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k}$ be the per unit cycle time obtained by executing cycle $S''_{\mathcal{I}'}$ on instance $\mathcal{I}'$.

Claim 1: $\frac{T^{\mathcal{I}}(S''_{\mathcal{I}})}{k} \geq \frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'}$.

Proof of Claim 1: Since $p'_j \leq p_j, \forall j \in M$, we have $\frac{T^{\mathcal{I}}(S''_{\mathcal{I}})}{k} \geq \frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k} \geq \frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'}$. $\square$

From equations (1) and (2) in the main paper and $\theta > 2\delta$, we have $\frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'} \geq (m+2)\delta + 2(m+1)\epsilon + |B'_1|\delta + |B'_2|\delta + \sum_{j \in B'_3} p'_j$. Observe that the only change in this bound and the one proposed in the statement of Property A is that the third term is $\frac{4}{3}|B'_1|\delta$ instead of $|B'_1|\delta$. As a consequence of Claim 1, it is sufficient to prove that $\frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'} \geq (m+2)\delta + 2(m+1)\epsilon + \frac{4}{3}|B'_1|\delta + |B'_2|\delta + \sum_{j \in B'_3} p'_j$. Thus, we focus on problem instance $\mathcal{I}'$ and cycle $S''_{\mathcal{I}'}$ in the rest of the proof.

Before we proceed further, it is important to understand the machine loadings and unloadings in a k' -unit cycle. Note that a k' -unit cycle can be represented by two consecutive loadings and unloadings of machine M_i of the form $\sigma(i, r) = (M_{i,2r-1}^l, \sigma_1, M_{i,2r-1}^u, \sigma_2, M_{i,2r}^l, \sigma_3, M_{i,2r}^u, \sigma_4)$, where $i = 0, 1, \dots, m+1$, $r = 1, \dots, \frac{k'}{2}$, and k' is even. $M_{i,2r}^l$ (resp., $M_{i,2r}^u$) denotes $2r^{th}$ loading (resp., unloading) of machine M_i . In $\sigma(i, r)$ at least one of $A(\sigma_1)$, $A(\sigma_2)$, $A(\sigma_3)$, and $A(\sigma_4)$ is not empty, where $A(\sigma_h)$ is the set of activities in σ_h , $h = 1, 2, 3, 4$ (see Online Appendix A for details). We note that the total partial waiting is zero in cycle $S''_{\mathcal{I}'}$ because the processing time of all machines are less than or equal to 2δ . The lower bound on the robot activity time corresponding to the sequence $\sigma(i, r)$ is given in Table 8.

Claim 2: Cycle $S''_{\mathcal{I}'}$ cannot have Case 4 or Case 8.

Proof of Claim 2: Suppose that cycle $S''_{T'}$ has Case 8 at a machine. This instance of Case 8 can be replaced with Case 6 since in both cases the robot arrives at the machine with a part to be processed there and departs the machine with a processed part. Thus, the new cycle is feasible. In this new cycle the robot movement time remains the same, but the robot residence time is improved to $4\epsilon + 2p_i$ because $\theta > 2\delta$. This contradicts the fact that cycle $S''_{T'}$ is optimal. The proof is similar for Case 4. $\square$

Claim 3: There exists a single-gripper 1-unit cycle $\hat{S}'$ that has the same per unit cycle time as the k' -unit dual-gripper cycle $S''_{T'}$.

Proof of Claim 3: We first assume k' is even, then we extend the proof to k' odd. The idea behind the proof is as follows. We decompose cycle $S''_{T'}$ into $k'/2$ sequences of two consecutive loadings and unloadings of a machine M_i of the form $\sigma(i, r) = (M_{i,2r-1}^l, \sigma_1, M_{i,2r-1}^u, \sigma_2, M_{i,2r}^l, \sigma_3, M_{i,2r}^u, \sigma_4)$, where $i = 0, 1, \dots, m+1$, $r = 1, \dots, \frac{k'}{2}$, for k' even. For each r , we then form two single-gripper 1-unit cycles using the sequences $\sigma(i, r)$, $i = 0, 1, \dots, m+1$, for all $(m+2)$ machines. That is, we use the first sequence of two consecutive loadings and unloadings of $(m+2)$ machines to form the first two 1-unit cycles, the next sequence of two consecutive loadings and unloadings of $(m+2)$ machines to form the next two 1-unit cycles, and so on. Finally, we use the $(k'/2)^{\text{th}}$ sequence of two consecutive loadings and unloadings of $(m+2)$ machines to form the last two 1-unit cycles. We will show that each pair of 1-unit cycles formed will have cycle times not larger than the lower bound on the original cycle time of cycle $S''_{T'}$, where this lower bound corresponds to the sequence $\sigma(i, r)$ and its lower bounds given in Table 8. (The lower bounds in Table 8 are very crude approximations and will be adjusted depending on the different scenarios described below.) Table 9 shows how the times in Table 8 are distributed to the pair of 1-unit cycles formed from $\sigma(i, r)$ assuming that $\theta = 2\delta$. Note that the total time in Table 9 is less or equal to that in Table 8 because $\theta > 2\delta$. We divide Case 3 into two subcases: (i) Case 3a, in which the robot performs full waiting in its first visit, and (ii) Case 3b, in which the robot performs full waiting in its second visit. Similarly we divide Case 5 (resp., Case 7) into two subcases: (i) Case 5a (resp., Case 7a), in which the robot performs full waiting in its first visit, and (ii) Case 5b (resp., Case 7b), in which it performs full waiting in its second visit.

Cases for $\sigma(i, r)$	1-unit cycle, $\hat{S}'$ Primary time+Residence time	1-unit cycle, $\hat{S}'$ Secondary time	1-unit cycle, $\bar{S}'$ Primary time+Residence time	1-unit cycle, $\bar{S}'$ Secondary time
Case 0	$\epsilon + \delta$	-	$\epsilon + \delta$	-
Case 0	$\epsilon + \delta$	-	$\epsilon + \delta$	-
Case 1	$\epsilon + \delta$	-	$\epsilon + \delta$	-
Case 1	$\epsilon + \delta$	-	$\epsilon + \delta$	-
Case 2	$2\epsilon + \delta$	δ	$2\epsilon + \delta$	δ
Case 3a	$p_i + 2\epsilon + \delta$	-	$2\epsilon + \delta$	δ
Case 3b	$2\epsilon + \delta$	δ	$p_i + 2\epsilon + \delta$	-
Case 5a	$p_i + 2\epsilon + \delta$	-	$2\epsilon + \delta$	2δ
Case 5b	$2\epsilon + \delta$	2δ	$p_i + 2\epsilon + \delta$	-
Case 6	$p_i + 2\epsilon + \delta$	-	$p_i + 2\epsilon + \delta$	-
Case 7a	$p_i + 2\epsilon + \delta$	-	$2\epsilon + \delta$	2δ
Case 7b	$2\epsilon + \delta$	2δ	$p_i + 2\epsilon + \delta$	-
Case 9	$p_i + 2\epsilon + \delta$	-	$p_i + 2\epsilon + \delta$	-

Table 9: Robot residence times and movement times for the constructed pair of 1-unit cycles $(\hat{S}', \bar{S}')$.

Forming single-gripper 1-unit cycles: In 1-unit cycles for a single-gripper cell, Dawande et al. (2010) identify that a robot can load and unload a machine two distinct ways, referring to these as *usages*, namely, usages $\mathcal{U}_1$ and $\mathcal{U}_3$ (defined in Online Appendix A). Using the information in Table 9 to form 1-unit cycles, we make an initial assignment of usages $\mathcal{U}_1$ and $\mathcal{U}_3$ to machines according to their respective cases as shown in Table 10. For example, if a machine is in Case 9, then this machine is assigned usage $\mathcal{U}_1$ in each pair of 1-unit cycles because it performs full waiting when processing each part.

Cases	First 1-unit cycle, $\hat{S}'$	Second 1-unit cycle, $\bar{S}'$
Case 2	$\mathcal{U}_3$	$\mathcal{U}_3$
Case 3a	$\mathcal{U}_1$	$\mathcal{U}_3$
Case 3b	$\mathcal{U}_3$	$\mathcal{U}_1$
Case 5a	$\mathcal{U}_1$	$\mathcal{U}_3$
Case 5b	$\mathcal{U}_3$	$\mathcal{U}_1$
Case 6	$\mathcal{U}_1$	$\mathcal{U}_1$
Case 7a	$\mathcal{U}_1$	$\mathcal{U}_3$
Case 7b	$\mathcal{U}_3$	$\mathcal{U}_1$
Case 9	$\mathcal{U}_1$	$\mathcal{U}_1$

Table 10: Initial usage assignment for machines.

Assume that 1-unit cycle $\hat{S}'$ (resp., 1-unit cycle $\bar{S}'$) is formed by the first (resp., second) loading and unloading sequences of machines in $\sigma(i, r)$, namely $\sigma^1(i, r)$ (resp., $\sigma^2(i, r)$). Note that the formation of $\hat{S}'$ and $\bar{S}'$ is guided by the times in Table 9 and by the machine usage assignment in Table 10. We show how to form $\hat{S}'$; the procedure is similar for $\bar{S}'$. The procedure below divides the machines into three sets $(\mathcal{M}_j^1, \mathcal{M}_j^2, \mathcal{M}_j^3)$ for $\hat{S}'$. We then define cycle $\hat{S}'$ by making a final revised assignment of appropriate usages ($\mathcal{U}_1$ or $\mathcal{U}_3$) to the machines in those sets.

Procedure:

Step 1: Let $M = \{M_1, M_2, \dots, M_m\}$ be an ordered set in which machines are arranged in increasing order of their indices. Start with M_1 and scan the machines in a forward pass through M_m . Group the machines in set M into disjoint subsets $\mathcal{M}_j^1$ such that $|\mathcal{M}_j^1| = 3$, $j = 1, 2, \dots, s_1$, and each $\mathcal{M}_j^1$ contains consecutive machines having usage $\mathcal{U}_3$.

Step 2: Let $\hat{M} = M - \{\mathcal{M}_j^1, j = 1, 2, \dots, s_1\}$ be an ordered set in which machines are arranged in increasing order of their indices. Start with the first machine in $\hat{M}$ and scan the machines in a forward pass through the last machine in $\hat{M}$. Group the machines in set $\hat{M}$ into disjoint subsets $\mathcal{M}_j^2$ such that $|\mathcal{M}_j^2| = 2$, $j = 1, 2, \dots, s_2$, and each $\mathcal{M}_j^2$ contains consecutive machines having usage $\mathcal{U}_3$.

Step 3: Let $\bar{M} = \hat{M} - \{\mathcal{M}_j^2, j = 1, 2, \dots, s_2\}$ be an ordered set in which machines are arranged in increasing order of their indices. Start with the first machine in $\bar{M}$ and scan the machines in a forward pass through the last machine in $\bar{M}$. Identify the machines in set $\bar{M}$ and place them into disjoint subsets $\mathcal{M}_j^3$, $j = 1, 2, \dots, s_3$, each with a single machine having usage $\mathcal{U}_3$. That is, $|\mathcal{M}_j^3| = 1$, $j = 1, 2, \dots, s_3$.

Suppose that the case assignments of cycle S''_T imply that all machines in $\hat{S}'$ are assigned usage $\mathcal{U}_3$

(see Table 10). $\hat{S}'$ can be formed by using the lower bounds on times given in Table 9 (each instance of $\mathcal{U}_3$ in Table 10 corresponds to time $2\epsilon + 2\delta$ or more in Table 9). In this case, the resulting cycle $\hat{S}'$ is the Odd-Even Cycle, and the total secondary time is at least $(m+2)\delta$. If all 1-unit cycles formed are the Odd-Even Cycle S'_c , then $\frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'} \geq 2(m+2)\delta + 2(m+1)\epsilon$. This bound for $\frac{T^{\mathcal{I}'}(S''_{\mathcal{I}'})}{k'}$ is larger than the one stated in Property A, so Property A is satisfied in this case.

Consequently, we now consider the case in which there exists at least one cycle among the formed 1-unit cycles (say, $\hat{S}'$) that has total secondary time and full waiting time less than $(m+2)\delta$. Observe that the secondary travel times in Table 9 are given based on times in Table 8 for Cases 2, 3b, 5b, and 7b in $\sigma^1(i, r)$. Moreover, note that the three machines in $\mathcal{M}_j^1$ have a minimum of 3δ collective secondary travel time (Table 9). If at least one machine in $\mathcal{M}_j^1$ belongs to Case 5b or 7b, then the three machines in $\mathcal{M}_j^1$ have a minimum of 4δ collective secondary travel time (Table 9). Otherwise all three machines in $\mathcal{M}_j^1$ belong to Case 2 or 3b. In this case we show below that the three consecutive machines in $\mathcal{M}_j^1$ must have generated at least 4δ secondary travel time across those machines in $\sigma^1(i, r)$ due to the robot movements in Cases 2 and 3b. Thus, the secondary travel time for this case must be adjusted from 3δ to 4δ . Table 11 shows the revised lower bounds on the secondary travel time.

Number of usage $\mathcal{U}_3$	Minimum Secondary travel time in Table 9	Adjusted Lower bound on Secondary travel time
$ \mathcal{M}_j^1 = 3$	3δ	4δ
$ \mathcal{M}_j^2 = 2$	2δ	4δ
$ \mathcal{M}_j^3 = 1$	δ	2δ

Table 11: Revised lower bound on the secondary travel time.

Let $\mathcal{M}_j^1 = \{M_{q^*}, M_{q^*+1}, M_{q^*+2}\}$. Note that the robot loads machine M_{q^*} , performs some other activities, then returns to M_{q^*} and unloads the part. This creates secondary travel time at machine M_{q^*} . Let $\mathcal{P}_{q^*}$ denote the robot's travel path between $M_{q^*}^l$ and $M_{q^*}^u$. Note that the robot's travel in $\mathcal{P}_{q^*}$ can be decomposed into primary and secondary passes across adjacent machines. The following two possible subcases may occur.

Case A: In $\mathcal{P}_{q^*}$, the robot traverses all pairs of adjacent machines in set $\bar{M}$, where $\bar{M} = \{M_{q^*}, M_{q^*+1}, M_{q^*+2}, \dots, M_m, M_{m+1}, M_0, M_1, \dots, M_{q^*-2}, M_{q^*-1}, M_{q^*}\}$ and the number of the robot's primary and secondary passes across two adjacent machines is at least $m+2$.

Case B: Otherwise.

We now show that Case A cannot occur in sequence $\sigma^1(i, r)$. In $\mathcal{P}_{q^*}$, the number of secondary passes across any two adjacent machines M_r and M_{r+1} in $\bar{M}$ is at least one. Here, the robot travels across all $(m+2)$ machines between the activities $M_{q^*}^l$ and $M_{q^*}^u$. Therefore, when the robot completes sequence $\sigma^1(i, r)$, the total secondary time is at least $(m+2)\delta$, which contradicts our assumption. Hence the robot path $\mathcal{P}_{q^*}$ does not belong to Case A. Hence, $\sigma^1(i, r)$ provides at an even number of secondary times across machines in $\mathcal{M}_j^1$ due to the robot movements in Cases 2 and 3b. Thus, the machines in $\mathcal{M}_j^1$, $j = 1, \dots, s_1$, will generate at least 4δ secondary travel times across them. Consequently, the secondary travel time for this case must be adjusted from 3δ to 4δ (see Table 11).

Similarly, machines in $\mathcal{M}_j^2$, $j = 1, \dots, s_2$, generate at least 4δ secondary travel time among those machines in sequence $\sigma^1(i, r)$ because at least one more machine outside $\mathcal{M}_j^2$ must be involved. Hence, we set machines in $\mathcal{M}_j^2$ having usage $\mathcal{U}_3$ to usage $\mathcal{U}_1$, which results in an equivalent residence time of 4δ or less because $p_i \leq 2\delta$. Similarly, the machine in $\mathcal{M}_j^3$, $j = 1, \dots, s_3$, generates at least 2δ secondary travel time, so we set that machine usage to $\mathcal{U}_1$, which results in an equivalent residence time of 2δ or less. Table 12 provides the final assignment of machine usages in $\hat{S}'$. Now we provide the machine loading/unloading sequence for $\hat{S}'$: (i) for machines in set $\mathcal{M}_j^1 = \{M_i, M_{i+1}, M_{i+2}\}$, use the machine loading sequence $(M_i^l, M_{i+1}^u, M_{i+2}^l, M_i^u, M_{i+1}^l, M_{i+2}^u)$, and (ii) for a machine M_j in set $M - \{\mathcal{M}_j^1, j = 1, 2, \dots, s_1\}$, use the machine loading/unloading sequence (M_j^l, M_j^u) . The above machine loading/unloading sequences fully define the single-gripper 1-unit cycle $\hat{S}'$. Note that $\hat{S}'$ uses the same lower bound on times in Table 8 or the revised lower bound on the secondary travel times in Table 11, if required. We form all k 1-unit cycles from sequences $\sigma(i, r)$ similarly.

Machines	1-unit cycle, $\hat{S}'$
Machines in $\mathcal{M}_j^1$	$\mathcal{U}_3$
Machines in $M - \{\mathcal{M}_j^1, j = 1, 2, \dots, s_1\}$	$\mathcal{U}_1$

Table 12: Final usage assignment for machines in $\hat{S}'$.

Now we show that all k 1-unit cycles formed have the same cycle time as the per unit cycle time of cycle $S''_{\mathcal{T}'}$. Suppose the cycle time of any constructed 1-unit cycle is less than the per unit cycle time of cycle $S''_{\mathcal{T}'}$. This contradicts the fact that cycle $S''_{\mathcal{T}'}$ is optimal. Moreover, each constructed 1-unit cycle cannot have cycle time more than the per unit cycle time of cycle $S''_{\mathcal{T}'}$ because each is constructed using the corresponding robot activity times in sequences $\sigma(i, r)$. Thus, all k 1-unit cycles formed must have the same cycle time as the per unit cycle time of cycle $S''_{\mathcal{T}'}$. The proof is similar for k' odd, as the last sequence $\sigma(i, r)$ of cycle $S''_{\mathcal{T}'}$ corresponds to a 1-unit cycle. $\square$

Note that $\hat{S}'$ has the same per unit cycle time as the k' -unit dual-gripper cycle $S''_{\mathcal{T}'}$. Moreover, $\hat{S}'$ is an optimal single-gripper 1-unit cycle for problem instance $\mathcal{I}'$.

Claim 4: If $[\frac{4}{3}|B'_1|\delta + |B'_2|\delta + \sum_{j \in B'_3} p'_j] < (m+2)\delta$, then $T(\hat{S}') \geq (m+2)\delta + 2(m+1)\epsilon + \frac{4}{3}|B'_1|\delta + |B'_2|\delta + \sum_{j \in B'_3} p'_j$.

Proof of Claim 4: From Claim 3, there exists a single-gripper 1-unit cycle $\hat{S}'$ for problem instance $\mathcal{I}'$ that has the same per unit cycle time as the k' -unit dual-gripper cycle $S''_{\mathcal{T}'}$. Thus, the proof follows from Rajapakshe et al.'s (2011) proof of this property for 1-unit cycles. $\square$

This completes the proof of Property A. $\blacksquare$

We now illustrate the construction of 1-unit cycles from a 2-unit cycle using Example B.

Example B: Consider cycle $S''_{\mathcal{T}'}$ for $m = 6$ and $k' = 2$, where cycle $S''_{\mathcal{T}'} = (M_0^u, M_0^u, M_1^l, M_1^u, M_1^l, M_1^u, M_2^l, M_2^u, M_2^l, M_3^l, M_3^u, M_3^l, M_4^l, M_4^u, M_4^l, M_5^u, M_5^l, M_5^u, M_6^l, M_6^u, M_6^l, M_6^u, M_7^l, M_7^l)$ (see Figure 4(a)). Let $p_i \leq 2\delta$ for $i = 1, 2, \dots, 6$. $T(S''_{\mathcal{T}'}) = 16\delta + 28\epsilon + 6\theta + 2p_1 + p_2 + p_4 + p_5 + 2p_6$, where the total full waiting time is $2p_1 + p_2 + p_4 + p_5 + 2p_6$ and the total partial waiting time is zero.

Note that $\sigma(i, r) = S''_{T'}$ because cycle $S''_{T'}$ is a 2-unit cycle. Based on $\sigma(i, r)$, we identify the cases for machines in Table 13, which also provides lower bounds on the robot residence times and movement times in $\sigma(i, r)$. Note that 1-unit cycles $\hat{S}'$ and $\bar{S}'$ are formed by the loading and unloading sequences of machines in $\sigma(i, r)$. Table 14 shows how the times in Table 13 are distributed to the pair of 1-unit cycles formed. The total of the lower bounds in Table 14 is $LB = 23\delta + 28\epsilon + 2p_1 + p_2 + p_4 + p_5 + 2p_6$. In the formation of $\hat{S}'$ and $\bar{S}'$, we use the times in Table 14 and the initial machine usage assignment in Table 15. Consider the formation of $\hat{S}'$: $\mathcal{M}_1^1 = \{M_3, M_4, M_5\}$, $s_1 = 1$, $\mathcal{M}_1^2 = \emptyset$, $\mathcal{M}_1^3 = \emptyset$. Thus, there is no revision to the machine usage assignment for $\hat{S}'$ (see Table 16), so $\hat{S}' = (M_1^l, M_2^l, M_3^l, M_5^l, M_4^l, M_6^l, M_7^l)$ and $T(\hat{S}') = 12\delta + 14\epsilon + p_1 + p_2 + p_6$ (see Figure 4 (b)). Now consider the formation of $\bar{S}'$. $\mathcal{M}_1^1 = \emptyset$, $\mathcal{M}_1^2 = \{M_2, M_3\}$, $s_2 = 1$, $\mathcal{M}_1^3 = \emptyset$. Thus, there is a revision to the machine usage assignment for $\bar{S}'$ (see Table 16): $\bar{S}' = (M_1^l, M_2^l, M_3^l, M_4^l, M_5^l, M_6^l, M_7^l)$ and $T(\bar{S}') = 8\delta + 14\epsilon + p_1 + p_2 + p_3 + p_4 + p_5 + p_6$ (see Figure 4(c)).

It follows that $T(\hat{S}') + T(\bar{S}') = 20\delta + 28\epsilon + 2p_1 + p_2 + p_4 + p_5 + 2p_6 + (p_2 + p_3) \leq 24\delta + 28\epsilon + 2p_1 + p_2 + p_4 + p_5 + 2p_6 = LB + \delta$. Therefore we use at the most $LB + \delta$ time in the formation of $\hat{S}'$ and $\bar{S}'$. That is, we need one δ more than the time in LB . This extra δ is added to the lower bound when revising the machine usage assignment for $\bar{S}'$. Note that the total travel time in cycle $S''_{T'}$ is 16δ . The two instances of 2δ secondary time in Table 14 are obtained from switch time θ for Cases 5a and 7b, respectively. Thus, note that $T(S''_{T'}) = 16\delta + 28\epsilon + 6\theta + 2p_1 + p_2 + p_4 + p_5 + 2p_6 \geq 28\delta + 28\epsilon + 2p_1 + p_2 + p_4 + p_5 + 2p_6 \geq LB + \delta = T(\hat{S}') + T(\bar{S}')$.

Online Appendix B.2: Proofs of Results from Section 5

Let Ω' and $\mathcal{L}(\Omega')$ (resp., Ω'' and $\mathcal{L}(\Omega'')$) be the optimum per-unit cycle time and its lower bound for a single-gripper (resp., dual-gripper) cell.

Lemma 1 *If $p_{\max} \leq \delta \leq \theta$, then cycle S'_u is optimal.*

Proof: If $p_{\max} \leq \delta \leq \theta$, then, by (1) and (2),

$$\min\{\Omega', \Omega''\} \geq (m+2)\delta + 2(m+1)\epsilon + \sum_{i=1}^m \min\{p_i, \delta\} \geq T(S'_u). \quad \blacksquare$$

Figure 4: (a) Cycle $S''_{T'}$, (b) Cycle $\hat{S}'$, and (c) Cycle $\bar{S}'$.

Machine	Cases	Residence time	Movement times
I	Case 0	$2\epsilon + \theta$	δ
O	Case 1	$2\epsilon + \theta$	δ
M_1	Case 9	$4\epsilon + \theta + 2p_1$	δ
M_2	Case 5	$4\epsilon + \theta + p_2$	2δ
M_3	Case 2	4ϵ	4δ
M_4	Case 3	$4\epsilon + p_4$	3δ
M_5	Case 7	$4\epsilon + \theta + p_5$	2δ
M_6	Case 9	$4\epsilon + \theta + 2p_6$	δ

Table 13: Robot residence times and movement times in $\sigma(i, r)$ for Example B.

(Machine, Case)	1-unit cycle, $\hat{S}'$ Primary time+ Residence time	1-unit cycle, $\hat{S}'$ Secondary time	1-unit cycle, $\bar{S}'$ Primary time+Residence time	1-unit cycle, $\bar{S}'$ Secondary time
I , Case 0	$\epsilon + \delta$	-	$\epsilon + \delta$	-
O , Case 1	$\epsilon + \delta$	-	$\epsilon + \delta$	-
M_1 , Case 9	$p_1 + 2\epsilon + \delta$	-	$p_1 + 2\epsilon + \delta$	-
M_2 , Case 5a	$p_2 + 2\epsilon + \delta$	-	$2\epsilon + \delta$	2δ
M_3 , Case 2	$2\epsilon + \delta$	δ	$2\epsilon + \delta$	δ
M_4 , Case 3b	$2\epsilon + \delta$	δ	$p_4 + 2\epsilon + \delta$	-
M_5 , Case 7b	$2\epsilon + \delta$	2δ	$p_5 + 2\epsilon + \delta$	-
M_6 , Case 9	$p_6 + 2\epsilon + \delta$	-	$p_6 + 2\epsilon + \delta$	-
Total	$p_1 + p_2 + p_6 + 14\epsilon + 8\delta$	4δ	$p_1 + p_4 + p_5 + p_6 + 14\epsilon + 8\delta$	3δ

Table 14: Robot residence times and movement times for the pair of 1-unit cycles $(\hat{S}', \bar{S}')$ constructed in Example B.

Machine, Cases	First 1-unit cycle, $\hat{S}'$	Second 1-unit cycle, $\bar{S}'$
M_1 , Case 9	$\mathcal{U}_1$	$\mathcal{U}_1$
M_2 , Case 5a	$\mathcal{U}_1$	$\mathcal{U}_3$
M_3 , Case 2	$\mathcal{U}_3$	$\mathcal{U}_3$
M_5 , Case 3b	$\mathcal{U}_3$	$\mathcal{U}_1$
M_5 , Case 7b	$\mathcal{U}_3$	$\mathcal{U}_1$
M_6 , Case 9	$\mathcal{U}_1$	$\mathcal{U}_1$

Table 15: Initial usage assignment for machines in Example B.

Machine	First 1-unit cycle, $\hat{S}'$	Second 1-unit cycle, $\bar{S}'$
M_1	$\mathcal{U}_1$	$\mathcal{U}_1$
M_2	$\mathcal{U}_1$	$\mathcal{U}_1$
M_3	$\mathcal{U}_3$	$\mathcal{U}_1$
M_5	$\mathcal{U}_3$	$\mathcal{U}_1$
M_5	$\mathcal{U}_3$	$\mathcal{U}_1$
M_6	$\mathcal{U}_1$	$\mathcal{U}_1$
Total time	$p_1 + p_2 + p_6 + 14\epsilon + 12\delta$	$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + 14\epsilon + 8\delta$

Table 16: Revised usage assignment for machines in Example B.

Lemma 2 *If $\theta \geq 4\delta + 2\epsilon$ and $p_{\max} \geq \min\{(m+2)\delta + 2m\epsilon + (m-1)\theta, 3m\delta + 2(m-1)\epsilon\}$, then cycle S'_d is optimal.*

Proof: If $\theta \geq 4\delta + 2\epsilon$ and $p_{\max} \geq \min\{(m+2)\delta + 2m\epsilon + (m-1)\theta, 3m\delta + 2(m-1)\epsilon\}$, then $T(S'_d) = p_{\max} + 4\delta + 4\epsilon \leq \Omega'$ by (1). In addition, the condition $\theta \geq 4\delta + 2\epsilon$ ensures that $p_{\max} + 4\delta + 4\epsilon \leq p_{\max} + 2\epsilon + \theta \leq \Omega''$ by (3). ■

Lemma 3 *If $\theta > 2\delta$ and $p_{\max} \geq (m+2)\delta + (m-1)\epsilon$, then the best of cycles $\{S'_d, S'_c, S'_\ell\}$ provides a $5/3$ -approximation.*

Proof: $\theta > 2\delta$ implies $\mathcal{L}(\Omega'') = \max\{(m+1)\delta + 2(m+1)\epsilon + \sum_{i=1}^m \min\{p_i, \delta\}, p_{\max} + 2\epsilon + \theta\}$ and $\mathcal{L}(\Omega') = \max\{(m+1)\delta + 2(m+1)\epsilon + \sum_{i=1}^m \min\{p_i, \delta\}, p_{\max} + 4\delta + 4\epsilon\}$. Note that if $\theta \leq 4\delta + 2\epsilon$, then $\mathcal{L}(\Omega'') \leq \mathcal{L}(\Omega')$, otherwise $\mathcal{L}(\Omega'') > \mathcal{L}(\Omega')$. Let $\mathcal{L}^* = (m+2)\delta + 2(m+1)\epsilon + \frac{4}{3}|B_1|\delta + |B_2|\delta + \sum_{j \in B_3} p_j$. Rajapakshe et al. (2011) use lower bounds for $\mathcal{L}(\Omega')$ and $\mathcal{L}^*$ to derive a $5/3$ bound for single-gripper 1-unit cycles using the best of cycles $\{S'_d, S'_c, S'_\ell\}$. One may easily verify this approximation using lower bounds for $\mathcal{L}(\Omega'')$ and $\mathcal{L}^*$. Therefore, the $5/3$ performance bound holds for dual-gripper cells. ■

Lemma 4 *If $\theta > 2\delta$ and $p_{\max} \leq (m+2)\delta + (m-1)\epsilon$, then the better of cycles $\{S'_c, S'_\ell\}$ provides a $3/2$ -approximation.*

Proof: Define $\mathcal{L}(\Omega')$, $\mathcal{L}(\Omega'')$, and $\mathcal{L}^*$ as in the proof of Lemma 3. Rajapakshe et al. (2011) use lower bounds for $\mathcal{L}(\Omega')$ and $\mathcal{L}^*$ and derive a $3/2$ performance bound for single-gripper 1-unit cycles using the better of cycles $\{S'_c, S'_\ell\}$. One may easily verify this approximation using lower bounds for $\mathcal{L}(\Omega'')$ and $\mathcal{L}^*$. Thus, the $3/2$ -bound holds for the dual-gripper cells. ■

Lemma 5 *If $\theta \leq \delta$ and $p_{\max} \leq \frac{\delta+\theta}{2}$, then cycle S''_0 is optimal.*

Proof: If $p_{\max} \leq \frac{\delta+\theta}{2}$ and $\theta \leq \delta$, then

$$\begin{aligned} \Omega'' &\geq (m+1)(\delta + 2\epsilon + \theta) + \sum_{i=1}^m \min\left\{0, p_i - \frac{\theta + \delta}{2}\right\} \geq (m+1)(\delta + 2\epsilon + \theta) + \sum_{i=1}^m p_i - m\frac{\theta + \delta}{2} \\ &\geq (m+2)\frac{\delta}{2} + 2(m+1)\epsilon + (m+2)\frac{\theta}{2} + \sum_{i=1}^m p_i = \frac{T(S''_0)}{2}. \end{aligned}$$

Additionally,

$$\frac{T(S''_0)}{2} \leq (m+2)\delta + 2(m+1)\epsilon + \sum_{i=1}^m p_i \leq \Omega'. \quad \blacksquare$$

Lemma 6 *Cycle S''_m is optimal under either of the following sets of conditions:*

- (a) $\theta \leq 4\delta + 2\epsilon$ and $p_{\max} \geq (m+2)\delta + 2m\epsilon + (m-1)\theta$,
- (b) $p_{\min} \geq \delta \geq \theta$.

Proof: (a) If $p_{\max} \geq (m+2)\delta + 2m\epsilon + (m-1)\theta$, then $T(S''_m) = p_{\max} + 2\epsilon + \theta \leq \Omega''$ by (3). The condition $\theta \leq 4\delta + 2\epsilon$ ensures that $p_{\max} + 2\epsilon + \theta \leq p_{\max} + 4\delta + 4\epsilon \leq \Omega'$ by (1).

(b) That $p_{\min} \geq \delta \geq \theta$ implies $T(S''_m) = \Omega''$ by (5). Furthermore, if $p_{\max} \leq (m+2)\delta + 2m\epsilon + (m-1)\theta$, then $T(S''_m) = (m+2)\delta + 2(m+1)\epsilon + m\theta \leq 2(m+1)\delta + 2(m+1)\epsilon \leq \Omega'$ by (1). ■

Lemma 7 *Cycle S''_q provides a $3/2$ -approximation under any of the following sets of conditions:*

- (a) $\delta < \theta \leq 2\delta$,
- (b) $\theta \leq \delta$ and $p_{\max} \geq (m+2)\delta + (m-1)\epsilon$,
- (c) $\theta \leq \delta$ and $\theta \geq \frac{\delta}{3}$,
- (d) $\theta \leq \delta$ and $r_2 \leq \frac{2}{3}(m-1)$, where $R_2 = \{i : p_i < \theta, 1 \leq i \leq m\}$ and $r_2 = |R_2|$.

Proof: (a) Let $B_1 = \{j | p_j \geq 2\delta\}$, $B_2 = \{j | \delta \leq p_j < 2\delta\}$, and $B_3 = \{j | p_j < \delta\}$, so $|B_1| + |B_2| + |B_3| = m$. That $\theta > \delta$ implies

$$\min\{\Omega', \Omega''\} \geq (m+2)\delta + 2(m+1)\epsilon + \sum_{i=1}^m \min\{\delta, p_i\} \geq (m+2)\delta + 2(m+1)\epsilon + |B_1|\delta + |B_2|\delta + \sum_{j \in B_3} p_j.$$

If $T(S''_q) = p_{\max} + 2\epsilon + \theta$, then cycle S''_q is optimal. Otherwise,

$$T(S''_q) \leq (m+2)\delta + 2(m+1)\epsilon + |B_1|\theta + \sum_{j \in B_2} p_j + \sum_{j \in B_3} p_j \leq (m+2)\delta + 2(m+1)\epsilon + |B_1|2\delta + |B_2|2\delta + \sum_{j \in B_3} p_j.$$

Thus,

$$\begin{aligned} \frac{T(S''_q)}{\Omega''} &\leq \frac{(m+2)\delta + 2(m+1)\epsilon + |B_1|2\delta + |B_2|2\delta + \sum_{j \in B_3} p_j}{(m+2)\delta + 2(m+1)\epsilon + |B_1|\delta + |B_2|\delta + \sum_{j \in B_3} p_j} \\ &\leq \frac{(m+2)\delta + |B_1|2\delta + |B_2|2\delta}{(m+2)\delta + |B_1|\delta + |B_2|\delta} \leq 1 + \frac{|B_1| + |B_2|}{(m+2) + |B_1| + |B_2|} \leq \frac{3}{2}. \end{aligned}$$

(b) $p_{\max} \geq (m+2)\delta + (m-1)\epsilon$ and (3) together imply $\Omega'' \geq p_{\max} + 2\epsilon + \theta \geq (m+2)\delta + (m+1)\epsilon + \theta$.

Lower bound (4) becomes

$$\Omega'' \geq \left(m + 1 - \frac{r_2}{2}\right)(\delta + \theta) + 2(m+1)\epsilon + \sum_{i \in R_2} p_i,$$

where $R_2 = \{i : p_i < \theta, 1 \leq i \leq m\}$ and $r_2 = |R_2|$.

If $T(S''_q) = p_{\max} + 2\epsilon + \theta$, then cycle S''_q is optimal. Otherwise,

$$\begin{aligned} \frac{T(S''_q)}{\Omega''} &\leq \frac{(m+2)\delta + 2(m+1)\epsilon + (m-r_2)\theta + \sum_{i \in R_2} p_i}{\Omega''} \\ &\leq 1 + \frac{(m+1)\epsilon + (m-r_2-1)\theta + \sum_{i \in R_2} p_i}{\left(m + 1 - \frac{r_2}{2}\right)(\delta + \theta) + 2(m+1)\epsilon + \sum_{i \in R_2} p_i} \\ &\leq 1 + \frac{(m+1)\epsilon + (m-r_2-1)\theta + r_2\theta}{(2m+2-r_2)\theta + 2(m+1)\epsilon + r_2\theta} \\ &\leq 1 + \frac{(m+1)\epsilon + (m-1)\theta}{2(m+1)\theta + 2(m+1)\epsilon} \leq \frac{3}{2}. \end{aligned}$$

For the proofs of (c) and (d), let $R_3 = \{i : p_i < \frac{\delta+\theta}{2}, 1 \leq i \leq m\}$ and $r_3 = |R_3|$. Define cycle S''_{q_3} by

Begin

For $i = 1$ to m do

if $p_i \geq \frac{\delta+\theta}{2}$, then assign machine M_i the activity sequence (M_i^u, M_i^l) .

otherwise ($p_i < \frac{\delta+\theta}{2}$), assign machine M_i the activity sequence (M_i^l, M_i^u) .

Next i

End

Clearly, $T(S''_q) \leq T(S''_{q_3}) = \{(m+2)\delta + 2(m+1)\epsilon + (m-r_3)\theta + \sum_{i \in R_3} p_i, p_{\max} + 2\epsilon + \theta\}$.

If $T(S''_q) = p_{\max} + 2\epsilon + \theta$, then cycle S''_q is optimal. Otherwise, we have,

$$\frac{T(S''_q)}{\Omega''} \leq \frac{T(S''_{q_3})}{\Omega''} \leq \frac{(m+2)\delta + 2(m+1)\epsilon + (m-r_3)\theta + \sum_{i \in R_3} p_i}{(m+1 - \frac{r_3}{2})(\delta + \theta) + 2(m+1)\epsilon + \sum_{i \in R_3} p_i} \leq 1 + \frac{(r_3+2)(\delta - \theta)}{(2m-r_3+2)(\delta + \theta)}. \quad (\text{B.1})$$

This ratio is increasing in r_3 and decreasing in $\frac{\theta}{\delta}$, so its maximum values occur if either $r_3 = m$ or $\theta = 0$.

(c) If $r_3 = m$ and $\theta \geq \frac{\delta}{3}$, then $\frac{T(S''_q)}{\Omega''}$ is bound by $3/2$. That $r_2 \leq r_3$ completes the proof.

(d) If $\theta = 0$ and $r_3 \leq \frac{2}{3}(m-1)$, then $\frac{T(S''_q)}{\Omega''}$ is bound by $3/2$. That $r_2 \leq r_3$ completes the proof. ■

Online Appendix C: Illustration of Procedures Grouping and Transform

The two procedures are illustrated using the 2-unit cycle $S''_{T'}$ (see Figure 5(a)) given in Example B in Online Appendix B.1. The data for the example is the same as before except that $p_3 < \theta$, $p_4 \geq \theta$, and $\theta \leq \delta$. Procedure Grouping starts by scanning the machines from M_0 to M_7 and forms a subset $\mathcal{M}_1 = \{M_2, M_3, M_4, M_5\}$ since there are at least two robot passes across each pair of adjacent machines in $\mathcal{M}_1$. The remaining machines belong to Case 9.

We now illustrate Procedure Transform: machines M_2 and M_5 are assigned to Case 5 and Case 7, respectively; machines M_3 and M_4 are assigned Case 6 and Case 8, respectively; the case assignments of the remaining machines are the same as before (i.e, Case 9). The resulted cycle is an E-cycle, $\overline{S''_{T'}}$ = $(M_0^u, M_0^u, M_1^l, M_1^u, M_1^l, M_1^u, M_2^l, M_2^u, M_2^l, M_3^l, M_3^u, M_4^l, M_4^u, M_5^l, M_5^u, M_6^l, M_6^u, M_7^l, M_7^u, M_7^l, M_7^u, M_6^l, M_6^u, M_5^l, M_5^u, M_4^l, M_4^u, M_3^l, M_3^u, M_2^l, M_2^u, M_1^l, M_1^u, M_0^l, M_0^u)$ (See Figure 5(b)).

Figure 5: (a) Cycle $S''_{T'}$ and (b) Cycle $\overline{S''_{T'}}$.

Online Appendix D: Proofs of Results from Section 6

Lemma 8 *If $\theta \leq \delta$ and $p_{\max} \leq (m+2)\delta + (m-1)\epsilon$, then for given sets $\mathcal{M}_j$, $j = 1, \dots, t$, the following lower bounds apply to the cycle time of any 2-unit cycle S''_T :*

$$T(S''_T) \geq \mathcal{L}_1 \stackrel{\text{def}}{=} (m+2n+2)\delta + 4(m+1)\epsilon + \sum_{j=1}^t \sum_{s=0}^{n_j} 2 \min\{p_{i_j+s}, \theta\} + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta) \quad (7)$$

$$T(S''_T) \geq \mathcal{L}_2 \stackrel{\text{def}}{=} \sum_{j=1}^{t_1} (p_{i_j} + p_{i_j+1}) + \sum_{j=t_1+1}^t p_{i_j+h_j} + (m+2t_1+2)\delta + (4m-4n+6t-2t_1+4)\epsilon \\ + (m-n+3t-3t_1+2)\theta + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} 2p_s. \quad (8)$$

Proof: That lower bound (7) applies to $T(S''_T)$ follows directly from Property P_1 and Table 4. For lower bound (8), we first show that for any $\mathcal{M}_j$ with $n_j \geq 2$,

$$\hat{T}(\mathcal{M}_j) \geq p_{i_j+h_j} + n_j\delta + 10\epsilon + \min\{4\delta, 2\theta + p_{i_j}\} + \min\{4\delta, 2\theta + p_{i_j+n_j}\}, \quad j = 1, \dots, t. \quad (9)$$

If $p_{i_j+h_j} \leq 2n_j\delta + (4n_j-6)\epsilon - \min\{4\delta, 2\theta + p_{i_j}\} - \min\{4\delta, 2\theta + p_{i_j+n_j}\}$, then the lower bound (9) is trivial by Property P_1 . Otherwise, suppose that activity sequence $\widetilde{\mathcal{M}}_j$ serves the machines of $\mathcal{M}_j$ in minimal time and that $\hat{T}(\widetilde{\mathcal{M}}_j) < p_{i_j+h_j} + n_j\delta + 10\epsilon + \min\{4\delta, 2\theta + p_{i_j}\} + \min\{4\delta, 2\theta + p_{i_j+n_j}\}$. These imply that the operations of $\widetilde{\mathcal{M}}_j$ can be represented as $(\sigma_1^j, M_{i_j+h_j}^u, \sigma_2^j, M_{i_j+h_j}^\ell, \sigma_3^j, M_{i_j+h_j}^u, \sigma_4^j, M_{i_j+h_j}^\ell, \sigma_5^j)$, where $\hat{T}(\sigma_3^j) = p_{i_j+h_j}$ and $\hat{T}(\sigma_1^j) + \hat{T}(\sigma_2^j) + \hat{T}(\sigma_4^j) + \hat{T}(\sigma_5^j)$ is minimized.

In $\widetilde{\mathcal{M}}_j$, after the robot's one visit to $M_{i_j-1} \in U_9$, it arrives at machine M_{i_j} holding two parts, say A and B , each of whose next processing will be on M_{i_j} . It has two options for unloading and reloading $M_{i_j+h_j}$ as quickly as possible (minimizing $\hat{T}(\sigma_1^j) + \hat{T}(\sigma_2^j)$); one does not use its dual-gripper capability, the other does. The same holds for its exit from $\mathcal{M}_j$ after its second unloading and reloading of $M_{i_j+h_j}$ (minimizing $\hat{T}(\sigma_4^j) + \hat{T}(\sigma_5^j)$):

1. The robot loads A onto M_{i_j} and then moves directly to $M_{i_j+h_j}$ to unload and reload it while still holding B . This requires the robot move sequence $(M_{i_j}^\ell, M_{i_j+h_j}^u, M_{i_j+h_j+1}^\ell, M_{i_j+h_j-1}^u, M_{i_j+h_j}^\ell)$, which consumes time $(h_j+4)\delta + 5\epsilon$. After $M_{i_j+h_j}$ finishes processing (time $p_{i_j+h_j}$), the fewest actions that the robot can perform in $\widetilde{\mathcal{M}}_j$ are to unload and reload $M_{i_j+h_j}$, then move to $M_{i_j+n_j}$ and unload it: $(M_{i_j+h_j}^u, M_{i_j+h_j+1}^\ell, M_{i_j+h_j-1}^u, M_{i_j+h_j}^\ell, M_{i_j+n_j}^u)$, which consumes time $(n_j-h_j+4)\delta + 5\epsilon$. It performs this sequence while holding a part whose next processing is on $M_{i_j+n_j+1}$, which allows it to start the actions of Case 9 on that machine as soon as possible. This argument also holds if $h_j = 1$ or $h_j = n_j - 1$.
2. Alternatively, the robot can maximize throughput while using its dual-gripper capability. To do so, the robot must have a part that is ready to be processed on $M_{i_j+h_j}$ when it unloads $M_{i_j+h_j}$ ($i_j+h_j \in U_8$). Thus, the robot must hold no part when it unloads $M_{i_j+h_j-1}$. Its sequence is: load A onto M_{i_j} , wait for it to complete, unload A from M_{i_j} , and load B onto M_{i_j} . The robot

then moves to M_{i_j+1} and loads A onto it. Next, it travels to $M_{i_j+h_j-1}$ (with both grippers empty), unloads it, moves to $M_{i_j+h_j}$, unloads it, and reloads it. The robot move sequence for these actions is $(M_{i_j}^\ell, M_{i_j}^u, M_{i_j}^\ell, M_{i_j+1}^\ell, M_{i_j+h_j-1}^u, M_{i_j+h_j}^\ell, M_{i_j+h_j}^u)$, which consumes time $p_{i_j} + h_j\delta + 7\epsilon + 2\theta$. If $h_j = 1$, then remove $(M_{i_j+1}^\ell, M_{i_j+h_j-1}^u)$ from the activity sequence and subtract 2ϵ from its execution time. After $M_{i_j+h_j}$ finishes processing (time $p_{i_j+h_j}$), a symmetric set of actions is performed at the end of $\widetilde{\mathcal{M}}_j$: $(M_{i_j+h_j}^u, M_{i_j+h_j}^\ell, M_{i_j+h_j+1}^\ell, M_{i_j+n_j-1}^u, M_{i_j+n_j}^\ell, M_{i_j+n_j}^u, M_{i_j+n_j}^\ell)$, which consumes time $p_{i_j+n_j} + (n_j - h_j)\delta + 7\epsilon + 2\theta$. If $h_j = n_j - 1$, then remove $(M_{i_j+h_j+1}^\ell, M_{i_j+n_j-1}^u)$ from the activity sequence and subtract 2ϵ from its execution time.

Note that an implementation could use single-gripper actions for the first half and dual-gripper actions for the second half, or vice versa. Hence, $\hat{T}(\widetilde{\mathcal{M}}_j)$ cannot be less than the proposed bound (9) if $n_j \geq 2$, and this lower bound for $\hat{T}(\mathcal{M}_j)$ is proven.

Because all machines not in $\bigcup_{j=1}^t \mathcal{M}_j$ belong to Case 9, the time required to serve these machines is $(m - n + 2)\delta + 4(m - n - t + 1)\epsilon + (m - n - t + 2)\theta + 2 \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} p_s$. Adding this and bounds (6) and (9) yields the following bound.

$$\begin{aligned}
T(S''_{\mathcal{I}}) &\geq \sum_{j=1}^{t_1} (\min\{p_{i_j}, p_{i_j+1}, \theta\} + p_{i_j} + p_{i_j+1}) + \sum_{j=t_1+1}^t p_{i_j+h_j} + (m + 2t_1 + 2)\delta + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} 2p_s \\
&\quad + (4m - 4n + 6t - 2t_1 + 4)\epsilon + (m - n - t + t_1 + 2)\theta + \sum_{j=t_1+1}^t (\min\{4\delta, 2\theta + p_{i_j}\} + \min\{4\delta, 2\theta + p_{i_j+n_j}\}) \\
&\geq \sum_{j=1}^{t_1} (p_{i_j} + p_{i_j+1}) + \sum_{j=t_1+1}^t p_{i_j+h_j} + (m + 2t_1 + 2)\delta + (4m - 4n + 6t - 2t_1 + 4)\epsilon \\
&\quad + (m - n + 3t - 3t_1 + 2)\theta + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} 2p_s. \quad \blacksquare
\end{aligned}$$

Corollary 1 *If $\theta \leq \delta$ and $p_{\max} \leq (m + 2)\delta + (m - 1)\epsilon$, then either cycle S''_q is optimal over all k -unit cyclic solutions, or the following lower bound applies to the per unit cycle time $\frac{T(S''(k))}{k}$ of any k -unit cycle $S''(k)$ ($k \geq 1$):*

$$\frac{2T(S''(k))}{k} \geq \min_{\mathbb{M}} \{\max\{\mathcal{L}_1, \mathcal{L}_2\}\}.$$

Proof: Let $S''_{\mathcal{I}^*}(k)$ be an optimal k -unit cycle for problem instance $\mathcal{I}$. First we assume k is even; we then extend the proof for k odd. We decompose $S''_{\mathcal{I}^*}(k)$ into $k/2$ sequences of two consecutive loadings and unloadings of a machine M_i of the form $\sigma(i, r) = (M_{i,2r-1}^\ell, \sigma_1, M_{i,2r-1}^u, \sigma_2, M_{i,2r}^\ell, \sigma_3, M_{i,2r}^u, \sigma_4)$, for some $i \in \{1, \dots, m\}$ and all $r = 1, \dots, \frac{k}{2}$. Procedure Grouping can be applied to each $\sigma(i, r)$, which gives each $\sigma(i, r)$ its own collection of $\mathcal{M}$ -sets, so each $\sigma(i, r)$ satisfies its respective lower bounds (7) and (8), which are denoted by $\mathcal{L}_1(\sigma(i, r))$ and $\mathcal{L}_2(\sigma(i, r))$. Define

$$r^* = \operatorname{argmin}_{1 \leq r \leq \frac{k}{2}} \{\max\{\mathcal{L}_1(\sigma(i, r)), \mathcal{L}_2(\sigma(i, r))\}\}.$$

It follows that

$$\frac{2T(S''(k))}{k} \geq \max\{\mathcal{L}_1(\sigma(i, r^*)), \mathcal{L}_2(\sigma(i, r^*))\}. \quad (10)$$

For k odd, sequences $\sigma(i, r)$, for $r = 1, \dots, \frac{k-1}{2}$, are as above, and the last sequence $\sigma(i, \frac{k+1}{2})$ of $S''_{\mathcal{I}^*}(k)$ corresponds to a sequence of one loading and one unloading of a machine M_i : $\sigma(i, \frac{k+1}{2}) = \left(M_{i, \frac{k+1}{2}}^\ell, \sigma_1, M_{i, \frac{k+1}{2}}^u, \sigma_2 \right)$, for some $i \in \{1, \dots, m\}$. Hence, we have the following additional lower bound from $\sigma(i, \frac{k+1}{2})$, since $\theta \leq \delta$ and $p_{\max} \leq (m+2)\delta + (m-1)\epsilon$:

$$\frac{T(S''(k))}{k} \geq (m+2)\delta + 2(m+1)\epsilon + (m-r_2)\theta + \sum_{i \in R_2} p_i \stackrel{\text{def}}{=} \widehat{\mathcal{L}}.$$

$\widehat{\mathcal{L}}$ is a bound for 1-unit cycles (Drobouchevitch et al. 2006). If $2\widehat{\mathcal{L}} \leq \max\{\mathcal{L}_1(\sigma(i, r^*)), \mathcal{L}_2(\sigma(i, r^*))\}$, then cycle S''_q is optimal among all k -unit cycles. Otherwise, the bound in (10) holds. ■

Corollary 2 *Let $\theta \leq \delta$ and $p_{\max} \leq (m+2)\delta + (m-1)\epsilon$. Given sets $\mathcal{M}_j$, $j = 1, \dots, t$, if $n \geq \frac{m+2}{6}$, then cycle S''_q provides a 3/2-approximation.*

Proof: From equation (7), we have

$$\frac{2T(S''_q)}{\mathcal{L}_1} \leq \frac{2(m+2)\delta + 4(m+1)\epsilon + 2(m-r_2)\theta + 2\sum_{s \in R_2} p_s}{(m+2n+2)\delta + 4(m+1)\epsilon + \sum_{j=1}^t \sum_{s=0}^{n_j} 2\min\{p_{i_j+s}, \theta\} + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta)}.$$

Consider the two sums in the denominator:

$$\begin{aligned} \sum_{j=1}^t \sum_{s=0}^{n_j} 2\min\{p_{i_j+s}, \theta\} + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta) &\geq \sum_{s=1}^m 2\min\{p_s, \theta\} + (m-n-t)\theta \\ &\geq \sum_{s \in R_2} 2\min\{p_s, \theta\} + \sum_{s \notin R_2} 2\min\{p_s, \theta\} + (m-n-t)\theta \\ &\geq \sum_{s \in R_2} 2p_s + [2(m-r_2) + m-n-t]\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{2T(S''_q)}{\mathcal{L}_1} &\leq \frac{2(m+2)\delta + 2(m-r_2)\theta}{(m+2n+2)\delta + [2(m-r_2) + (m-n-t)]\theta} \\ &\leq \frac{2(m+2)\delta + \frac{2}{3}(m+2)\theta}{(m+2n+2)\delta + \frac{1}{3}(5m-3n-3t+4)\theta}, \end{aligned}$$

since we may assume $r_2 > \frac{2}{3}(m-1)$ from Lemma 7(d). This last ratio is maximized by minimizing both θ and n . Substituting $n = \frac{m+2}{6}$ and $\theta = 0$ yields

$$\frac{2T(S''_q)}{\mathcal{L}_1} \leq \frac{2(m+2)}{\frac{4}{3}(m+2)} \leq \frac{3}{2}. \quad \blacksquare$$

Theorem 1 *If $\theta \leq 2\delta$, then the better of 1-unit cycle S''_q and 2-unit E-Cycle $\overline{S''_{\mathcal{I}}}$ has per unit cycle time that provides a 3/2-approximation to the optimum per unit cycle time. This bound is tight.*

Proof: Let $S''_{\mathcal{T}*}(k)$ be an optimal k -unit cycle. To simplify our comparison, we define $\tilde{T}(S''_{\mathcal{T}*}(k)) \stackrel{\text{def}}{=} 2T(S''_{\mathcal{T}*}(k))/k$. We determine the collection of $\mathcal{M}_j$ -sets for $\sigma(i, r^*)$, where r^* is defined in the proof of Corollary 1. Form $\overline{S''_{\mathcal{T}}}$ using these $\mathcal{M}_j$ -sets by Procedure Transform. We assume that the first t' epicycles in $\overline{S''_{\mathcal{T}}}$ do not have positive partial waiting; let $\bar{n}' = \sum_{j=1}^{t'} \bar{n}_j$. As before, this assumption causes no loss of generality because the epicycles' executions are independent of each other. The outline of this proof is

1. Develop expressions for $T(\overline{S''_{\mathcal{T}}})$
2. Develop a formula for a lower bound on the optimum 2-unit cycle time
3. Show that the ratio of these two cannot exceed $3/2$.

1) If $\overline{S''_{\mathcal{T}}}$ has no positive partial waiting, then its cycle time is

$$T(\overline{S''_{\mathcal{T}}}) = (m+2\bar{n}+2)\delta + 4(m+1)\epsilon + \sum_{j=1}^t \sum_{s=1}^{\bar{n}_j-1} 2 \min\{p_{\bar{i}_j+s}, \theta\} + \sum_{j=1}^t (p_{\bar{i}_j} + p_{\bar{i}_j+\bar{n}_j}) + 2t\theta + \sum_{s \notin \bigcup_{j=1}^t \overline{\mathcal{M}}_j} (2p_s + \theta). \quad (11)$$

If epicycle $\overline{\mathcal{M}}_j$ has positive partial waiting, then it has the following execution time (recall $n_j \geq 2$ by definition):

$$\hat{T}(\overline{\mathcal{M}}_j) = p_{\bar{i}_j} + p_{\bar{i}_j+\bar{h}_j} + p_{\bar{i}_j+\bar{n}_j} + \bar{n}_j\delta + 2(\bar{n}_j+4)\epsilon + 3\theta + \sum_{s=1}^{\bar{n}_j-1} \min\{p_{\bar{i}_j+s}, \theta\}, \quad j = t' + 1, \dots, t. \quad (12)$$

The 3θ term arises from one gripper rotation each at $M_{\bar{i}_j}$ and $M_{\bar{i}_j+\bar{n}_j}$, and the two gripper rotations at $M_{\bar{i}_j+\bar{h}_j}$, with one of these included in the summation.

The execution time of an epicycle without positive partial waiting is

$$\hat{T}(\overline{\mathcal{M}}_j) = p_{\bar{i}_j} + p_{\bar{i}_j+\bar{n}_j} + 3\bar{n}_j\delta + 4(\bar{n}_j+1)\epsilon + 2\theta + \sum_{s=1}^{\bar{n}_j-1} 2 \min\{p_{\bar{i}_j+s}, \theta\}, \quad (13)$$

if $n_j \geq 2$ and slightly less if $n_j = 1$. Hence, we subtract (13) from (12), sum the difference over $t' + 1, \dots, t$, and add the result to (11) to obtain an upper bound for $T(\overline{S''_{\mathcal{T}}})$ if there is positive partial waiting in epicycles $t' + 1, \dots, t$:

$$(12) - (13) \leq p_{\bar{i}_j+\bar{h}_j} - 2\bar{n}_j\delta - 2(\bar{n}_j-2)\epsilon + \theta - \sum_{s=1}^{\bar{n}_j-1} \min\{p_{\bar{i}_j+s}, \theta\}.$$

This difference is summed over $t = t' + 1, \dots, t$ and added to (11). Observe that $\sum_{t'=1}^t \bar{n}_j = \bar{n} - \bar{n}'$:

$$\begin{aligned} T(\overline{S''_{\mathcal{T}}}) &\leq (m+2\bar{n}'+2)\delta + [4(m+1) - 2(\bar{n} - \bar{n}') + 4(t-t')]\epsilon + (3t-t')\theta + \sum_{s \notin \bigcup_{j=1}^t \overline{\mathcal{M}}_j} (2p_s + \theta) \\ &\quad + \sum_{j=1}^{t'} \left(\sum_{s=1}^{\bar{n}_j-1} 2 \min\{p_{\bar{i}_j+s}, \theta\} + p_{\bar{i}_j} + p_{\bar{i}_j+\bar{n}_j} \right) + \sum_{j=t'+1}^t \left(\sum_{s=1}^{\bar{n}_j-1} \min\{p_{\bar{i}_j+s}, \theta\} + p_{\bar{i}_j} + p_{\bar{i}_j+\bar{h}_j} + p_{\bar{i}_j+\bar{n}_j} \right). \quad (14) \end{aligned}$$

2) To find a lower bound on the time $\hat{T}(\sigma(i, r^*))$ for sequence $\sigma(i, r^*)$, we first recall our two lower bounds on the execution times of $\mathcal{M}_j$ -sets:

$$\hat{T}(\mathcal{M}_j) \geq 3n_j\delta + 4(n_j + 1)\epsilon + \sum_{s=0}^{n_j} 2 \min\{p_{i_j+s}, \theta\}, \quad j = 1, \dots, t, \quad (15)$$

$$\hat{T}(\mathcal{M}_j) \geq p_{i_j+h_j} + n_j\delta + 10\epsilon + \min\{4\delta, 2\theta + p_{i_j}\} + \min\{4\delta, 2\theta + p_{i_j+n_j}\}, \quad j = 1, \dots, t. \quad (16)$$

These come from Property P_1 and inequality (9), respectively.

To get a lower bound on the optimum cycle time for $S''_{\mathcal{I}^*}(k)$, we apply lower bound (15) to the first t' $\mathcal{M}_j$ -sets, and lower bound (16) to the last $t - t'$, and add these to a lower bound on the robot's actions in the remainder of the cell $((m - n + 2)\delta + 4(m - n - t + 1)\epsilon + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta))$:

$$\begin{aligned} \tilde{T}(S''_{\mathcal{I}^*}(k)) &\geq (m + 2n' + 2)\delta + [4(m + 1) - 4(n - n') + 6(t - t')]\epsilon + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta) \\ &+ \sum_{j=1}^{t'} \sum_{s=0}^{n_j} 2 \min\{p_{i_j+s}, \theta\} + \sum_{j=t'+1}^t (p_{i_j+h_j} + \min\{4\delta, 2\theta + p_{i_j}\} + \min\{4\delta, 2\theta + p_{i_j+n_j}\}). \end{aligned} \quad (17)$$

3) To simplify the algebra, we compute the bounding ratio $T(\overline{S''_{\mathcal{I}^*}})/\tilde{T}(S''_{\mathcal{I}^*}(k))$ as $1 + [T(\overline{S''_{\mathcal{I}^*}}) - \tilde{T}(S''_{\mathcal{I}^*}(k))]/\tilde{T}(S''_{\mathcal{I}^*}(k))$, using (14) and (17):

$$\begin{aligned} T(\overline{S''_{\mathcal{I}^*}}) - \tilde{T}(S''_{\mathcal{I}^*}(k)) &\leq 2(\bar{n}' - n')\delta + 2(n - n' - t + t')\epsilon + (3t - t')\theta - \sum_{s \in \bigcup_{j=1}^t \overline{\mathcal{M}_j} \setminus \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta) \\ &+ \sum_{j=1}^{t'} \left(\sum_{s=1}^{\bar{n}_j-1} 2 \min\{p_{\bar{i}_j+s}, \theta\} - \sum_{s=0}^{n_j} 2 \min\{p_{i_j+s}, \theta\} + p_{\bar{i}_j} + p_{\bar{i}_j+\bar{n}_j} \right) \\ &+ \sum_{j=t'+1}^t \left(\sum_{s=1}^{\bar{n}_j-1} \min\{p_{\bar{i}_j+s}, \theta\} + p_{\bar{i}_j} + p_{\bar{i}_j+\bar{n}_j} - \min\{4\delta, 2\theta + p_{i_j}\} - \min\{4\delta, 2\theta + p_{i_j+n_j}\} \right) \\ &\leq 4t'\delta + 2(n - n' - t + t')\epsilon + (n - n' - t + t')\theta. \end{aligned} \quad (18)$$

We explain the simplifications that lead to (18), starting within the summation of $j = 1, \dots, t'$, and the derivation of the coefficients of δ and θ :

- For each $j \in \{1, \dots, t'\}$, Procedure Transform may expand $\mathcal{M}_j$ by adding a machine to either end. If $M_{\bar{i}_j} = M_{i_{j-1}}$ is added to $\mathcal{M}_j$, then $p_{\bar{i}_j+1} = p_{i_j}$, $\bar{i}_j \in \bigcup_{j=1}^t \overline{\mathcal{M}_j} \setminus \bigcup_{j=1}^t \mathcal{M}_j$, $(\bar{n}' - n')$ increases by one, and $(p_{\bar{i}_j} + \theta)$ is subtracted. If $M_{\bar{i}_j}$ is not added to $\mathcal{M}_j$, then $\bar{i}_j = i_j$, $p_{\bar{i}_j} \leq 2\delta + \theta$, and, at worst, $p_{\bar{i}_j} - 2\theta$ is added to (18), where $p_{\bar{i}_j} - 2\theta \leq 2\delta - \theta$. Furthermore, $(\bar{n}' - n')$ does not increase if $M_{\bar{i}_j}$ is not added to $\mathcal{M}_j$. Similar arguments apply to machine $M_{\bar{i}_j+\bar{n}_j}$. Hence, the coefficient of δ increments by two for each machine M_{i_j} or $M_{i_j+n_j}$, $j = 1, \dots, t'$, for a total of $4t'\delta$, and the coefficient of θ decrements for each M_{i_j} or $M_{i_j+n_j}$, $j = 1, \dots, t'$, for a total of $(-2t'\theta)$.
- If Procedure Transform expands $\mathcal{M}_j$, $j \in \{t' + 1, \dots, t\}$, by adding machine $M_{\bar{i}_j} = M_{i_{j-1}}$, then $\bar{i}_j \in \bigcup_{j=1}^t \overline{\mathcal{M}_j} \setminus \bigcup_{j=1}^t \mathcal{M}_j$, so $p_{\bar{i}_j}$ disappears. Otherwise, $\bar{i}_j = i_j$, which implies that $p_{\bar{i}_j} < 2\delta + \theta$, so

$p_{\bar{i}_j}$ is removed by $\min\{4\delta, 2\theta + p_{i_j}\}$. In either case 2θ is subtracted for each $M_{\bar{i}_j}$. Similar arguments apply to machine $M_{\bar{i}_j + \bar{n}_j}$. Combining this with the double sum $\sum_{j=t'+1}^t \sum_{s=1}^{\bar{n}_j-1} \min\{p_{\bar{i}_j+s}, \theta\}$ adds $[n - n' - 4(t - t')]\theta$.

Now, for the denominator, return to $\tilde{T}(S''_{\mathcal{I}*}(k))$ in (17). We expand $\sum_{j=t'+1}^t p_{i_j+h_j}$ by deriving a lower bound on it. That there is positive partial waiting at $M_{i_j+h_j}$, $j=t'+1, \dots, t$, in $\overline{S''_{\mathcal{I}}}$ implies

$$\begin{aligned} p_{i_j+h_j} &> 2n_j\delta + 2(n_j - 1)\epsilon - \theta + \sum_{s=1}^{n_j-1} \min\{p_{i_j+s}, \theta\}, \quad j = t' + 1, \dots, t, \\ \text{so } \sum_{j=t'+1}^t p_{i_j+h_j} &> 2(n - n')\delta + 2(n - n' + t - t')\epsilon - (t - t')\theta + \sum_{j=t'+1}^t \sum_{s=1}^{n_j-1} \min\{p_{i_j+s}, \theta\}. \end{aligned} \quad (19)$$

Using (19), lower bound (17) now becomes

$$\begin{aligned} \tilde{T}(S''_{\mathcal{I}*}(k)) &\geq (m+2n+2)\delta + [4(m+1) - 2(n - n') + 8(t - t')]\epsilon + 3(t - t')\theta + \sum_{s \notin \bigcup_{j=1}^t \mathcal{M}_j} (2p_s + \theta) \\ &+ \sum_{j=1}^{t'} \sum_{s=0}^{n_j} 2 \min\{p_{i_j+s}, \theta\} + \sum_{j=t'+1}^t \left(\sum_{s=1}^{n_j-1} \min\{p_{i_j+s}, \theta\} + p_{i_j} + p_{i_j+n_j} \right) \\ &\geq (m+2n+2)\delta + [4(m+1) - 2(n - n') + 8(t - t')]\epsilon + (3t - 3t')\theta. \end{aligned} \quad (20)$$

Our bounding ratio compares upper bound (18) to lower bound (20):

$$\frac{T(\overline{S''_{\mathcal{I}}})}{\tilde{T}(S''_{\mathcal{I}*}(k))} \leq 1 + \frac{4t'\delta + 2(n - n' - t + t')\epsilon + (n - n' - t + t')\theta}{(m+2n+2)\delta + [4(m+1) - 2(n - n') + 8(t - t')]\epsilon + (3t - 3t')\theta}. \quad (21)$$

To bound this fraction, first consider the δ and the θ terms. Recall that we may assume $\theta \leq \frac{\delta}{3}$ by Lemma 7(c). We must show that

$$\begin{aligned} \frac{12t' + (n - n' - t + t')}{3(m+2n+2) + (3t - 3t')\theta} &\leq \frac{1}{2} : \\ \frac{12t' + (n - n' - t + t')}{3(m+2n+2) + (3t - 3t')\theta} &\leq \frac{13t' + n - n' - t}{3m + 6n + 6 + 3t - 3t'} \leq \frac{1}{2} \\ \Leftrightarrow 26t' + 2n - 2n' - 2t &\leq 3m + 6n + 3t - 3t' + 6 \\ 29t' &\leq 3m + 4n + 2n' + 5t + 6 \end{aligned}$$

Since $t' \leq \min\{n, n', t\}$, the above holds if $18t' \leq 3m + 6$ or $t' \leq \frac{m+2}{6}$, which we may assume by Corollary 2 and $t' \leq t \leq n$.

Proving the bound for the ϵ terms requires

$$\frac{n - n' - t + t'}{2(m+1) - (n - n') + 4(t - t')\theta} \leq \frac{1}{2} \Leftrightarrow 3(n - n') - 6(t - t') \leq 2(m+1),$$

which also holds because we may assume $n \leq \frac{m+2}{6}$ by Corollary 2.

The bound in equation (21) is tight for a cell with $m = 10$, $p_2 = p_3 = p_7 = p_8 = 2\delta$, $p_k = 0$ otherwise, and $\epsilon = \theta = 0$: $\mathcal{M}_1 = \{2, 3\}$, $\mathcal{M}_2 = \{7, 8\}$, so $4t' = 8$ and $m + 2n + 2 = 16$. Furthermore, $T(S''_q) = (m + 2)\delta$, so $2T(S''_q)/\tilde{T}(S''_{\mathcal{I}*}(k)) = 3/2$. $\blacksquare$

Lemma 9 *If $\theta \leq \delta$ and $p_{\max} \leq (m+2)\delta + (m-1)\epsilon$, then there exists an E-cycle $\overline{S''_{\mathcal{T}}}$ that has the maximum throughput within $\mathcal{E}$ whose total partial waiting during each epicycle $\mathcal{M}_j$, $j = 1, \dots, t$, is less than 2δ .*

Proof: (For clarity of discourse, we omit overlines, except for cycle $\overline{S''_{\mathcal{T}}}$.) Suppose that $\overline{S''_{\mathcal{T}}}$ has the maximum throughput within $\mathcal{E}$ and that the total partial waiting during some epicycle $\mathcal{M}_j$ is greater than 2δ . The upper bound on $p_{\max}$ implies that there is no positive partial waiting at the first visit to any machine: $w_{s,1} = 0, \forall s$. Because $p_{i_j-1} + p_{i_j} < 2\delta + \theta$ and $p_{i_j+n_j} + p_{i_j+n_j+1} < 2\delta + \theta$, if $n_j \geq 2$, then $w_{i_j,2} = w_{i_j+n_j,2} = 0, j = 1, \dots, t$. If $n_j = 1$, then the total partial waiting time at M_{i_j} and M_{i_j+1} is at most $\max\{0, \max\{p_{i_j}, p_{i_j+1}\} - 2\delta - \epsilon\} \leq \max\{0, \theta - \epsilon\}$.

Now consider the internal machines of $\mathcal{M}_j$. Let W_j represent the total partial waiting during epicycle $\mathcal{M}_j$. We construct cycle $\overline{S''_{\mathcal{T}_1}}$ in which $W_j \leq 2\delta$ by iteratively reducing the total partial waiting time in $\mathcal{M}_j$ by extending $\mathcal{M}_j$ to include whichever has the larger processing time between M_{i_j-1} and $M_{i_j+n_j+1}$.

Cycle $\overline{S''_{\mathcal{T}_1}}$

Do Until $W_j \leq 2\delta$

 If $p_{i_j-1} \geq p_{i_j+n_j+1}$ Then

Case A: $M_{i_j-1} \in U_9$ or $M_{i_j-1} = I$.

 Create a new cycle $\overline{S''_{\mathcal{T}_1}}$ from $\overline{S''_{\mathcal{T}}}$ as follows: place machine M_{i_j-1} into set $\mathcal{M}_j$; if $M_{i_j-1} \neq I$, then place M_{i_j-1} into U_5 . Place M_{i_j} into U_8 if $p_{i_j} \geq \theta$, into U_6 otherwise.

Case B: $M_{i_j-1} \in \mathcal{M}_{j-1}$.

 Create a new cycle $\overline{S''_{\mathcal{T}_1}}$ from $\overline{S''_{\mathcal{T}}}$ by joining the two sets $\mathcal{M}_j$ and $\mathcal{M}_{j-1}$ into one epicycle: $\mathcal{M}_{j-1} = \mathcal{M}_{j-1} \cup \mathcal{M}_j$. Place M_{i_j} into U_8 if $p_{i_j} \geq \theta$, into U_6 otherwise. Place M_{i_j-1} into U_8 if $p_{i_j-1} \geq \theta$, into U_6 otherwise.

 Else

 A similar process occurs for $M_{i_j+n_j+1}$

 End If

Loop

It is easy to see that the partial waiting time in $\mathcal{M}_j$ and, hence, the cycle time are reduced at each step in each case. This contradicts the hypothesis that $\overline{S''_{\mathcal{T}}}$ is optimal. ■

Lemma 10 *If $\theta \leq \delta$ and $p_{\max} \leq (m+2)\delta + (m-1)\epsilon$, then Algorithm Epi-path finds cycle S''_{epi} in polynomial time $O(m^2)$.*

Proof: The directed acyclic network G constructed by Algorithm Epi-path has $O(m^2)$ arcs. In addition, the cost of each arc can be calculated in time $O(1)$. Thus, given a directed acyclic network G , we can find the shortest path in time $O(m^2)$. Also, it is clear from the construction that each feasible solution to finding an optimal 2-unit cycle corresponds to a path in G and vice-versa. The result follows. ■