An exact optimization method based on dominance properties for the design of AS/RSs

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ABSTRACT

Optimizing the design of storage systems is of crucial importance for operational excellence. The aim of this paper is to explore the optimization of the design for automated storage systems. In this latter, we are interested in three different systems. First, we highlight the different methods used to deal with this problem and point out their limits. Thereafter, we present the different corresponding cycle times models. Then, we identify and demonstrate several mathematical properties, which we use as dominance properties to implement an efficient resolution algorithm. Our algorithm greatly surpasses the existing optimization methods used for the design optimization problems. The proposed approach allows us to obtain the optimal solution ten times faster than the commonly used methods, as illustrated by a comparative numerical study.

Keywords: AS/RS, Optimization, Warehousing, Material handling, Logistics.

1. Introduction

For decades, distribution centers and warehouses have been fundamental links in any supply chain. Technological development has resulted in many of the activities related to these logistic installations being modernized as they progress. AS/RS (Automated Storage and Retrieval Systems) are the result of this automation journey and are available in different types and shapes to meet different needs.

Typical AS/RS are composed of storage racks, storage/retrieval (S/R) machines and pickup/delivery (P/D) stations. Different types are identified based on their shapes, S/R devices, operating principle, etc. These include the multi-aisles, mobile-rack, man-on-board, carrousel, deep-lane, shuttle-based systems and flow-rack systems (Sari, et al., 2005).

The current competitive environment compels companies to stand out at all levels to optimize their performances, particularly those related to warehouses operations. As a result, numerous questions arise both at the design (strategic) and the control (tactical and operational) levels (Gagliardi, et al., 2012). In this paper, we are interested in finding the best design dimensions of different AS/RSs (strategic level) where the system's cycle time is the considered criterion to minimize. Existing solving methods present drawbacks in precision (continuous methods) or in solving time (long run time). The choice of this criterion is dictated by the specificities of the studied systems and their parameters. First, the studied systems (as it is the case for the majority of AS/RSs) have a fixed number of S/R machines and P/D stations (as explained in the systems description section below). Therefore, these parameters do not come into play in any design optimization. This also means that there is no waiting time within the system for the machines neither any auto-blocking or congestion, and the only affected time performance is the cycle time. However, this does not mean that there is no waiting times and congestions for the products being stored or retrieved, but these performance factors are function of the service time (in this case the cycle time), the flow rate and the adopted storage policy. Since we are interested in a strategic issue, which is the server's time optimization (namely the AS/RS), the cycle time is the only criterion that is considered. Knowingly, the cycle time minimization is the necessary but not sufficient condition to the optimization of waiting times and congestions. The other conditions are the implementation of optimal storage policy, interleaving rules, batching rules, etc., which are tactical and operational issues that need to be studied downstream the design optimization of the AS/RS. Moreover, the main decision variables in the optimization of the design of these storage systems are the systems dimensions (in terms of unit load locations) in height, length and depth. This whole approach, i.e. taking into account the system's dimensions, has been used for several systems in the literature, as in (Yang, et al., 2015; Yang, et al., 2017; Xu, et al., 2018b) for the multi-deep AS/RS, in (De Koster, et al., 2008; Xu, et al., 2018a) for the 3D compact rack AS/RS, and in (Kouloughli and Sari, 2015) for the multi-aisle. However, it should be noticed that this is not true for shuttle-based, autonomous vehicles storage/retrieval systems and classical warehouses with cranes or forklifts-based material handling equipment. Since the racks' layout and the number of shuttles/vehicles/cranes/lifts/forklift affect the congestion and the waiting time of the material handling devices within the system.

Two types of approaches exist in the literature to solve the problem of the design of AS/RS. On the one hand, there are fast resolution methods offering good solutions, but without any optimality insurance. These methods are mostly relying on relaxation approaches to approximate the best dimensions. On the other hand, exact resolution methods provide optimal solutions but require high computation time. Moreover, no theorical complexity studies have been conducted in the literature for this problem to determine the reasons of this high computation time when dealing with systems with large sizes. A detailed overview on existing approaches is provided in the literature review section. This motivated us to investigate new resolution methods that may reconcile the advantages of the two types of methods without their drawbacks. The main contributions of this paper consist in identifying dominance properties in the cycle time formulas for three different AS/RSs. The identified properties are used to introduce an efficient exact resolution method. We test the new approach on a set of AS/RS systems of different sizes and show that our approach provides the optimal solution while being ten times faster than existing exact approaches.

Before going deeper into the exposition of our study, we would like to highlight some notions and concepts that our work is based on. Firstly, the cycle time is the average time required for the system to complete a given transaction. This transaction can be a simple storage, a simple retrieval or a combination of both. As a result, we identify two types of cycles, the simple storage or retrieval cycle (in some cases they are equal and in others not), and the dual cycle, which is a retrieval associated with a storage (the machine, after carrying out a storage, proceeds to a retrieval without returning to the dwell point position between the two transactions). Some types of systems can also perform quadruple and sextuple cycles; this is possible when the system has S/R machines that can contain two or three unit loads, or for man-onboard AS/RSs.

The calculation of the average cycle time requires the consideration of all possible transactions for a given system. Next, according to the storage policy to be considered in the long run, an average cycle time is computed using the appropriate probability distribution. It is therefore necessary to consider all the unitary movements that S/R machines make in order to model the total path they travel and thus the required time for a complete cycle. Most of the systems in the industry are equipped with stacker cranes. The latter being provided with independent motors for horizontal and vertical movements, they perform diagonal paths according to a Tchebychev distance (Bozer and White, 1984). However, not all the automated storage systems are concerned by this characteristic. The shuttle-based systems (Lerher, et al., 2015) and the split-platform AS/RS (Liu, et al., 2016) do not use stacker cranes as handling machines (as highlighted in the next section). For this reason, we are interested in three different systems, the bidirectional flow-rack AS/RS (high density system, i.e. compact rack), the multi-aisles (low density system, i.e. lot of free space) and the mobile-rack AS/RS (medium density system), since these systems are composed of S/R machines that move according to a Tchebychev travel. More details about the systems architecture and running will be given in the corresponding section.

The Tchebychev travel (or Tchebychev distance), also called maximum metric, is a metric defined on a vector space where the distance between two vectors is the maximum of their differences along any coordinate dimension. Since our stacker cranes move in a two-dimensional space, we associate to this plane (rack face) a Cartesian coordinate system, where each unit-load location is identified by the coordinates $(x\ ;\ y)$. This way, the required travel time for the S/R machine to move from the position $(x\ ;\ y)$ to $(x'\ ;\ y')$ is given by:

$$T_1 = max(|x - x'|t'_h\ ; |y - y'|t'_v) \quad (1)$$

Where t'_h and t'_v are the unit displacement times between adjacent bins horizontally and vertically respectively.

The literature contains two different approaches for the cycle time calculation. As the systems are composed of a number of storage segments, the first one considers the system rack as a denumerable number of storage segments. Discrete cycle time models are therefore proposed as discrete functions. These models, when considered, require using discrete optimization techniques to find the best dimensions in order to optimize the cycle time. The second approach is a relaxation of the first models in which the rack face is considered a surface of an infinity of points, where each point can be reached by the S/R machine to perform a storage or retrieval operation. Naturally, this modelling is not faithful to the real system behavior. However, it gives the possibility to come out with easy-to-use cycle time models for further investigations such as the usage of continuous optimization methods.

To the best of our knowledge, no study has been conducted to develop efficient exact resolution methods other than full enumeration. Thus, through our study, we aim to develop an exact and efficient resolution method for this problem. To this end, several steps have been taken. First, we introduce mathematical dominance properties that help eliminate dominated configurations. Several properties are demonstrated in this paper for the classical unit-load AS/RS (Appendix A) and the threepreviously mentioned storage systems (Section 4). Second, we investigate the problem's complexity and propose a resolution algorithm based on the introduced dominance properties.

Therefore, in order to exhibit this, we deploy our paper as follows. In the sequel, we will present a brief literature review where works related to cycle times calculation and design optimization of AS/RS will be highlighted. Then, the third section will present the studied systems, their cycle time formulas along with the assumptions they are based on. The fourth section will expose the addressed optimization problems and present the identified mathematical properties, where subsections are devoted to a complexity study of the problem in a first place, then a proposition of a resolution algorithm that is based on the demonstrated dominance properties. Finally, the proposed approach is compared to other resolution methods used in the literature in the numerical study section. We show that the dominance properties allow not only to identify special cases for some AS/RS but also to reduce the search space and accelerate problem solving. We end up the study with a discussion and conclusion section.

2. Literature review

This section highlights the current studies on warehouse design, the different types of AS/RSs and the existing optimization models, as well as numerous resolution methods that have been used to deal with the studied problem. This is intended to highlight the limits that arise and the gap that this study aims to fill while situating our contribution in the existing literature.

2.1. Warehouse design

What is meant by warehouse design is the identification of the values of a parameter set to optimize one or many performance factors of the warehouse. These parameters can be the racks layout and disposition, the number and shapes of aisles, the type and the number of handling equipment (or employees), the degree of automation, etc. This is done in order to optimize a set of criteria such as cycle time, congestion, waiting time, total and usage cost, energy consumption, carbon footprint, throughput, etc.

Pohl, et al. (2009) developed expected travel distance expressions for dual cycle operations. This was performed for three common warehouse designs. They could show that the best of the three is the one least commonly found in practice while demonstrating that the optimal placement of a “middle cross aisle” in the most common design was, in fact, not in the middle. In (Gue, et al., 2012), the authors used Mathematica to implement a numerical method based on travel-time models they developed and presented in the same paper. The aim was to optimize the design of a unit-load system by finding out which of a ’Flying-V’ or ‘Inverted-V’ cross-aisles design was the best. Öztürkoglu and Hoser (2018) developed new warehouse designs that provide a reduction in travel distance for the order-picking operation. For this purpose, they proposed a new layout problem called the “discrete cross aisle warehouse design”. In this problem, a linear middle cross aisle is divided into segments called tunnels on each picking aisle. In order to calculate the average tour length for the proposed design problem, they developed an efficient algorithm that solves the order-picking problem optimally. Bortolini, et al. (2018) provided the analytical model to best design a non-traditional warehouse for unit-load with diagonal cross-aisles and storage policy according to the class-based storage strategy. Their model minimizes the average single-command cycle time, while choosing the best sizes of classes, their shapes, and the position/numbers of additional aisles. Tutam and White (2019) developed discrete formulas of the expected traveled distance in a unit-load warehouse with a middle-cross-aisle. Their results revealed that the number of dock doors and the number of storage/retrieval locations significantly affect the optimal shape factor regardless of the orientation of storage/retrieval locations. They also provided recommendations regarding conditions in which each layout configuration minimizes the expected traveled distance. Yener and Yazgan (2019) investigated the effectiveness of designing warehouses to determine average order picking time and travelled distance using a data mining technique.

2.2. AS/RSs types and mathematical modelling

There are several types of automated storage systems in the industry and their related problems are studied in the literature. In addition to simulation, mathematical modelling is one of the most used tools to address these problems. Mathematical models are formulas that allow the user to study and evaluate the behavior of the system prior to its installation. One of the most modelled performance factors is the cycle time (or the throughput). These cycle time models can be either used for performance evaluation and systems comparison, or for further investigations as design optimization. In the sequel, we are presenting the existing models and modelling approaches. In Section 2.3, the related addressed decision problems and the used resolution methods will be presented.

Bozer and White (1984) were among the first researchers to be interested in cycle time calculation for the unit-load AS/RS, where they presented a statistical approach by considering the rack face as a continuous surface of an infinity of points. Subsequently, other types of systems were studied, where one or both approaches were used (continuous rack face assumption approach and the discrete rack face assumption approach).

- Flow-rack and 3D compact rack AS/RSs

Several cycle time models were developed in (Sari, et al., 2005) for the classical flow-rack AS/RS, while Hamzaoui and Sari (2015) and Sari and Hamzaoui (2016) were interested in the single machine flow-rack AS/RS. Considering different modelling approaches, Chen, et al. (2015) and Hamzaoui and Sari (2019) provided cycle time models for the bidirectional flow-rack. Ghomri and Sari (2017) were interested in the classical flow-rack AS/RS, and could come up with new cycle time models considering the variety of items stored in the system and their proportions. The free-fall flow-rack system is considered in (Metahri and Hachemi, 2018), where the authors developed several cycle time models and evaluated their precision through simulation.

Dealing with the high density AS/RSs, De Koster, et al. (2008) presented the first cycle time models for the 3D compact-rack AS/RS, while Xu, et al. (2018a) and Xu, et al. (2019) proposed new models considering different dwell-point positions policies for the S/R machine.

Dealing with the multi-deep AS/RS, which is a system similar to the flow-rack, Xu, et al. (2018b) developed new cycle time models under the assumption of a class-based storage policy.

- Multi-aisle and mobile-rack AS/RSs

Eldemir, et al. (2004) proposed an analytical cycle time model for multi-aisle AS/RS when using dedicated storage and computationally efficient procedures for estimating space requirements with systems using randomized or class-based storage. Ghomri, et al. (2009) presented continuous cycle time models for the single and dual cycle of this same system. On the strength of Hwang and Lee (1990) work on the unit-load AS/RS, Lerher, et al. (2010) came up with new cycle time models considering the operating characteristics of the storage and retrieval machine such as acceleration and deceleration and the maximum velocity. Gamberi, et al. (2012) developed analytical and numerical models for the aisle captive crane AS/RS while considering a class-based storage policy. Ouhoud, et al. (2016) proceeded to a comparative study between continuous and discrete models of the single cycle time for a multi-aisle automated system with class-based storage policy.

Dealing with the mobile-rack AS/RS, Guezzen, et al. (2013) developed continuous cycle time models and evaluated their accuracy via simulation. By taking an interest in the classical unit-load AS/RS (which is in fact an aisle-captive multi-aisle), Schenone, et al. (2018) proposed cycle time models when using dual-shuttle handling machines whereas Schenone, et al. (2020) proposed for this same system an approach to compute the travel time in a class-based storage environment. They completed a regression analysis in order to define the importance of the key predictors taken into account.

- Autonomous vehicle, shuttle-based and split-platform storage systems

The autonomous-vehicle-based storage/retrieval systems (AVS/RS) use autonomous handling devices (vehicles) to carry out storage/retrieval operations. These systems are also equipped with lifts to provide vertical movements. D'Antonio and Chiabert (2019) developed analytical models for the cycle time and throughput evaluation that are capable of assessing the performance of a tier-to-tier, multi-shuttle AVS/RS, while taking into account the ability of the vehicles to simultaneously perform different tasks. D'Antonio, et al. (2019) were interested in the evaluation of the energy consumption for the same system using analytical models, which exhibit a higher flexibility and a lower energy consumption compared to the traditional AS/RS. Dealing with the design of AVS/RS, Ekren (2020a) was interested in both cycle time and energy consumption.

Shuttle-based S/R systems are a variation of the AVS/RS, since they use little motorized shuttles as handling equipment to deal with relatively light loads, in addition to being tier-captive in the majority of cases. Lerher, et al. (2015) presented an analytical travel time model for the computation cycle time for shuttle-based storage and retrieval systems, considering the operating characteristics of the elevators, lifting tables and the shuttle carrier, such as acceleration and deceleration and the maximum velocity. Tappia, et al. (2016) developed novel queuing network models to estimate the performance of both single-tier and multitier shuttle-based S/R systems. Ekren (2020b) developed mathematical models for energy consumption and regeneration for this same system.

The split-platform (SP) AS/RS is a relatively new material handling system that is composed of one vertical platform for each rack and N horizontal platforms to serve N tiers. The vertical platform provides the vertical link among different tiers of the AS/RS rack, whereas the horizontal platforms access the storage cells on a given tier (Vasili, et al., 2006). Hu, et al. (2005) presented a travel-time model under the stay dwell point policy, i.e. the platforms remain where they are after completing a storage/retrieval operation. Their model was validated by computer simulations. The results showed that the developed model was reliable for the design and analysis of this system. In (Liu, et al., 2016), the authors were the first to present a continuous travel time model for the DC in the SP-AS/RS under input and output (I/O) dwell point policy. Liu, et al. (2018) studied two dual command travel time models for the split-platform AS/RS. They validated these models by computer simulation, which gave accurate results.

It must be noticed that these three systems (AVS/RS, SBS/RS and SP-AS/RS) are equipped with handling devices that cannot move in different axes simultaneously unlike the previously presented systems, which can move in the two axes at the same time, which is mathematically equivalent to a Tchebychev travel (see section 1). This is why in this paper; we are interested in three different AS/RSs whose handling devices move according to a Tchebychev travel: the multi-aisles (low density system), the mobile-rack (medium density) and the bidirectional flow-rack (high density). This particularity allowed us to identify mathematical properties (presented in Section 4) that characterize this type of displacement.

2.3. Design optimization of AS/RSs and used resolution methods

In the literature, design optimization in AS/RSs is the process that aim at identifying upstream parameters that optimize the performances of the system in steady state. In the vast majority of cases, these parameters are the three spatial dimensions (in storage locations) of the system. The performance criteria to optimize are (mainly) the cycle time or throughput, energy consumption, usage cost, carbon foot-print, etc. However, for AVS/RSs (including SBS/RSs), in addition to the three spatial dimensions, the number of vehicles and lifts are also design parameters that must be considered in this optimization process.

We can see that, in the literature, several resolution methods were used regardless of the problem's complexity. In this section, we are presenting the main conducted studies and used resolution methods for different AS/RSs, and then discuss their appropriate use in each situation.

When dealing with mono-objective optimization design, the cycle time is the most considered criterion to minimize. An optimal system layout has to be defined for a minimum cycle time in the long run. In general, the adopted methods are based on the continuous cycle time models (based on the continuous rack face assumption), where optimal ratios between the different system dimensions are found by solving the optimization problem using Lagrange multipliers. De Koster, et al. (2008) used this method to identify the optimal layout of the 3D-compact rack AS/RS for minimum single and dual cycles. Considering the same system, Xu, et al. (2018a) and Xu, et al. (2019) found its optimal design for minimum cycle time considering different dwell-point policies.

By taking an interest in the multi-aisles AS/RS, Kouloughli and Sari (2015) were able to identify the optimal layout of the system in addition to a "useful region", where small modifications in the system design do not significantly affect the cycle time. Yang, et al. (2015) and Yang, et al. (2017) used the same method to investigate the multi-deep automated storage system while considering the operational characteristics of the S/R machines. Hamzaoui, et al. (2019) investigated the bidirectional flow-rack AS/RS and adapted this same method for the design optimization.

However, these methods do not ensure the optimality of the obtained solutions. Indeed, the obtained solution (values of the three spatial dimensions) is a continuous and not a discrete one. Thus, the solution does not take into consideration the fact that the rack is composed of a discrete number of unit-load storage emplacements. Therefore, an approximation has to be made in order to obtain integer solutions (number of storage emplacement in each dimension), which does not provide any guarantee regarding the optimality of the obtained solutions. In other papers, heuristic or numerical methods were used. In (Hwang, et al., 2002), a heuristic algorithm was proposed to solve the design optimization problem for the mini-load AS/RS combined with the AGV cell-sizing problem. In (Lerher, et al., 2010), the design of unit-load and multi-aisle AS/RS was addressed. The authors presented a mathematical optimization model where the objective was to determine the best system configuration for a minimum total cost. A genetic algorithm was then introduced and implemented to solve this optimization problem. In (Marchet, et al., 2013), the AVS/RS was studied for the cost optimization. The authors proposed a method to find the number of aisles, tiers and columns that minimizes the cost while considering a throughput target constraint. These heuristics and metaheuristics also do not give any guarantee on the optimality of the obtained solutions.

Furthermore, several methods for multi-objective optimization were used in the AS/RS design. In (Borovinšek, et al., 2017), a multi-objective optimization model was presented, where the throughput, the energy consumption and the total cost were the criteria to be minimized. The model was formulated for the shuttle-based system, and the objective was to determine its optimal layout to optimize the cited criteria. Finally, this decision problem was solved using NSGA II. Other papers addressed the multi-objective design optimization for AS/RS. In (Rajković, et al., 2017; Accorsi, et al., 2017), the cycle time, the carbon emission and the cost were the considered criteria. The former used NSGA II to solve the problem while the latter used another proposed heuristic. Tostani, et al. (2020) proposed a bi-level and bi-objective model, in addition to a Modified

Cooperative Coevolutionary Algorithm for Bi-Level Optimization (MCoBRA), which was presented then used to solve the proposed model. Ekren (2020a) was interested in the minimization of two conflicting performance measures – the average cycle time and the average energy consumption – using a hierarchical approach to solve the optimization problem.

Moreover, as far as we know, the only exact used method for this specific design optimization problem is the full enumeration. This is due to the nonlinearity of the problem (MINLP). Even if there are some solvers (mainly based on a branch-and-bound algorithm, e.g. Couenne, SCIP, LINDO, etc.) that can solve some nonlinear problems, they cannot solve all of them, especially the ones related to AS/RSs design optimization for the minimum cycle time. The cycle time formulas generally include decision variables in the sum's upper limits (as the cycle time formulas presented in this paper), which is binding for the internal solver resolution algorithm (mainly branch-and-bound). Based on this, the studies that were interested in the exact resolution of this problem have used full enumeration as a resolution method, as in (Sari, 2003) for the classical flow-rack AS/RS and in (Hamzaoui and Sari, 2015) for the single machine flow-rack AS/RS. In these two papers, an enumeration algorithm was used to determine optimal integer dimensions, in order to see the impact of the load rate variation on the optimal configurations and show how far this latter degrades the average cycle time value for single and dual cycles while varying dwell-point positions. Xu, et al. (2018b) also used an enumeration method while applying constraints on the maximum dimensions of the system, which for small instances, significantly reduced the research space, while Metahri (2019) used this same method to deal with the free-fall flow-rack AS/RS.

We have been through warehouse design, different AS/RSs types, mathematical modelling especially cycle time models, and finally AS/RS design optimization as well as main used resolution methods. In the light of this literature review, we could see that even if no formal complexity study was conducted for this problem (the design optimization problem), in the vast majority of cases, approached methods were used to solve this optimization problem. Even though these methods have the advantage of being fast, they have no guarantee of optimality. Moreover, the full enumeration, which obviously gives the optimal solution, is very far from being computationally efficient. Therefore, which methods are the most adapted to this problem? Is it relevant to use such approached methods even if we do not know the computational complexity of the problem? Is it relevant to use exact but time-consuming methods to solve this problem? Can we know a priori that one design is better than another and therefore reduce the search space? Can we find out or develop adapted, time efficient and exact methods to solve the AS/RS design optimization problem? To answer these research questions we aim to fill several gaps: (1) Conduct a complexity study in order to determine the "Design optimization problem" complexity; (2) Address the above mentioned compromise (time/precision) by developing an exact and computationally efficient optimization method (3) Highlight the time gain when using the proposed algorithm on the one hand, and the loss in precision when using the commonly used approached methods on the other hand. To do so, thanks to the conducted study and through the several sections of the paper, we are introducing the following contributions:

- Conduct a formal complexity study and study the relevance of using approached methods.
- Identify mathematical properties by studying different previously developed cycle time models.
- Set up an efficient resolution algorithm using the demonstrated mathematical properties as dominance properties.
- Show the loss in precision of the commonly used approximate methods compared with an exact resolution of the problem.

In the sequel, we are presenting the three studied AS/RSs and their cycle time models. Then, we proceed to the formulation of the optimization problem. This is followed by the dominance properties identification and demonstration; a brief complexity study; and the resolution algorithm presentation. Finally, we end up with a numerical study, followed by a discussion and conclusion section.

3. Studied systems

A typical AS/RS is composed of some basic components; the racks (the three-dimensional structures that will contain the stored *unit-loads*), the S/R machines, the pickup/delivery stations, the control and management software package (Sari, 2003). Depending on the AS/RS type and size, a *unit-load* can be a pallet, pallet-sized load or smaller loads. Small systems as mini-load AS/RS allow selection of items in totes, trays or cartons (this is what makes the discrete modelling more accurate).

In this section, we are presenting the studied systems and their related cycle time models. These models have been developed by colleagues in previous works and presented in the papers cited below. In their modelling process, they followed a number of assumptions:

A_1 – The rack face is composed of a discrete number of unit-load storage emplacements.

A_2 – The S/R machines move according to a Tchebychev displacement.

A_3 – The S/R machines velocities are considered constant.

A_4 – Storage and retrieval locations are considered to be uniformly distributed (random storage policy).

A_5 – The machines dwell points are located at the P/D stations, which are situated at the bottom corner of the rack.

For what will follow, the following notation has been used:

Notation

$\boldsymbol{ERC1}$	Average retrieval cycle time for the bidirectional flow-rack.	$\boldsymbol{ERC2}$	Average retrieval cycle time for the multi-aisles.
$\boldsymbol{ERC3}$	Average retrieval cycle time for the mobile-rack.	$(\boldsymbol{H};\boldsymbol{V};\boldsymbol{M})$	Number of unit-load emplacements horizontally, vertically and in depth respectively.
$\boldsymbol{t'_p}$	Required time for displacement between adjacent bins (unit displacement) or in the transversal aisle.	$\boldsymbol{t'_r}$	Required time for the racks displacement.
$\boldsymbol{t'_h}$	Required time for displacement between adjacent bins (horizontally).	t'_v	Required time for displacement between adjacent bins (vertically).
$\boldsymbol{N}$	Number of unit-load emplacements in the whole system (Volume).	$\boldsymbol{\rho}$	Load rate.

3.1. The bi-directional flow-rack AS/RS

The bi-directional flow-rack (BFR) system belongs to the flow-rack systems family. These systems are high-density systems, which means that they comprise only one compact rack where transversal movements are ensured by gravitational conveyors within the rack.

This system has been invented and patented by (Southeast University, 2013) and presented for the first time in the literature in (Chen, et al., 2015), whereas discrete and continuous cycle times have been presented in

(Hamzaoui and Sari, 2019). This system is composed of a multitude of storage bins distributed horizontally and vertically. Each bin is equipped with a gravitational conveyor and can contain a given number of unit-loads. The bins are tilted alternately to one side or the other on the two rack faces, so that the bins in the same column are tilted to the same side (see Fig. 1). The system is equipped with two S/R machines, one on each rack face, so that the two machines can perform both storage and retrieval operations. Each one of the machines can store in one column out of two and retrieve on the others. However, the two machines work in different manners according to the operation to be performed. Since in the case of a storage or a retrieval of a load in the first position, each machine can work independently of the other, but when it comes to a retrieval of a load which is not on the first position the two machines work together (see Fig. 2).

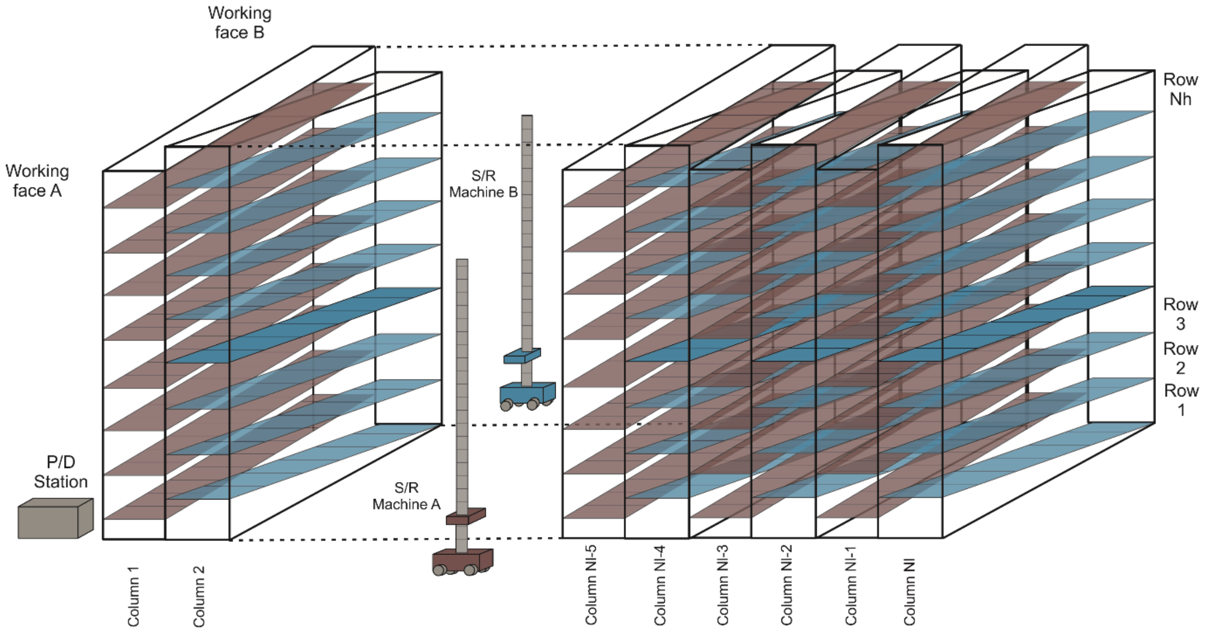

Fig. 1. Bidirectional flow-rack AS/RS (Hamzaoui, et al., 2019)

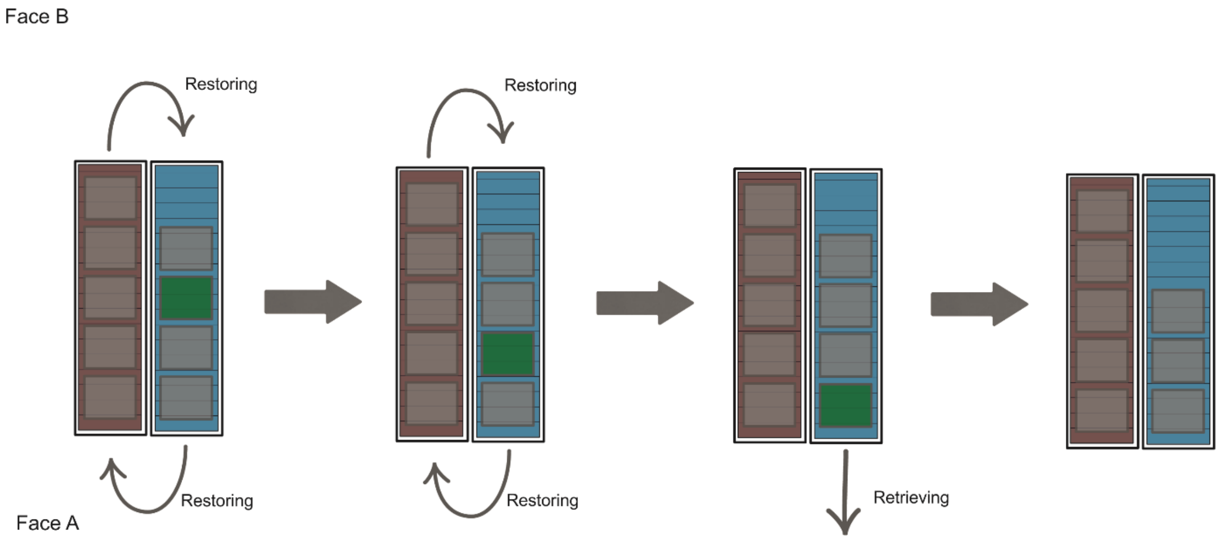

Fig. 2. Retrieval operation in the BFR AS/RS (Hamzaoui, et al., 2019)

The single cycle retrieval time model was presented in (Hamzaoui and Sari, 2019). Two different cycle time formulas were developed, one based on the continuous rack face approximation, and the other based on the exact discrete rack face consideration as mentioned before. In this paper, we consider the second modelling approach to formulate our decision problem (Assumption A_1). On the one hand, this has never been considered to address this kind of problems, and on the other hand, it is more precise than the first cited modelling approach.

In (Hamzaoui and Sari, 2019), for $> \frac{1}{\rho}$, the retrieval single cycle time is given by:

$$ERC1 = \frac{1}{\rho N}\left(\frac{4}{\frac{1}{S_1}+\frac{1}{S_2}} + (\rho M - 1)(2(S_1+S_2) + \rho N.max(t'_h; t'_v)) \right) \tag{2}$$

Where:

$$S_1 = \sum_{i=1}^{\frac{H}{2}} \sum_{j=1}^{V} max\big((2i-1)t'_h; jt'_v\big) \tag{3}$$

$$S_2 = \sum_{i=1}^{\frac{H}{2}} \sum_{j=1}^{V} max(2it'_h; jt'_v) \tag{4}$$

3.2. The multi-aisle AS/RS

A multi-aisle AS/RS is composed of a set of fixed parallel racks, arranged in pairs and separated by aisles. A common aisle, perpendicular to racks, links all serving aisles. A single storage/retrieval machine serves all racks. The S/R machine moves simultaneously in both vertical and horizontal directions (Tchebychev travel) on serving aisles as well as on the common aisle. Since the dwell point of the S/R machine is the pickup/delivery (P/D) station, the S/R machine always starts from the P/D station located in the lower left corner of the system and returns to it at the end of each cycle for both storage and retrieval operations (Kouloughli and Sari, 2015). A discrete retrieval cycle time for the multi-aisle AS/RS was presented in (Ghomri, et al., 2009):

$$ERC2 = \frac{4}{N} \sum_{k=1}^{\frac{M}{2}} \sum_{i=1}^{H} \sum_{j=1}^{V} max\left((it'_h + kt'_p)\,;\, jt'_v \right) \tag{5}$$

3.3. The mobile-rack AS/RS

This system is a variation of the multi-aisle AS/RS. This latter is composed of racks that move literally on rails so that one can open an aisle between any two adjacent racks. The S/R machine enters then the aisle to perform storage or retrieval operations (Guezzen, et al., 2013). The mobile-rack is therefore a variant of the multi-aisle while having the advantages of space reduction as the compact rack storage systems. A discrete retrieval cycle time for the mobile-rack AS/RS was presented in (Guezzen, et al., 2013):

$$ERC3 = \frac{2}{N} \sum_{k=1}^{\frac{M}{2}} \sum_{i=1}^{H} \sum_{j=1}^{V} max\big(it'_h + max(t'_r\,; kt'_p)\,; jt'_v\big) + max\left((it'_h + kt'_p)\,;\, jt'_v \right) \tag{6}$$

Note that, for Equations (5) and (6), a small modification has been made. In (Ghomri, et al., 2009) and (Guezzen, et al., 2013), the dwell point for the S/R machine in considered to be perfectly in front of the first storage bin. This is not the case in (Hamzaoui and Sari, 2019) for the bidirectional flow-rack in Equation (2), where the dwell points are located at the P/D stations. These stations are located at the lower corner of the rack, and offset one unit in length horizontally and vertically from the first bin (which is closer to reality). This is why we replaced $(k-1)$ by k and $(j-1)$ by j in these equations, in order to study the three cycle time formulas in a coherent way. In the next section, we are going to formalize our design optimization problems, where the retrieval cycle time is the objective function to minimize. From there, a study of the problem will be conducted in order to identify mathematical properties and propose an efficient exact resolution method.

4. The addressed optimization problems

4.1. Optimization problems formulation

As mentioned before, the addressed optimization problems in this paper concern the design optimization of AS/RS for minimum retrieval cycle time. We consider three optimization problems (P1), (P2) and (P3) for the BFR, multi-aisle and mobile-rack AS/RS respectively.

For (P1), (P2) and (P3), $(H; V; M)$ is the representation of the decision variables, where length, height and depth (in unit-load emplacements) are represented respectively.

Problem (P1):

$$Minimize \;\; ERC1 \tag{7}$$

$$Subject\ to: \begin{cases} H \cdot V \cdot M = N \quad \forall\ (H; V; M) \in \mathbb{N}^{*3} & (8) \\ M > \dfrac{1}{\rho} & (9) \\ H = 2k \; with \; k \in \mathbb{N}^{*} & (10) \\ H > 1, V > 1, M > 1 & (11) \\ (H; V; M) \in \mathbb{N}^{*3} & (12) \end{cases}$$

The problem (P1) can be interpreted as follows. Equation (7) is the objective function to optimize for a given system volume, which is restricted by Equation (8) that fixes the total number of unit-load storage locations regardless of its dimensions. Equation (9) is the condition for which the cycle time formula can be used. The architectural constraint of the system is ensured by Equations (10) and (11). Equation (10) guarantees that the number of bins in length is always even, while Equation (11) sets the lower limits of the three system dimensions. Finally, Equation (12) is the integrality constraint.

Problem (P2):

$$Minimize \;\; ERC2 \tag{13}$$

$$Subject\ to: \begin{cases} H \cdot V \cdot M = N \quad \forall\ (H; V; M) \in \mathbb{N}^{*3} & (14) \\ H > 1, V > 1, M > 1 & (15) \\ M = 2q \quad with \; q \in \mathbb{Z}^{*} & (16) \\ (H; V; M) \in \mathbb{N}^{*3} & (17) \end{cases}$$

The problem (P2) can be interpreted as follows. Equation (13) is the objective function to optimize for a given system volume, which is restricted by Equation (14) that fixes the total number of unit-load storage locations regardless of its dimensions. Equation (15) sets the lower limits of the three system dimensions whereas Equation (16) guarantees that the number of bins in length is always even. Finally, Equation (17) is the integrality constraint.

Problem (P3):

$$Minimize \;\; ERC3 \tag{18}$$

$$Subject\ to: \begin{cases} H \cdot V \cdot M = N \quad \forall\ (H; V; M) \in \mathbb{N}^{*3} & (19) \\ H > 1, V > 1, M > 1 & (20) \\ M = 2q \quad with \; q \in \mathbb{Z}^{*} & (21) \\ (H; V; M) \in \mathbb{N}^{*3} & (22) \end{cases}$$

The problem (P3) can be interpreted as follows. Equation (18) is the objective function to optimize for a given system volume, which is restricted by Equation (19) that fixes the total number of unit-load storage locations regardless of its dimensions. The architectural constraint of the system is ensured by Equations (20) and (21). Equation (20) sets the lower limits of the three system dimensions while Equation (21) guarantees that the number of bins in length is always even. Finally, Equation (22) is the integrality constraint.

4.2. Dominance properties

Searching for the best system's sizes can be a tedious task. Some configurations (length, height and depth) are dominated by others since they provide a worse cycle time. To accelerate the search and efficiently find the optimal solution, it is thus worth identifying the dominated configurations and eliminating them from the search space to accelerate the exploration process.

For the three addressed problems, thanks to Equation (8), Equation (14) and Equation (19), it is easy to note that the variable M can be deduced from the two other variables. This means that (P1), (P2) and (P3) can be reduced to two-decision-variables optimization problems. Thus, a solution can be represented with the values of the two remaining variables. Let $(H; V)$ be a solution in which the rack has a length of H and a height of V unit load emplacements. If we consider that $(H; V)$ and $(V; H)$ are two candidate solutions, we would like to investigate whether one of these two solutions dominate the other one. Since we consider a Tchebychev travel, when $t'_h \neq t'_v$, the most intuitive answer to this question is to associate the largest dimension with the highest speed (i.e. lowest unit travel time) and vice versa. It means that when $t'_h < t'_v$, $(max(H; V); min(H; V))$ would dominate $(min(H; V); max(H; V))$ and conversely when $t'_h > t'_v$. This intuition is always verified for the classical unit-load AS/RS (see proof in appendix A). **However, this does not hold for the three systems that we are studying.** In the sequel, when looking at three different storage systems, we show through some dominance properties **when this is verified.**

Notice that, for (P1), if one or both of V and H are odd, one or both of the two candidate solutions will be eliminated (because of Equation (10)). The previously asked question can arise only when V and H are both even.

4.2.1. The bidirectional flow-rack AS/RS

As mentioned above, the problem (P1) is associated with the bidirectional flow-rack. After mathematically investigating the cycle time equation of this system, we introduce the following two properties.

<u>Property 1</u>

Let $(H; V)$ and $(V; H)$ be two candidate solutions, where H and V are two different even integers, and $t'_h < t'_v$.

If $min(H; V) \geq \frac{t'_h}{t'_v - t'_h} - 1$ then:

$$\left(max(H; V); min(H; V)\right) \quad dominates \quad (min(H; V); max(H; V))$$

<u>Property 2</u>

Let $(H; V)$ and $(V; H)$ be two candidate solutions, where H and V are two different even integers, and $t'_h > t'_v$.

If $min(H; V) \geq \frac{t'_v}{t'_h - t'_v} - 1$ then:

$$\left(min(H; V); max(H; V)\right) \quad dominates \quad \left(max(H; V); min(H; V)\right)$$

<u>Proof of properties 1 and 2</u>

Let $(H; V)$ and $(V; H)$ be two candidate solutions, where H and V are two different even integers, and $t'_h \neq t'_v$. We aim to compare between $ERC1$ (objective function for the solution $(H; V)$) and $ERC1'$ (objective function for the solution $(V; H)$), where:

$$ERC1' = \frac{1}{\rho N}\left(\frac{4}{\frac{1}{S_1'}+\frac{1}{S_2'}} + (\rho M - 1)(2(S_1' + S'_2) + \rho N.max(t'_h; t'_v))\right) \quad (23)$$

Where:

$$S_1' = \sum_{i=1}^{\frac{V}{2}}\sum_{j=1}^{H} max\big((2i-1)t'_h; jt'_v\big) \quad (24)$$

$$S_2' = \sum_{i=1}^{\frac{V}{2}}\sum_{j=1}^{H} max(2it'_h; jt'_v) \quad (25)$$

It is obvious that the case $H = V$ is not addressed. Furthermore, comparing between $(H; V)$ and $(V; H)$ for $V < H$ is equivalent to comparing between $(V; H)$ and $(H; V)$ for $H < V$. **Thus, only the case $V < H$ is addressed.**

Case 1: $V < H$:

To compare between $ERC1$ and $ERC1'$, we aim to compare S_1 to S'_1 and S_2 to S'_2.

Comparing S_1 to S'_1 :

$$S_1 - S'_1 = \sum_{i=1}^{\frac{H}{2}}\sum_{j=1}^{V} max\big((2i-1)t'_h; jt'_v\big) - \sum_{i=1}^{\frac{V}{2}}\sum_{j=1}^{H} max\big((2i-1)t'_h; jt'_v\big)$$

Let's split up the first sum of index i and the second sum of index j:

$$S_1 - S'_1 = \sum_{i=1}^{\frac{V}{2}}\sum_{j=1}^{V} max\big((2i-1)t'_h; jt'_v\big) + \sum_{i=\frac{V}{2}+1}^{\frac{H}{2}}\sum_{j=1}^{V} max\big((2i-1)t'_h; jt'_v\big)$$
$$- \sum_{i=1}^{\frac{V}{2}}\sum_{j=1}^{V} max\big((2i-1)t'_h; jt'_v\big) - \sum_{i=1}^{\frac{V}{2}}\sum_{j=V+1}^{H} max\big((2i-1)t'_h; jt'_v\big)$$

The two blue sums eliminate each other:

$$S_1 - S'_1 = \sum_{i=\frac{V}{2}+1}^{\frac{H}{2}}\sum_{j=1}^{V} max\big((2i-1)t'_h; jt'_v\big) - \sum_{i=1}^{\frac{V}{2}}\sum_{j=V+1}^{H} max\big((2i-1)t'_h; jt'_v\big)$$

Let's make an index change $i' = \frac{2i-V}{2}$ and $j' = j - V$, so:

$$S_1 - S'_1 = \sum_{i'=1}^{\frac{H-V}{2}}\sum_{j=1}^{V} max\big((V+2i'-1)t'_h; jt'_v\big) - \sum_{i=1}^{\frac{V}{2}}\sum_{j'=1}^{H-V} max\big((2i-1)t'_h; (V+j')t'_v\big)$$

Let's split up the two second sums (each term of the sum is split into two terms):

$$S_1 - S'_1 = \sum_{i'=1}^{\frac{H-V}{2}} \sum_{j=1}^{\frac{V}{2}} (max\big((V+2i'-1)t'_h; (2j-1)t'_v\big) + max\big((V+2i'-1)t'_h; 2jt'_v\big)) - \sum_{i=1}^{\frac{V}{2}} \sum_{j'=1}^{\frac{H-V}{2}} (max\big((2i-1)t'_h; (V+2j'-1)t'_v\big) + max\big((2i-1)t'_h; (V+2j')t'_v\big))$$

So by making another index change we can write all the terms in the same two sums:

$$S_1 - S'_1 = \sum_{i=1}^{\frac{H-V}{2}} \sum_{j=1}^{\frac{V}{2}} \Big(max\big((V+2i-1)t'_h; (2j-1)t'_v\big) + max\big((V+2i-1)t'_h; 2jt'_v\big) - max\big((2j-1)t'_h; (V+2i-1)t'_v\big) - max\big((2j-1)t'_h; (V+2i)t'_v\big)\Big) \quad (26)$$

Comparing S_2 to S'_2:

$$S_2 - S'_2 = \sum_{i=1}^{\frac{H}{2}} \sum_{j=1}^{V} max(2it'_h; jt'_v) - \sum_{i=1}^{\frac{V}{2}} \sum_{j=1}^{H} max(2it'_h; jt'_v)$$

Let's split up the first sum of index i and the second sum of index j:

$$S_2 - S'_2 = \sum_{i=1}^{\frac{V}{2}} \sum_{j=1}^{V} max(2it'_h; jt'_v) + \sum_{i=\frac{V}{2}+1}^{\frac{H}{2}} \sum_{j=1}^{V} max(2it'_h; jt'_v) - \sum_{i=1}^{\frac{V}{2}} \sum_{j=1}^{V} max(2it'_h; jt'_v) - \sum_{i=1}^{\frac{V}{2}} \sum_{j=V+1}^{H} max(2it'_h; jt'_v)$$

The two blue sums eliminate each other:

$$S_2 - S'_2 = \sum_{i=\frac{V}{2}+1}^{\frac{H}{2}} \sum_{j=1}^{V} max(2it'_h; jt'_v) - \sum_{i=1}^{\frac{V}{2}} \sum_{j=V+1}^{H} max(2it'_h; jt'_v)$$

Let's make an index change $i' = \frac{2i-V}{2}$ and $j' = j - V$, so:

$$S_2 - S'_2 = \sum_{i'=1}^{\frac{H-V}{2}} \sum_{j=1}^{V} max\big((V+2i')t'_h; jt'_v\big) - \sum_{i=1}^{\frac{V}{2}} \sum_{j'=1}^{H-V} max(2it'_h; (V+j')t'_v)$$

Let's split up the two second sums (each term of the sum is split into two terms):

$$S_2 - S'_2 = \sum_{i'=1}^{\frac{H-V}{2}} \sum_{j=1}^{\frac{V}{2}} (max((V+2i')t'_h; (2j-1)t'_v) + max((V+2i')t'_h; 2jt'_v)) - \sum_{i=1}^{\frac{V}{2}} \sum_{j'=1}^{\frac{H-V}{2}} (max(2it'_h; (V+2j'-1)t'_v) + max(2it'_h; (V+2j')t'_v))$$

So by making another index change we can write all the terms in the same two sums:

$$S_2 - S'_2 = \sum_{i=1}^{\frac{H-V}{2}} \sum_{j=1}^{\frac{V}{2}} \Big(max((V+2i)t'_h; (2j-1)t'_v) + max((V+2i)t'_h; 2jt'_v) - max(2jt'_h; (V+2i-1)t'_v) - max(2jt'_h; (V+2i)t'_v)\Big) \quad (27)$$

At this level, in order to know the sign of $S_1 - S'_1$ and $S_2 - S'_2$, we are going to compare between the different terms of these sums. In other words:

For $(S_1 - S'_1)$, we compare between $\boldsymbol{max((2j-1)t'_h; (V+2i-1)t'_v)}$ and $\boldsymbol{max((V+2i-1)t'_h; (2j-1)t'_v)}$ as well as between $\boldsymbol{max((2j-1)t'_h; (V+2i)t'_v)}$ and $\boldsymbol{max((V+2i-1)t'_h; 2jt'_v)}$.

For $(S_2 - S'_2)$, we compare between $\boldsymbol{max(2jt'_h; (V+2i)t'_v)}$ and $\boldsymbol{max((V+2i)t'_h; 2jt'_v)}$ as well as between $\boldsymbol{max(2jt'_h; (V+2i-1)t'_v)}$ and $\boldsymbol{max((V+2i)t'_h; (2j-1)t'_v)}$.

Case 1.1 $\boldsymbol{t'_h < t'_v}$ $\boldsymbol{(property\ 1)}$:

We have:

$$\forall\, i = 1..\frac{H-V}{2}: \qquad V + 2i - 1 \geq V + 1$$

$$\forall\, j = 1..\frac{V}{2}: \qquad 2j \leq V$$

By transitivity:

$$\forall\, i = 1..\frac{H-V}{2} \quad and \quad \forall\, j = 1..\frac{V}{2}: \qquad V + 2i - 1 > 2j$$

The inference is:

$$\forall\, i = 1..\frac{H-V}{2} \quad and \quad \forall\, j = 1..\frac{V}{2}: \qquad \boldsymbol{V+2i} > \boldsymbol{V+2i-1} > \boldsymbol{2j} > \boldsymbol{2j-1}$$

So, we can conclude the following (note that in **case 1.1** we have $\boldsymbol{t'_h < t'_v}$):

$$\forall\, i = 1..\frac{H-V}{2}\ and\ \forall\, j = 1..\frac{V}{2}:$$

$$max((\boldsymbol{2j-1})t'_h; (\boldsymbol{V+2i-1})t'_v) > max((\boldsymbol{V+2i-1})t'_h; (\boldsymbol{2j-1})t'_v) \quad (28)$$

And

$$max((\boldsymbol{2j-1})t'_h; (\boldsymbol{V+2i})t'_v) > max((\boldsymbol{V+2i-1})t'_h; \boldsymbol{2j}t'_v) \quad (29)$$

Consequently, thanks to Equations (28) and (29), and by looking at Equation (26), we deduce:

$$S_1 - S'_1 < 0 \tag{30}$$

On the other hand, we have:

$\forall\, i = 1..\dfrac{H-V}{2}: \qquad V + 2i - 1 \geq V + 1$

$\forall\, j = 1..\dfrac{V}{2}: \qquad 2j \leq V$

By transitivity:

$\forall\, i = 1..\dfrac{H-V}{2}\ \ and\ \ \forall\, j = 1..\dfrac{V}{2}: \qquad V + 2i - 1 > 2j$

The inference is:

$\forall\, i = 1..\dfrac{H-V}{2}\ and\ \forall\, j = 1..\dfrac{V}{2}: \qquad \boldsymbol{V+2i} > \boldsymbol{V+2i-1} > \boldsymbol{2j} > \boldsymbol{2j-1}$

So, we can conclude the following (note that in **case 1.1** we have $\boldsymbol{t'_h < t'_v}$)

$\forall\, i = 1..\frac{H-V}{2}\ and\ \forall\, j = 1..\frac{V}{2}$:

$$max(\boldsymbol{2j}t'_h; (\boldsymbol{V+2i})t'_v) > max\big((\boldsymbol{V+2i})t'_h; \boldsymbol{2j}t'_v\big) \tag{31}$$

And

$$max(\boldsymbol{2j}t'_h; (\boldsymbol{V+2i-1})t'_v) \geq max\big((\boldsymbol{V+2i})t'_h; (\boldsymbol{2j-1})t'_v\big) \tag{32}$$

Note that the Equation (32) is true thanks to the condition $min(H;V) \geq \frac{t'_h}{t'_v - t'_h} - 1$ (property 1) which leads to $(V+1)t'_v \geq (V+2)t'_h$ (because, as mentioned above, in **case 1** we have $\boldsymbol{min(H;V) = V}$)

Consequently, thanks to Equations (31) and (32), and by looking at Equation (27), we deduce:

$$S_2 - S'_2 < 0 \tag{33}$$

Conclusion 1

Thanks to Equations (30) and (33), we have $ERC1 < ERC1'$. Which means that:

$(H;V)\ dominates\ (V;H)$, in this case, it is the equivalent of saying

$\boldsymbol{(max(H;V); min(H;V))\ dominates\ (min(H;V); max(H;V))}$ Q.E.D.

Case 1.2 $\boldsymbol{t'_h > t'_v\ (property\ 2)}$:

This case can be studied as in case 1.1 and leads to conclusion 2.

Conclusion 2

Following the same process, we would have $ERC1 > ERC1'$. Which means that:

$(V;H)\ dominates\ (H;V)$, in this case, it is the equivalent of saying

$\boldsymbol{(min(H;V); max(H;V))\ dominates\ (max(H;V); min(H;V))}$ Q.E.D.

By symmetry, **case 2** ($\boldsymbol{V > H}$) leads to the same conclusions.

4.2.2. The multi-aisle AS/RS:

The problem (P2) is associated with the multi-aisle AS/RS. After mathematically investigating the cycle time equation of this system, we draw the following property.

Property 3

Let $(H;V)$ and $(V;H)$ be two candidate solutions, where H and V are two different integers.

If $t'_h \geq t'_v$ then:

$$(min(H;V);max(H;V))\ dominates\ (max(H;V);min(H;V))$$

If $(min(H;V)+1)t'_v - max(H;V)t'_h - Mt'_p > 0$ then:

$$(max(H;V);min(H;V))\ dominates\ (min(H;V);max(H;V))$$

Proof of property 3

Property 3 is demonstrated using the same reasoning as in proof of property 1.

4.2.3. The mobile-rack AS/RS:

The problem (P3) is associated with the mobile-rack AS/RS. After mathematically investigating the cycle time formulas of this system, we introduce the following property.

Property 4

Let $(H;V)$ and $(V;H)$ be two candidate solutions, where H and V are two different integers.

If $t'_h \geq t'_v$ then:

$$(min(H;V);max(H;V))\ dominates\ (max(H;V);min(H;V))$$

If $(min(V;H)+1)t'_v - max(V;H)t'_h - max(t'_r\,;Mt'_p) > 0$ then:

$$(max(H;V);min(H;V))\ dominates\ (min(H;V);max(H;V))$$

Proof of property 4

Property 4 is demonstrated using the same reasoning as in proof of property 1.

4.3. Problem resolution

4.3.1. Complexity study

As mentioned in the literature review section, different resolution methods were used to deal with the AS/RS design optimization problem. We have highlighted different methods, either approximate or exact ones. However, regardless of the computational issues, no complexity study was conducted to justify the use of either methods. In this section, we proceed to a brief complexity study of the addressed optimization problems.

Theorem 1

The problems (P1), (P2) and (P3) are polynomial.

Proof of theorem 1

First, we focus on the problem (P1). As mentioned in the problem formulation section, the goal is to find the triplet $(H;\ V;M)$ that minimizes the objective function $ERC1$. On the one hand, each solution can be evaluated in a polynomial time, which is at the most equal to $O(\frac{N}{2})$ (see the sums S_1 and S_2). On the other hand finding all the possible cases requires $O(N^3)$, which means that the optimal solution can be obtained at the most in

$O(\frac{N^4}{2})$ using enumeration. Consequently, the problem is polynomial. The complexity of the problems (P2) and (P3) is obtained following the same reasoning. Q.E.D.

Remark

Even if the addressed problem is polynomial in $O(N^4)$, when N is big (for large systems), the computing time becomes constraining. This is why, as seen in the literature review, heuristics and metaheuristics were used to solve this problem without even studying its complexity beforehand. For this reason, in the sequel, we introduce an exact resolution algorithm that is based on the previously identified mathematical properties to solve the problem in a much efficient manner even if N is large.

4.3.2. Resolution algorithm

In this section, we present the exact resolution method (for the problem (P1)) that uses the previously presented dominance properties to solve the addressed problem efficiently. In addition to the dominance properties that help us to prune the search space, we can notice that this latter can be further reduced since it is not necessary to go through the entire set $\{1,2,3 \dots N\}$ for each one of the three dimensions thanks to Equation (8) (for the problem (P1)). According to this Equation (i.e. $H \cdot V \cdot M = N$), the value of each one of the dimensions is a divisor of N. It is therefore easy to notice that one needs to only investigate the set $\{1,2,3 \dots \sqrt{N}\}$, and this for the following reason:

Let d, d' and D be positive integers, with d and d' two divisors of D, so that: $dd' = D$.

If $d \leq \sqrt{D}$ then $\frac{D}{d'} \leq \sqrt{D}$ and therefore $d' \geq \sqrt{D}$ (The reverse is also true)

Therefore, the division quotient of each divisor less than or equal to $\sqrt{D}$ is greater than or equal to $\sqrt{D}$ and vice versa. Consequently finding all the divisors of D is possible by finding only those that are less than or equal to $\sqrt{D}$.

As a result, we set up the following resolution method (Algorithms 1 and 2), whose algorithmic complexity is $O(N^2)$. Therefore, the design optimization problem of the AS/RS ((P1), (P2) and (P3)) is polynomial at most in $O(N^2)$. Consequently, the problem complexity is reduced from $O(N^4)$ to $O(N^2)$. We can see in Algorithm 1 that the For-loop goes at the most to $\sqrt{N}$. This loop comprises two Evaluate() functions (presented in Algorithm 2). We can see in Algorithm 2, that Evaluate() function has a For-loop which goes at the most to $\sqrt{\frac{N}{2}}$. This latter calls at the most two times the Update_Best() function (This function calculates the objective function value and updates the best current one, and as seen before, the evaluation of each solution requires $O(\frac{N}{2})$), which necessitates $O(N)$. The overall algorithm computation needs at most $O\left(2\sqrt{N}.\sqrt{\frac{N}{2}}.N\right) = O(\sqrt{2}N^2) \approx O(N^2)$. Moreover, even if the complexity is at most in $O(N^2)$, the algorithm is more efficient than that. Since the identified dominance properties (when the conditions are verified) further reduce the search space by half.

Algorithm 1: *Proposed exact method*

$\textbf{Read}(t'_h, t'_v, N, \rho)$
$\textbf{Best}_{Solution} \leftarrow \{0; 0; 0; +\infty\}$
For i from 2 to $\sqrt{N}$ **:**
  If $mod(N; i) = 0$ **:**
    If $mod(i; 2) = 0$ **and** $i \neq 2$ **:**
      $\textbf{Best}_{Solution} \leftarrow Evaluate\left(i; \frac{N}{i}\right)$
    End If
    If $mod(\frac{N}{i}; 2) = 0$ **and** $\frac{N}{i} \neq 2$ **and** $\frac{N}{i} \neq i$**:**

```
        Best_Solution ← Evaluate(N/i; i)
      End If
    End If
End For

Return Best_Solution
```

Algorithm 2: *Algorithm of Evaluate(a, b) function*

```
Read(a, b)
For j from 2 to √a :
  If mod(a; j) = 0 then:
    If mod(a/j; 2) = 0 and mod(j; 2) = 0 then:
      If t'_h = t'_v then:
        S ← Update_Best(a/j; j; b)
        S ← Update_Best(j; a/j; b)
      Else:
        If t'_h < t'_v then:
          If j ≥ t'_h/(t'_v − t'_h) − 1 then:
            S ← Update_Best(a/j; j; b)
          Else:
            S ← Update_Best(a/j; j; b)
            S ← Update_Best(j; a/j; b)
          End If
        Else:
          If j ≥ t'_v/(t'_h − t'_v) − 1 then:
            S ← Update_Best(j; a/j; b)
          Else :
            S ← Update_Best(a/j; j; b)
            S ← Update_Best(j; a/j; b)
          End If
        End If
      End If
    Else:
      If mod(a/j; 2) = 0 then:
        S ← Update_Best(a/j; j; b)
      Else:
        If mod(j; 2) = 0 then:
          S ← Update_Best(j; a/j; b)
        End If
      Else If
    End If
  End If
End For
```

36: Return S

4.3.3. Numerical comparative study

Through this section, in the first place we compare our resolution method to the full enumeration method for the reasons mentioned in the literature review section. This is done in order to evaluate the time computing gain while looking for exact solutions. Moreover, in order to validate our algorithm, we have evaluated it through two testing phases. The first one was by running the algorithm on small instances, which can be trivially calculated (by hand) while varying the parameters, whereas the second one was conducted for large instances where the results (the optimal solutions) were compared to the ones found via full enumeration. Both conducted optimization methods (proposed algorithm and full enumeration) were implemented under C++ programs. A C++ instruction (*clock()*) was used to display the average computation time at the end of the execution.

The numerical study was performed for 7 system sizes while varying the storage volume with a tolerance percentage. Regarding the parameters, we used five combinations for machines velocities according to (Hamzaoui and Sari, 2015). This covers a wide and coherent spectrum of the relationship between horizontal and vertical velocities. Moreover, even if we varied the load rate from 0.6 to 0.9, we present the results for a load rate of 0.8 (*zone of actual operation* as defined in (Sari, et al., 2005)). The results of the design optimization and the two methods comparison are displayed in Tables 1-3.

Table 1
Optimization results and performances comparison of the two methods (Bidirectional flow-rack)

Size	Variation (%)	$(t'_h;\ t'_v)$															Computation time		
		(1,378;0,316)			(0,316;1,378)			(1;1)			(1,274;0,615)			(0,615;1,274)			Proposed method (s)	Enumeration method (s)	Increase %
		H	*V*	*M*	*H*	*V*	*M*	*H*	*V*	*M*	*H*	*V*	*M*	*H*	*V*	*M*			
10000	±5	14	68	10	68	14	10	24	22	18	18	38	14	38	18	14	5.63	61.82	+998.04%
30000	±5	20	95	15	90	20	16	32	33	27	26	55	20	56	27	19	56.81	618.09	+987.99%
60000	±3	26	112	20	112	26	20	42	42	33	32	70	26	70	32	26	148.15	1577.14	+ 964.56%
100000	±2	30	131	25	132	31	24	50	49	40	40	82	30	84	39	30	271.54	3058.96	+1026.52%
200000	±2	38	172	30	172	38	30	62	62	51	50	106	37	100	49	40	1234.10	12980.52	+ 951.82%
400000	±1	48	212	39	212	48	39	78	77	66	62	128	50	130	61	50	3141.01	36306.27	+1055.87%
800000	±0.75	62	267	48	266	61	49	98	99	82	78	167	61	164	77	63	7935.04	-	-

Table 2
Optimization results and performances comparison of the two methods (Multi-aisles)

Size	Variation (%)	$(t'_h;\ t'_v)$															Computation time		
		(1,378;0,316)			(0,316;1,378)			(1;1)			(1,274;0,615)			(0,615;1,274)			Proposed method (s)	Enumeration method (s)	Increase %
		H	*V*	*M*	*H*	*V*	*M*	*H*	*V*	*M*	*H*	*V*	*M*	*H*	*V*	*M*			
10000	±5	9	59	18	24	8	50	15	23	28	11	36	24	18	14	38	3.15	37.02	+1075.24%
30000	±5	13	85	26	35	12	68	22	31	42	16	53	34	28	19	54	34.89	405.02	+1060.84%
60000	±3	16	107	34	44	15	90	27	40	54	21	66	42	33	26	68	101.71	1120.65	+1001.81%
100000	±2	20	129	38	52	18	106	32	48	64	25	79	50	41	30	80	197.13	2235.09	+1033.81%
200000	±2	24	164	50	66	23	130	41	60	80	31	99	64	52	37	102	908.95	10070.01	+1007.87%
400000	±1	30	213	62	83	29	166	51	78	100	39	127	80	65	47	130	2081.28	22539.95	+982.98%
800000	±.75	40	255	78	106	36	210	64	97	128	50	159	100	84	57	166	7115.25	-	-

Table 3
Optimization results and performances comparison of the two methods (Mobile rack)

Size	Variation (%)	$(t'_h;\ t'_v)$															Computation time		
		(1,378;0,316)			(0,316;1,378)			(1;1)			(1,274;0,615)			(0,615;1,274)			Proposed method (s)	Enumeration method (s)	Increase %
		H	***V***	***M***	***H***	***V***	***M***	***H***	***V***	***M***	***H***	***V***	***M***	***H***	***V***	***M***			
10000	±5	9	59	18	24	8	50	15	23	28	11	36	24	18	14	38	6.19	71.23	+ 1050.73%
30000	±5	13	85	26	34	12	70	22	31	42	16	53	34	27	19	56	71.01	819.17	+ 1053.60%
60000	±3	17	107	32	44	15	90	27	40	54	21	66	42	33	26	68	195.02	2210.85	+ 1033.65%
100000	±2	19	129	40	53	18	104	32	48	64	25	79	50	40	30	82	392.02	4465.88	+ 1039.19%
200000	±2	24	164	50	65	23	132	41	60	80	31	99	64	51	37	104	2310.31	24775.13	+ 972.37%
400000	±1	31	213	60	83	29	166	51	78	100	40	127	78	65	47	130	5259.15	55632.37	+ 957.82%
800000	±.75	40	255	78	105	36	212	64	97	128	50	159	100	84	57	166	13769.07	-	-

In Tables 1-3, the optimal dimensions $(H; V; M)$ for each system size are displayed according to horizontal and vertical machines velocities. First and second columns show the different system sizes and the tolerated variation in sizes. Columns 3-7 show the three optimal dimensions according to the machines velocities. Finally, the last three columns are dedicated to the computation time. Columns 8 and 9 display the mean computation time (for each system, the two algorithms were run several times) for the proposed method and the full enumeration respectively, while the last column shows the gaps between the two methods computation times. It can be clearly seen that the proposed method's performance significantly exceeds that of the enumeration one, since the computing time increases at least by 950% when using enumeration method instead of the proposed algorithm. These performances can be clearly noticed through Fig. 3-5.

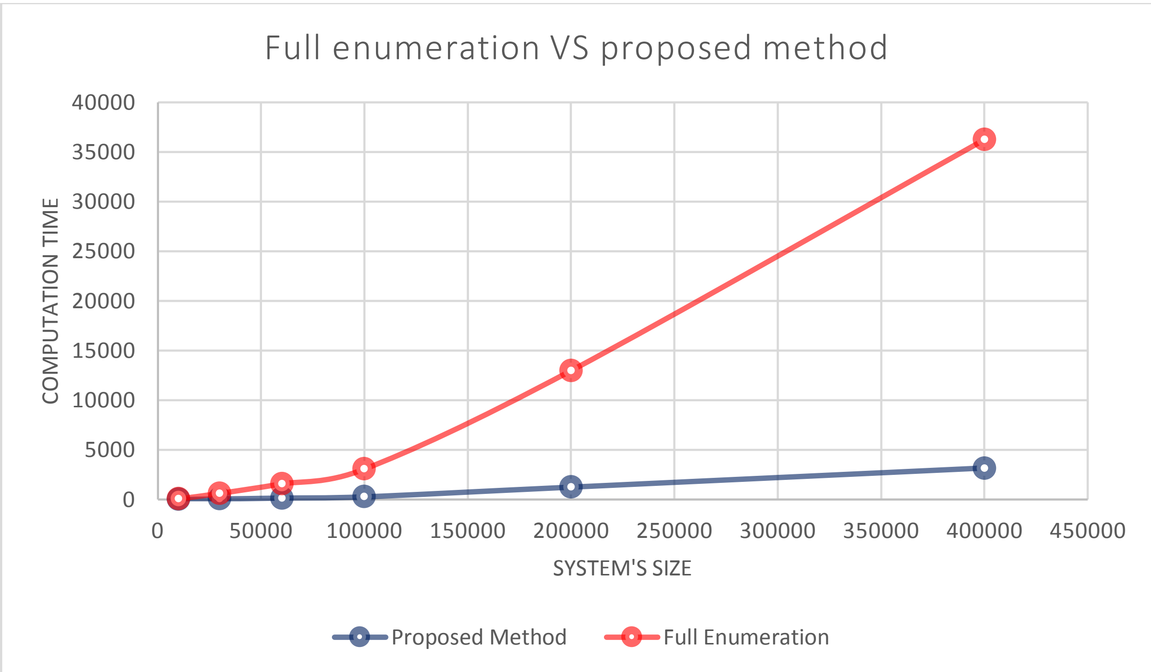

Fig. 3. Computation time comparison (bidirectional flow-rack)

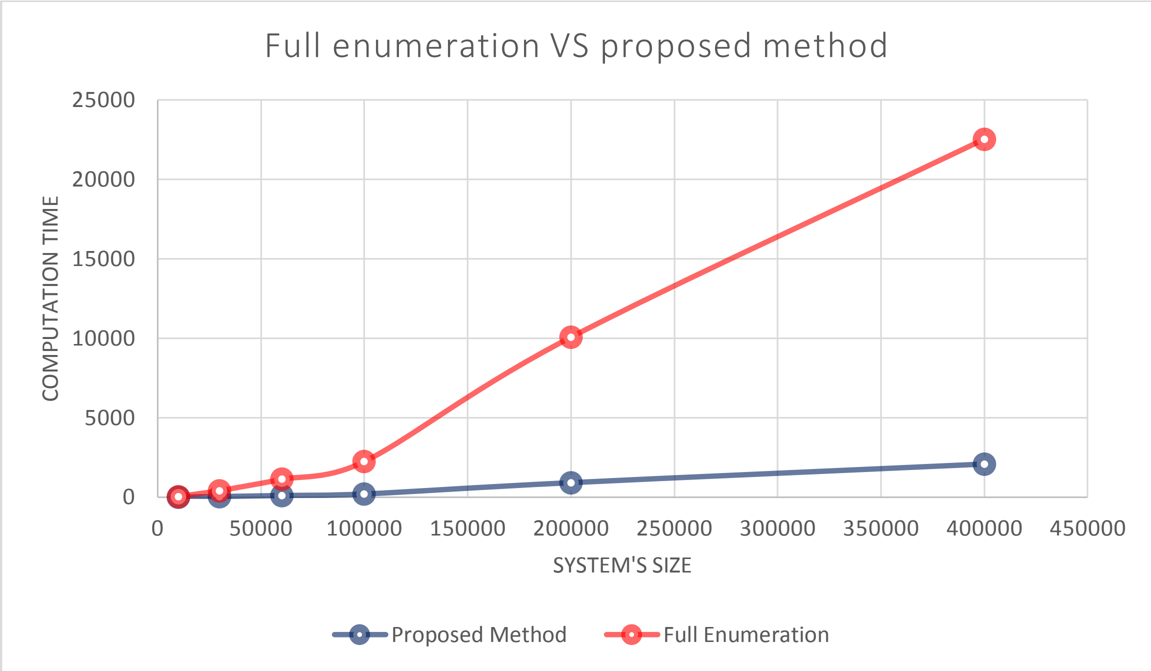

Fig. 4. Computation time comparison (multi-aisles)

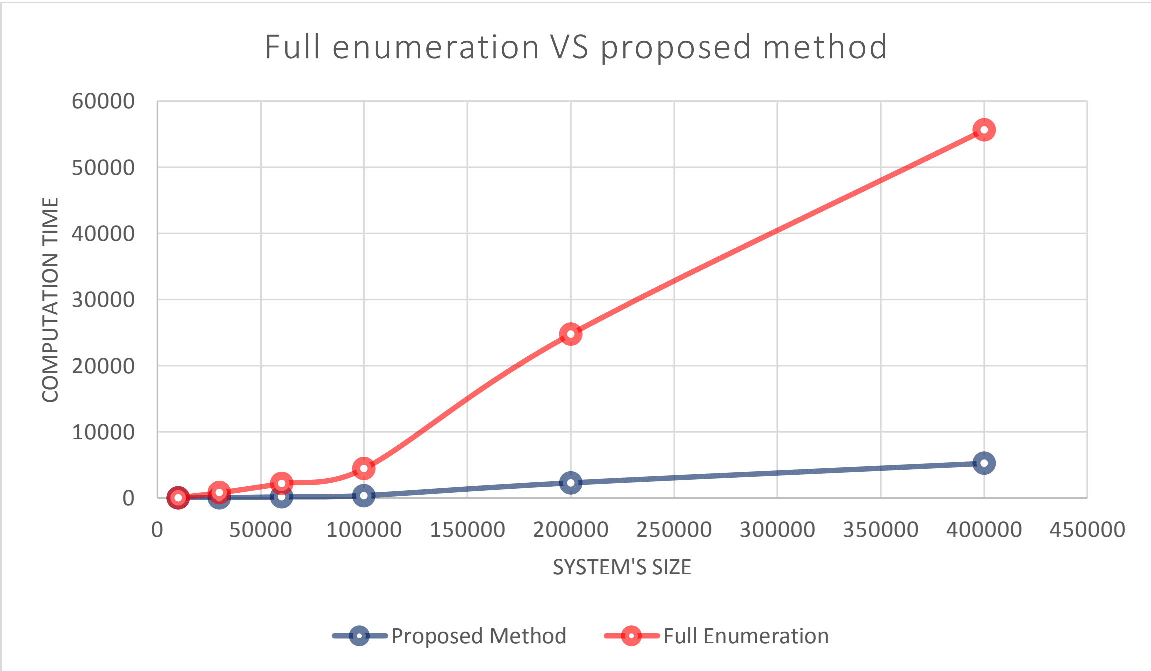

Fig. 5. Computation time comparison (mobile-rack)

In the second place, we compare our exact resolution method to a heuristic one. This latter is based on the resolution of the relaxed optimization problem using an average cycle time formula based on a continuous calculation approach (continuous rack face assumption) as mentioned in the literature review section. The comparison is drawn to evaluate the precision loss when using this kind of heuristic methods. As shown before, these methods are widely used for their short computational time but without being aware of the problem's complexity and the loss of precision. We showed that with a complexity study and mathematical properties, an exact and time-efficient method prevents us from having these losses of precision.

For the bidirectional flow-rack system, an average cycle time model was presented in (Hamzaoui and Sari, 2019), where $T = max(Ht'_h; Vt'_v)$ and $b = \frac{min(Ht'_h; Vt'_v)}{max(Ht'_h; Vt'_v)}$:

$$ERC4 = \frac{T}{\rho M}\left(\frac{b^2}{3}+1\right)(\frac{2\rho M-1}{2}) + (\rho M-1)t'_p \tag{34}$$

Concerning the multi-aisles AS/RS, continuous cycle time models were presented in (Kouloughli and Sari, 2015), while in (Guezzen, et al., 2013) we can find the ones related to the mobile-rack AS/RS.

This approximation method was conducted using Mathematica 11.0. The relaxed optimization problem is solved using the Lagrange multipliers method, and then the integer solutions are obtained by rounding to the nearest integers that respect the problem constraints. This method does not offer any optimality guaranties. However, it gives satisfactory results very quickly. The obtained results are displayed in Tables 4-6 for $\rho = 0.8$ and $(t'_h;\ t'_v) = (1.378; 0.316)$.

The first column represents the studied system sizes, which vary from small to medium and large systems. Columns t_h and t_v represent values obtained using the analytical optimization, with $t_h = Ht'_h$ and $t_v = Vt'_v$. It can be noticed that for the bidirectional flow-rack and the mobile-rack AS/RSs, in all the cases $t_h = t_v$which means that the system is square in time. However, after applying rounding on H, V and M (columns 4-6) the shape factor is reduced but stays very close to 1 in most cases (column 7). In columns 8-11, results of the exact resolution method are presented, we have the three dimensions H, V and M in addition to the related shape factor. This method gave systems that tend to be square in time ($b \approx 1$ in column 11) just as the results obtained analytically. Column 12 displays the cycle time value when using configurations obtained from the heuristic method, while column 13 shows the cycle time value for the optimal configurations obtained thanks to the proposed exact resolution method. The percentage gap between the two values is presented in the last column.

We can clearly notice that even if this approximated method is very fast and provides in some cases optimal solutions or near-optimal ones, it does not provide any guaranty despite their optimality since there is no precise way to assess a priori what the gap will be. In our example, it can reach 8%, 9% or even more for the mobile-rack. Our proposed method provides the exact optimal solution while remaining very fast.

Table 4
Numerical results and comparison for the bidirectional flow-rack

ρ=0.8 / $(t'_h;\ t'_v) = (1.378; 0.316)$													
N	t_h	t_v	H	V	M	**b**	H	V	M	**b**	**ERC(Heu)**	**ERC(Ext)**	**Gap(%)**
1000	9.052	9.052	8	25	5	0.717	10	50	2	0.872	11.3690	10.9572	3.7582
2000	10.88	10.88	10	40	5	0.917	10	40	5	0.917	15.3321	15.3321	0
3000	12.24	12.24	10	30	10	0.688	10	50	6	0.872	20.1641	18.4403	9.3480
4000	13.34	13.34	10	40	10	0.917	10	50	8	0.872	21.6455	21.0292	2.9306
5000	14.28	14.28	10	50	10	0.872	10	50	10	0.872	23.4646	23.4646	0
6000	15.1	15.1	10	50	12	0.872	12	50	10	0.955	25.8230	25.2391	2.3134
7000	15.84	15.84	10	50	14	0.872	14	50	10	0.819	28.1376	27.1718	3.5544
8000	16.51	16.51	10	50	16	0.872	16	50	10	0.717	30.4247	29.2535	4.0036
9000	17.13	17.13	12	50	15	0.955	12	50	15	0.955	31.0980	31.0980	0
10000	17.7	17.7	10	50	20	0.872	20	50	10	0.573	34.9496	33.6874	3.7468
20000	22.05	22.05	20	50	20	0.573	20	100	10	0.872	45.5137	41.872	8.6972
30000	25.11	25.11	20	75	20	0.86	20	100	15	0.872	49.0874	48.1003	2.0521
40000	27.55	27.55	20	80	25	0.917	20	100	20	0.872	55.6673	53.9404	3.2014
50000	29.61	29.61	20	100	25	0.872	20	100	25	0.872	59.6973	59.6973	0
60000	31.41	31.41	24	100	25	0.955	30	125	16	0.955	63.2650	63.1385	0.2003
70000	33.03	33.03	20	100	35	0.872	28	125	20	0.977	70.9669	66.0981	7.3660

Table 5
Numerical results and comparison for the multi-aisles

ρ=0.8 / $(t'_h; t'_v) = (1.378; 0.316)$													
N	t_h	t_v	H	V	M	b	H	V	M	b	ERC(Heu)	ERC(Ext)	Gap(%)
1000	7.38	11.01	5	35	6	0.623	4	30	8	0.581	15.9307	15.0639	5.7542
2000	9.30	13.87	7	46	6	0.624	5	38	10	0.574	19.9859	18.4242	8.4764
3000	10.65	15.88	8	49	8	0.712	6	40	12	0.654	22.4089	20.8925	7.2581
4000	11.72	17.48	9	55	8	0.714	7	46	12	0.664	24.5195	22.8512	7.3007
5000	12.62	18.82	8	60	10	0.581	7	49	14	0.623	25.4643	24.3716	4.4835
6000	13.42	20.00	10	62	10	0.703	8	51	14	0.684	27.7152	25.7554	7.6093
7000	14.12	21.06	10	67	10	0.651	8	52	16	0.671	28.7132	26.9969	6.3574
8000	14.77	22.02	11	70	10	0.685	8	60	16	0.581	30.2432	28.2231	7.1576
9000	15.36	22.90	11	71	12	0.676	9	60	16	0.654	31.2995	29.2595	6.9721
10000	15.91	23.72	11	75	12	0.640	9	59	18	0.665	32.1081	30.1873	6.3629
20000	20.04	29.88	15	95	14	0.689	11	79	22	0.607	40.7047	37.5197	8.4889
30000	22.94	34.21	17	109	16	0.680	13	85	26	0.667	46.2491	42.6821	8.3571
40000	25.25	37.65	19	119	18	0.696	14	97	28	0.629	50.9588	46.6565	9.2212
50000	27.20	40.56	20	128	20	0.681	16	99	30	0.705	54.5563	50.1679	8.7481
60000	28.90	43.10	21	136	20	0.673	16	105	34	0.664	57.1362	53.2179	7.3627
70000	30.43	45.37	22	144	22	0.666	17	116	34	0.639	60.5269	55.9546	8.1714

Table 6
Numerical results and comparison for the mobile-rack

ρ=0.8 / $(t'_h; t'_v) = (1.378; 0.316)$													
N	t_h	t_v	H	V	M	b	H	V	M	b	ERC(Heu)	ERC(Ext)	Gap(%)
1000	7.390	7.390	5	24	8	0.908	2	20	24	0.436	11.4203	9.4224	21.2071
2000	9.311	9.311	7	29	10	0.950	3	23	28	0.569	14.7619	11.4052	29.4313
3000	10.658	10.658	7	34	12	0.898	4	24	30	0.727	16.1299	12.9068	24.9721
4000	11.731	11.731	9	37	12	0.943	4	28	34	0.623	18.4320	13.9650	31.9871
5000	12.637	12.637	9	40	14	0.981	4	30	40	0.581	19.2065	15.0020	28.0263
6000	13.429	13.429	10	42	14	0.963	5	30	38	0.727	20.4543	15.8557	29.0028
7000	14.137	14.137	10	45	16	0.969	5	35	38	0.623	20.8345	16.6123	25.4161
8000	14.780	14.780	11	47	16	0.980	5	35	44	0.623	22.8176	17.3188	31.7505
9000	15.372	15.372	11	49	16	0.979	5	36	48	0.606	23.5321	17.9604	31.0221
10000	15.922	15.922	12	50	18	0.955	5	38	50	0.574	21.9183	18.5148	18.3826
20000	20.060	20.060	15	63	22	0.963	7	47	58	0.649	30.1263	22.9973	30.9993
30000	22.963	22.963	17	73	24	0.985	8	54	66	0.646	33.3458	26.1171	27.6780
40000	25.274	25.274	18	80	28	0.981	9	59	72	0.665	37.1884	28.6774	29.6784
50000	27.226	27.226	20	86	30	0.986	9	63	84	0.623	39.3139	30.7503	27.8488
60000	28.932	28.932	21	92	32	0.995	10	68	84	0.641	41.0543	32.5961	25.9485
70000	30.457	30.457	22	96	32	0.99	10	71	94	0.614	44.5112	34.2732	29.8717

5. Discussion and conclusion

In this paper, we were interested in the AS/RS design optimization problem for the minimization of the cycle time. In general, it is known that the strategic phase of the AS/RS design optimization is crucial for improving the performance of the system in steady state. This work is the first to introduce a complexity study for the optimization of the AS/RS design. We demonstrated that the studied problems are polynomial. Moreover, based on introduced dominance properties, we developed an exact and fast resolution method that solves the problem in $O(N^2)$ and highlighted the performance degradation when using existing approximate optimization methods. We believe that this work will set up a new agenda in the optimization of the design of AS/RS as it opens the path for better exact approaches for other types of AS/RSs.

On the practical and industrial level, we believe that our optimization method will be an important competitive approach for companies that offer services related to the design and the implementation of AS/RS. Indeed, it allows them to evaluate different technological solutions in a quick manner on different types of systems and different machines' speeds, as well as to evaluate the robustness of the solutions when facing the load rate variation. Furthermore, in addition to being computationally efficient, our method is precise (provides exact optimal solutions) avoiding thus a performance degradation that occurs when an approximate method is used. Consequently, preventing these performances degradation can be a key tool for improving the competitiveness index, avoiding the system overuse, and delaying the equipment degradation. Nevertheless, even if our study offers a disruptive approach to address the AS/RS design optimization problem, it is far from being complete and opens the way to a wide range of improvements and research perspectives. In addition to the three studied systems, future works may be interested in new systems such as AVS/RS and SBS/RS. Dominance properties can be extracted from other sophisticated models (such as those taking into account acceleration/deceleration and maximum speed). Besides, a strong interest can be given to other objective functions (carbon footprint and energy consumption as well as initial and operating costs) or even in multi-objective optimization.

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Appendix A. Dominance properties demonstration for the unit-load AS/RS

Bozer and White (1984) presented the discrete average cycle time for the classical unit-load AS/RS:

$$ERC = \frac{1}{HV}\sum_{i=1}^{H}\sum_{j=1}^{V} max(it'_h; jt'_v) \tag{A.1}$$

We set the following optimization problem (P4):

$$Minimize ERC \tag{A.2}$$

$$Subject\ to: \begin{cases} H \cdot V = N \quad \forall\, (H;V) \in \mathbb{N}^{*2} & \text{(A.3)} \\ H > 1, V > 1 & \text{(A.4)} \\ (H;V) \in \mathbb{N}^{*2} & \text{(A.5)} \end{cases}$$

Theorem 5

Let $(H;V)$ and $(V;H)$ be two candidate solutions, where $H\ and\ V$ are two different integers.

If $t'_h > t'_v$ then:

$$(min(H;V); max(H;V))\ \ dominates\ \ (max(H;V); min(H;V))$$

If $t'_h < t'_v$ then:

$$(max(H;V); min(H;V))\ \ dominates\ \ (min(H;V); max(H;V))$$

Proof of theorem 5

Let $(H;V)$ and $(V;H)$ be two candidate solutions, where $H\ and\ V$ are two different integers.

We aim to compare between ERC (Objective function for the solution $(H;V)$) and ERC' (Objective function for the solution $(V;H)$), where:

$$ERC' = \frac{1}{HV}\sum_{i=1}^{V}\sum_{j=1}^{H} max(it'_h; jt'_v) \tag{A.6}$$

It is obvious that the case $H = V$ is not addressed. Furthermore comparing between $(H;V)$ and $(V;H)$ for $V < H$ is equivalent to compare between $(V;H)$ *and* $(H;V)$ for $H < V$, **then only the case $V < H$ is addressed.**

<u>**Case 1: $V < H$:**</u>

We set:

$$ERC - ERC' = \frac{1}{HV}\sum_{i=1}^{H}\sum_{j=1}^{V} max(it'_h; jt'_v) - \frac{1}{HV}\sum_{i=1}^{V}\sum_{j=1}^{H} max(it'_h; jt'_v)$$

Let's split up the first sum of index i and the second sum of index j:

$$ERC - ERC' = \frac{1}{HV}\left(\sum_{i=1}^{V}\sum_{j=1}^{V} max(it'_h; jt'_v) + \sum_{i=V+1}^{H}\sum_{j=1}^{V} max(it'_h; jt'_v) - \sum_{i=1}^{V}\sum_{j=1}^{V} max(it'_h; jt'_v) - \sum_{i=1}^{V}\sum_{j=V+1}^{H} max\big((2i-1)t'_h; jt'_v\big)\right)$$

The two blue sums eliminate each other:

$$ERC - ERC' = \frac{1}{HV}\left(\sum_{i=V+1}^{H}\sum_{j=1}^{V} max(it'_h; jt'_v) - \sum_{i=1}^{V}\sum_{j=V+1}^{H} max(it'_h; jt'_v)\right)$$

By inverting the indices in the second sum:

$$ERC - ERC' = \frac{1}{HV}\sum_{i=V+1}^{H}\sum_{j=1}^{V}\left(max(it'_h; jt'_v) - max(jt'_h; it'_v)\right)$$

Therefore we have:

$$ERC - ERC' = \frac{1}{HV}\sum_{i=1}^{H-V}\sum_{j=1}^{V}\left(max((i+V)t'_h; jt'_v) - max(jt'_h; (i+V)t'_v)\right)$$

We have $\forall\, i = 1..(H-V)$ *and* $\forall\, j = 1..V$: $\boldsymbol{i + V > j}$

Therefore for each one of the two cases of theorem 5 we find:

<u>**Case 1.1 $t'_h > t'_v$:**</u>

$\forall\, i = 1..(H-V)$ *and* $\forall\, j = 1..V$

We have that $max\big((\boldsymbol{i+V})t'_h; \boldsymbol{j}t'_v\big) = (\boldsymbol{i+V})t'_h$

Moreover $(\boldsymbol{i+V})t'_h > \boldsymbol{j}t'_h$ and $(\boldsymbol{i+V})t'_h > (\boldsymbol{i+V})t'_v$

Therefore:

$$max((\boldsymbol{i}+\boldsymbol{V})t'_h; \boldsymbol{j}t'_v) > max(\boldsymbol{j}t'_h; (\boldsymbol{i}+\boldsymbol{V})t'_v) \tag{A.7}$$

Consequently:

$$ERC - ERC' > 0 \tag{A.8}$$

Conclusion 5.1:

Thanks to (A.8), $(V; H)$ $dominates$ $(H; V)$, which means in this case:
$(\boldsymbol{min}(\boldsymbol{H}; \boldsymbol{V}); \boldsymbol{max}(\boldsymbol{H}; \boldsymbol{V}))$ $\boldsymbol{dominates}$ $(\boldsymbol{max}(\boldsymbol{H}; \boldsymbol{V}); \boldsymbol{min}(\boldsymbol{H}; \boldsymbol{V}))$

Case 1.2 $t'_h < t'_v$:

$\forall\, i = 1..(H - V)$ and $\forall\, j = 1..V$

Following the same reasoning then before:

$$max((\boldsymbol{i}+\boldsymbol{V})t'_h; \boldsymbol{j}t'_v) < max(\boldsymbol{j}t'_h; (\boldsymbol{i}+\boldsymbol{V})t'_v) \tag{A.9}$$

Consequently:

$$ERC - ERC' < 0 \tag{A.10}$$

Conclusion 5.2:

Thanks to (A.10), $(H; V)$ $dominates$ $(V; H)$ which means in this case:
$(\boldsymbol{max}(\boldsymbol{H}; \boldsymbol{V}); \boldsymbol{min}(\boldsymbol{H}; \boldsymbol{V}))$ $\boldsymbol{dominates}$ $(\boldsymbol{min}(\boldsymbol{H}; \boldsymbol{V}); \boldsymbol{max}(\boldsymbol{H}; \boldsymbol{V}))$

By symmetry, the **Case 2** $(\boldsymbol{V} > \boldsymbol{H})$ leads to the same conclusions.

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