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Adaptive Equalizer Using a Mixture of Gaussians Based On-Line Density Estimator

Hao Chen, *Student Member, IEEE*, Yu Gong, *Member, IEEE*, Xia Hong, *Senior Member, IEEE*, and Sheng Chen, *Fellow, IEEE*

Abstract

This paper introduces a new adaptive nonlinear equalizer relying on a radial basis function model, which is designed based on the minimum bit error rate (MBER) criterion, in the system setting of the intersymbol interference channel plus a co-channel interference. Our proposed algorithm is referred to as the on-line mixture of Gaussians estimator aided MBER (OMG-MBER) equalizer. Specifically, a mixture of Gaussians based probability density function (PDF) estimator is used to model the PDF of the decision variable, for which a novel on-line PDF update algorithm is derived to track the incoming data. With the aid of this novel on-line mixture of Gaussians based sample-by-sample updated PDF estimator, our adaptive nonlinear equalizer is capable of updating its equalizer's parameters sample by sample to aim directly at minimizing the equalizer's achievable bit error rate (BER). The proposed OMG-MBER equalizer significantly outperforms the existing on-line nonlinear MBER equalizer, known as the least bit error rate equalizer, in terms of both the convergence speed and the achievable BER, as is confirmed in our simulation study.

Index Terms

Adaptive nonlinear equalizer, radial basis function, minimum bit error rate, probability density function, mixture of Gaussians

Hao Chen and Xia Hong are with the School of Systems Engineering, University of Reading, Reading RG6 6AY, UK (E-mails: hao.chen@pgr.reading.ac.uk, x.hong@reading.ac.uk).

Yu Gong is with the School of Electronic, Electrical and Systems Engineering, Loughborough University, Loughborough LE11 3TU, UK (E-mail: alex.yugong@gmail.com).

Sheng Chen is with Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK (Email: sqc@ecs.soton.ac.uk), and also with King Abdulaziz University, Jeddah 21589, Saudi Arabia.

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I. INTRODUCTION

Channel equalization as a standard approach in communications to combat the dispersive effects of a channel has been very well studied [1]–[4]. Both linear and nonlinear equalizers have been proposed [5]–[9], and the majority of them are based on the minimum mean-square-error (MMSE) criterion. The MMSE equalizer has the advantage of easy implementation with good performance. The MMSE linear equalizer also has a nature link to adaptive filters [10], and admits a very simple yet powerful on-line adaptation by the least mean square (LMS) algorithm [11]. However, the MMSE criterion is not equivalent to the minimum bit error rate (MBER) criterion, and the latter is the ultimate performance criterion for communication channel equalization [12]–[32]. Equalizers based on the MBER criterion are thus of great interests and have attracted considerable attentions [14], [20]–[23], [33]–[35]. The least bit error rate (LBER) nonlinear equalizer for binary transmission was proposed in [33], which was shown to achieve significant performance gain over both the adaptive linear and nonlinear MMSE equalizers.

The equalization problem can alternatively be regarded as a classification problem, where the task of the equalizer is to determine the decision boundaries for separating the different classes of data. It is shown in [33] that the adaptive MBER nonlinear equalizer achieves the decision boundary that is close to the optimum Bayesian equalizer, and so is its equalization performance. While both the batch and sample-by-sample adaptive MBER equalizers were proposed in [33], the latter, referred to as the LBER, is of particular interest since it is an on-line adaptive algorithm with very low computational complexity and is capable of tracking the channel variation well. In order to update the gradient of the bit error rate (BER) with the new coming data, which is required for adapting the MBER equalizer's parameters, it is necessary to adaptively estimate the probability density function (PDF) of the signed decision variable constructed based on the desired signal and the equalizer output. The LBER nonlinear equalizer [33] adopts a one-sample or single Gaussian kernel estimate for the PDF of the signed decision variable to realize sample-by-sample adaptation of the equalizer's parameters, in a manner similar to the LMS algorithm which uses a one sample estimate for the mean square error [10].

The performance of an adaptive MBER equalizer is to a large extent determined by the on-line PDF estimator employed. Despite of its computational simplicity and its superior performance over the adaptive MMSE equalizer, the LBER equalizer of [33] has a drawback, owing to the fact that it adopts a one-sample PDF estimate which is stochastic by nature and is very sensitive to the noise in the received signal sample. In statistical data learning, there exist large amount of works

[36]–[41] using the Gaussian mixture model to estimate PDF. These kernel density estimators based on a mixture of Gaussians however are batch learning algorithms by nature and they are unsuitable for on-line applications. In this paper, we also use a kernel density estimator, consisting of a small number of Gaussian kernels to estimate the PDF of the signed decision variable. Our novel contribution is to propose a new on-line mixture of Gaussians estimator, referred to as the OMG, to update the PDF estimate sample by sample. To be specific, a new Gaussian kernel is formed for every new data and it is then merged with the “nearest” existing Gaussian kernel in the kernel density estimator. With the aid of this OMG for on-line estimation of the signed decision variable’s PDF, the nonlinear equalizer’s parameters can be adapted sample by sample in the way similar to the LBER nonlinear equalizer of [33]. We refer to this new adaptive MBER nonlinear equalizer as the OMG aided MBER (OMG-MBER) algorithm. Because the proposed OMG-MBER equalizer relies on a much more accurate on-line PDF estimate, unlike the one-sample PDF estimator of the LBER algorithm, it outperforms the LBER nonlinear equalizer in terms of both the convergence speed and the equalizer’s achievable BER. A simulation study is carried out and the results obtained confirm that the OMG-MBER equalizer significantly improve the equalization performance, compared with the existing LBER equalizer.

This paper is organized as follows. Section II introduces the equalisation problem in the setting of the intersymbol interference channel plus a co-channel interference (CCI) as well as reviews the existing adaptive MMSE and MBER equalizers. Section III details our proposed novel OMG-MBER nonlinear equalizer. The simulation results are presented in Section IV, and our conclusions are given in Section V.

II. PRELIMINARIES

We first introduce the channel model and then review the adaptive equalizers based on the MMSE and MBER criteria.

A. System Model

We assume without loss of generality that the system suffers from one CCI. The received signal sample at symbol index k is given by [42]

$$\begin{aligned} r(k) &= \bar{r}(k) + n(k) = \bar{r}_0(k) + \bar{r}_1(k) + n(k) \\ &= \sum_{i=0}^{n_0-1} a_{0,i} b_0(k-i) + \sum_{i=0}^{n_1-1} a_{1,i} b_1(k-i) + n(k), \end{aligned} \quad (1)$$

where $n(k)$ is the additive white Gaussian noise (AWGN) with zero mean and variance $E[n^2(k)] = \sigma_n^2$, $\bar{r}(k)$, $\bar{r}_0(k)$ and $\bar{r}_1(k)$ are the noise-free received signal, desired signal and interfering signal, respectively, while $a_{0,i}$, $0 \leq i \leq n_0 - 1$, and $a_{1,i}$, $0 \leq i \leq n_1 - 1$, are the coefficients for the channel and co-channel, respectively, $b_0(k)$ and $b_1(k)$ are the uncorrelated binary desired and interfering data, respectively, taking value from the set $\{\pm 1\}$.

We assume that the equalizer length is m , namely, the equalization is based on the received vector of the m most recently received signal samples given by

$$\mathbf{r}(k) = [r(k) \ r(k-1) \cdots r(k-m+1)]^T. \quad (2)$$

Then (1) can be expressed in the matrix form as

$$\mathbf{r}(k) = \bar{\mathbf{r}}(k) + \mathbf{n}(k) = \bar{\mathbf{r}}_0(k) + \bar{\mathbf{r}}_1(k) + \mathbf{n}(k) = \mathbf{H}_0 \mathbf{b}_0(k) + \mathbf{H}_1 \mathbf{b}_1(k) + \mathbf{n}(k), \quad (3)$$

where $\mathbf{n}(k) = [n(k) \ n(k-1) \cdots n(k-m+1)]^T$ is the AWGN vector, $\mathbf{b}_0(k) = [b_0(k) \ b_0(k-1) \cdots b_0(k-n_0-m+2)]^T$ and $\mathbf{b}_1(k) = [b_1(k) \ b_1(k-1) \cdots b_1(k-n_1-m+2)]^T$ are the desired and interfering signal vectors, respectively, while $\mathbf{H}_0 \in \mathbb{R}^{m \times (m+n_0-1)}$ and $\mathbf{H}_1 \in \mathbb{R}^{m \times (m+n_1-1)}$ are the desired and interfering channel matrices, respectively, given by

$$\mathbf{H}_i = \begin{bmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,n_i-1} & 0 & \cdots & 0 \\ 0 & a_{i,0} & a_{i,1} & \cdots & a_{i,n_i-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & a_{i,0} & a_{i,1} & \cdots & a_{i,n_i-1} \end{bmatrix}, \quad (4)$$

for $i = 0$ and 1 . The equalization output can be expressed as

$$\hat{b}_0(k-d) = \text{sgn}(y(k)) \text{ with } y(k) = f(\mathbf{r}(k); \mathbf{w}), \quad (5)$$

where $f(\cdot; \cdot)$ denotes the equalizer function, $\mathbf{w}$ contains all the equalizer's adjustable parameters, and d is the decision delay.

As $b_0(k)$ and $b_1(k)$ are binary signals, there are $N_0 = 2^{m+n_0-1}$ and $N_1 = 2^{m+n_1-1}$ possible combinations or “states” for $\mathbf{b}_0(k)$ and $\mathbf{b}_1(k)$, respectively. The j -th state of $\mathbf{b}_0(k)$ is denoted as $\mathbf{b}_{0,j}$, where $1 \leq j \leq N_0$, while the l -th state of $\mathbf{b}_1(k)$ is denoted as $\mathbf{b}_{1,l}$, where $1 \leq l \leq N_1$. Accordingly, there are N_0 and N_1 states for $\bar{\mathbf{r}}_0(k)$ and $\bar{\mathbf{r}}_1(k)$, denoted as $\{\bar{\mathbf{r}}_{0,j} = \mathbf{H}_0 \mathbf{b}_{0,j}\}_{j=1}^{N_0}$ and $\{\bar{\mathbf{r}}_{1,l} = \mathbf{H}_1 \mathbf{b}_{1,l}\}_{l=1}^{N_1}$, respectively. Consequently, there are $N_r = N_0 \times N_1$ states for $\bar{\mathbf{r}}(k)$, which we denote as $\mathbf{R} = \{\bar{\mathbf{r}}_i\}_{i=1}^{N_r}$. $\mathbf{R}$ can be divided into two sub-sets, $\mathbf{R}_+$ and $\mathbf{R}_-$, corresponding to the d -th element of $\mathbf{b}_{0,j}$, denoted as $b_{0,j}^{(d)}$, being $+1$ and -1 , respectively. Thus $b_{0,j}^{(d)}$ serves as the

“class” label for $\mathbf{r}(k) \in \mathbb{R}^m$, and the equalization (5) is equivalent to a classification process, which partitions the m -dimensional observation space by the decision boundary defined as

$$f(\mathbf{r}; \mathbf{w}) = 0. \quad (6)$$

The optimal Bayesian equalizer $y_B(k) = f_B(\mathbf{r}(k); \mathbf{w})$, given the channel and co-channel, is expressed by [42], [43]:

$$\begin{aligned} f_B(\mathbf{r}(k); \mathbf{w}) &= \sum_{j=1}^{N_0} \sum_{l=1}^{N_1} \frac{b_{0,j}^{(d)}}{(2\pi\sigma_n^2)^{m/2} N_0 N_1} e^{-\frac{\|\mathbf{r}(k) - \bar{\mathbf{r}}_{0,j} - \bar{\mathbf{r}}_{1,l}\|^2}{2\sigma_n^2}} \\ &= \sum_{i=1}^{N_r} \frac{b_i^{(d)}}{(2\pi\sigma_n^2)^{m/2} N_r} e^{-\frac{\|\mathbf{r}(k) - \bar{\mathbf{r}}_i\|^2}{2\sigma_n^2}}, \end{aligned} \quad (7)$$

where $b_i^{(d)}$ is the class label for $\bar{\mathbf{r}}_i$. However, the computational complexity of this Bayesian solution is often too high, because the size of $\mathbf{R}$ is typically huge.

B. Equalizer Based on MMSE Criterion

Due to its mathematical tractability, the most widely adopted criterion for training an equalizer is the MMSE criterion, and the well-known LMS algorithm [11] offers an adaptive algorithm for training a linear equalizer based on the MMSE criterion. The LMS algorithm can also be extended to train a nonlinear equalizer according to

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \eta(b_0(k-d) - y(k)) \frac{\partial f(\mathbf{r}(k); \mathbf{w}(k-1))}{\partial \mathbf{w}}, \quad (8)$$

where η is a small positive step size.

The adaptive non-linear equalizer design based on the radial basis function (RBF) network was well studied [43], and the RBF equalizer output is given by

$$y(k) = f(\mathbf{r}(k); \mathbf{w}) = \sum_{j=1}^{n_c} \alpha_j e^{-\frac{\|\mathbf{r}(k) - \mathbf{c}_j\|^2}{\sigma_j^2}}, \quad (9)$$

where n_c is the number of RBF nodes employed, and the equalization parameter vector $\mathbf{w}$ consists of all the RBF weights α_j , variances σ_j^2 and centers $\mathbf{c}_j$. The dimension of $\mathbf{w}$ is therefore $n_c \times (m+2)$. The partial derivatives of the equalizer output with respect to the equalizer's

parameters $\mathbf{w}$ in (8) are given for $1 \leq j \leq n_c$ by

$$\left\{ \begin{array}{l} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \alpha_j} = e^{-\frac{\|\mathbf{r}(k) - \mathbf{c}_j\|^2}{\sigma_j^2}}, \\ \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \sigma_j^2} = \alpha_j e^{-\frac{\|\mathbf{r}(k) - \mathbf{c}_j\|^2}{\sigma_j^2}} \frac{\|\mathbf{r}(k) - \mathbf{c}_j\|^2}{\sigma_j^4}, \\ \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{c}_j} = 2\alpha_j e^{-\frac{\|\mathbf{r}(k) - \mathbf{c}_j\|^2}{\sigma_j^2}} \frac{\mathbf{r}(k) - \mathbf{c}_j}{\sigma_j^2}. \end{array} \right. \quad (10)$$

For the linear equalizer

$$y(k) = f_L(\mathbf{r}(k); \mathbf{w}) = \mathbf{w}^T \mathbf{r}(k), \quad (11)$$

(8) returns to its original LMS form

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \eta(b_0(k-d) - y(k))\mathbf{r}(k). \quad (12)$$

C. Equalizer Based on MBER Criterion

However, the ultimate goal of equalization is to minimize the BER [12]–[16]. Many studies [20]–[23], [33]–[35] have demonstrated that the MBER equalizer can significantly improve the performance compared with the traditional equalizer design based on the MMSE criterion. Specifically, for the linear equalizer (11), the MBER design has been extensively studied [13]–[16], [18], [20], and adaptive MBER linear equalizer has been proposed based on the LBER algorithm [21], [23]. Moreover, the MBER design and its adaptive LBER algorithm have been extended to the generic nonlinear equalizer of (5) [33], [34].

The error probability of the equalizer (5) can be obtained as [33]

$$P_E(\mathbf{w}) = \text{Prob}\{\text{sgn}(b_0(k-d))y(k) < 0\}. \quad (13)$$

Defining the signed decision variable as $y_s(k) = \text{sgn}(b_0(k-d))y(k)$, we have

$$P_E(\mathbf{w}) = \int_{-\infty}^0 p_y(y_s) dy_s, \quad (14)$$

where $p_y(y_s)$ is the PDF of $y_s(k)$. Linearizing the equalizer around $\bar{\mathbf{r}}(k)$ yields [33]

$$y(k) = f(\bar{\mathbf{r}}(k) + \mathbf{n}(k); \mathbf{w}) \approx f(\bar{\mathbf{r}}(k); \mathbf{w}) + e(k) = \bar{y}(k) + e(k), \quad (15)$$

where $e(k)$ is an equivalent zero-mean Gaussian noise with variance ρ^2 . Noting that the noise-free received signal $\bar{y}(k)$ takes value from the finite set, namely, $\bar{y}(k) \in Y_f = \{\bar{y}_i = f(\bar{\mathbf{r}}_i; \mathbf{w})\}_{i=1}^{N_r}$, the PDF of $y_s(k)$ can be approximated as [33]

$$p_y(y_s) \approx \frac{1}{N_r \sqrt{2\pi} \rho} \sum_{i=1}^{N_r} e^{-\frac{(y_s - \text{sgn}(b_i^{(d)}) \bar{y}_i)^2}{2\rho^2}}. \quad (16)$$

Substituting (16) into (14) then yields the following error probability of the equalizer

$$P_E(\mathbf{w}) \approx \frac{1}{N_r \sqrt{2\pi}} \sum_{i=1}^{N_r} \int_{\bar{g}_i(\mathbf{w})}^{\infty} e^{-\frac{x_i^2}{2}} dx_i = \frac{1}{N_r} \sum_{i=1}^{N_r} Q(\bar{g}_i(\mathbf{w})), \quad (17)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{y^2}{2}} dy, \quad (18)$$

$$\bar{g}_i(\mathbf{w}) = \frac{\text{sgn}(b_i^{(d)}) f(\bar{\mathbf{r}}_i; \mathbf{w})}{\rho}. \quad (19)$$

Then, the gradient of $P_E(\mathbf{w})$ can be approximated as

$$\nabla P_E(\mathbf{w}) \approx -\frac{1}{N_r \sqrt{2\pi}} \sum_{i=1}^{N_r} e^{-\frac{\bar{y}_i^2}{2\rho^2}} \frac{\partial \bar{g}_i(\mathbf{w})}{\partial \mathbf{w}} = -\frac{1}{N_r \sqrt{2\pi}} \sum_{i=1}^{N_r} e^{-\frac{\bar{y}_i^2}{2\rho^2}} \text{sgn}(b_i^{(d)}) \frac{\partial f(\bar{\mathbf{r}}_i; \mathbf{w})}{\partial \mathbf{w}}. \quad (20)$$

Thus, if the channel and co-channel are known, the equalizer parameters can be updated by minimizing the approximate BER in (17), leading to the MBER updating rule

$$\mathbf{w}(l) = \mathbf{w}(l-1) - \eta \nabla P_E(\mathbf{w}(l-1)). \quad (21)$$

In practice, the channel and co-channel are unknown. Thus, the set $\mathbf{R}$ and hence the set Y_f are unknown. The key to developing an effective adaptive algorithm for implementing the MBER design is to estimate the PDF $p_y(y_s)$ of the signed decision variable $y_s(k)$ based on the training data $D_K = \{\mathbf{r}(k), b_0(k-d)\}_{k=1}^K$ [33]. Specifically, given D_K , the Parzen window or kernel density estimate of the PDF $p_y(y_s)$ is given as [33]

$$\hat{p}_y(y_s) = \frac{1}{K \sqrt{2\pi} \rho} \sum_{k=1}^K e^{-\frac{(y_s - \text{sgn}(b_0(k-d)) y(k))^2}{2\rho^2}}. \quad (22)$$

Substituting (22) into (14) yields the estimated error probability $\hat{P}_E(\mathbf{w})$, the gradient of which is given by

$$\nabla \hat{P}_E(\mathbf{w}) = -\frac{1}{K \sqrt{2\pi} \rho} \sum_{k=1}^K e^{-\frac{y^2(k)}{2\rho^2}} \text{sgn}(b_0(k-d)) \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}. \quad (23)$$

This leads to the block adaptive MBER algorithm [33]

$$\begin{aligned}\mathbf{w}(l) &= \mathbf{w}(l-1) - \eta \nabla \hat{P}_E(\mathbf{w}(l-1)) \\ &= \mathbf{w}(l-1) + \frac{\eta}{K\sqrt{2\pi\rho}} \sum_{k=1}^K e^{-\frac{y^2(k)}{2\rho^2}} \text{sgn}(b_0(k-d)) \frac{\partial f(\mathbf{r}(k); \mathbf{w}(l-1))}{\partial \mathbf{w}}.\end{aligned}\quad (24)$$

To realize a sample by sample adaptation, the nonlinear LBER algorithm [33] only uses a single data point to estimate the PDF

$$\tilde{p}_y(y_s, k) = \frac{1}{\sqrt{2\pi\rho}} e^{-\frac{(y_s - \text{sgn}(b_0(k-d))y(k))^2}{2\rho^2}}. \quad (25)$$

With this “instantaneous” PDF estimate, we have a “one-sample” error probability estimate $\tilde{P}_E(\mathbf{w}, k)$ with the stochastic gradient given by

$$\nabla \tilde{P}_E(\mathbf{w}, k) = -\frac{1}{\sqrt{2\pi\rho}} e^{-\frac{y^2(k)}{2\rho^2}} \text{sgn}(b_0(k-d)) \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}. \quad (26)$$

This leads to the stochastic gradient adaptive algorithm known as the LBER algorithm [33]

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \frac{\eta}{\sqrt{2\pi\rho}} e^{-\frac{y^2(k)}{2\rho^2}} \text{sgn}(b_0(k-d)) \frac{\partial f(\mathbf{r}(k); \mathbf{w}(k-1))}{\partial \mathbf{w}}, \quad (27)$$

in which the equalizer parameter vector is adjusted sample by sample, in a similar manner to the LMS algorithm in (12).

III. ON-LINE MIXTURE OF GAUSSIANS ESTIMATOR AIDED MBER EQUALIZER

In order to realize a sample-by-sample adaptive MBER equalizer, the LBER algorithm [33] uses a single sample to estimate the PDF on-line. Our OMG-MBER adopts a novel mixture of Gaussians based on-line PDF estimator. Specifically, we derive the kernel PDF estimator based on a mixture of Gaussians with a small number of mixtures, which is capable of adapting the PDF estimate sample by sample. With the aid of this on-line kernel density estimation algorithm, our adaptive MBER equalizer is capable of updating its equalizer’s parameters sample by sample, similar to the LBER algorithm. Since our OMG-MBER algorithm relies on a much more accurate one-line PDF estimate, unlike the one-sample PDF of the LBER algorithm, we expect our OMG-MBER will outperform the LBER, at the expense of negligibly small increase in complexity.

A. On-Line Kernel Based PDF Estimator

Our goal is to derive an on-line PDF estimator that updates the estimate of $p_y(y_s)$ as each new data $y_s(k)$ is received. We consider the PDF estimator based on the mixture of M Gaussians [37] given by

$$\hat{p}_y(y_s) = \hat{p}^{(M)}(y_s; \boldsymbol{\lambda}_M, \boldsymbol{\mu}_M, \boldsymbol{\rho}_M) = \sum_{j=1}^M \lambda_j G(y_s; \mu_j, \rho_j), \quad (28)$$

$$\text{s.t. } \lambda_j > 0, 1 \leq j \leq M, \text{ and } \boldsymbol{\lambda}_M^T \mathbf{1}_M = 1, \quad (29)$$

where λ_j , μ_j and ρ_j are the weight, mean and standard deviation of the j -th Gaussian kernel, respectively, $\boldsymbol{\lambda}_M = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_M]^T$, $\boldsymbol{\mu}_M = [\mu_1 \ \mu_2 \ \cdots \ \mu_M]^T$, $\boldsymbol{\rho}_M = [\rho_1 \ \rho_2 \ \cdots \ \rho_M]^T$, $\mathbf{1}_M$ is the M -dimensional vector whose elements are all equal to one, and

$$G(y_s; \mu_j, \rho_j) = \frac{1}{\sqrt{2\pi}\rho_j} e^{-\frac{(y_s - \mu_j)^2}{2\rho_j^2}}. \quad (30)$$

At time $k = 0$, we have the initial estimate for $p_y(y_s)$ given as

$$\hat{p}^{(M)}(y_s; \boldsymbol{\lambda}_M(0), \boldsymbol{\mu}_M(0), \boldsymbol{\rho}_M(0)) = \sum_{j=1}^M \frac{1}{M} G(y_s; \mu_j(0), \rho_0), \quad (31)$$

where $\lambda_j(0) = \frac{1}{M}$ and $\rho_j(0) = \rho_0$ for $1 \leq j \leq M$, with ρ_0 being a predetermined kernel width, while $\mu_j(0)$ for $1 \leq j \leq M$ are some randomly drawn positive values. The initialization of the OMG-MBER based nonlinear equalizer will be discussed in more detail later in Subsection III-C.

At time step k , the new data point $y_s(k)$ is received, and the triplets $\{\boldsymbol{\lambda}_M, \boldsymbol{\mu}_M, \boldsymbol{\rho}_M\}$ in the PDF estimate

$$\hat{p}^{(M)}(y_s; \boldsymbol{\lambda}_M(k-1), \boldsymbol{\mu}_M(k-1), \boldsymbol{\rho}_M(k-1)) = \sum_{j=1}^M \lambda_j(k-1) G(y_s; \mu_j(k-1), \rho_j(k-1)) \quad (32)$$

need to be updated accordingly, while keeping the same number of mixtures M . A nature way is to place a Gaussian kernel on $y_s(k)$ and to merge this new kernel with its nearest existing mixture component $G(y_s; \mu_{j^*}(k-1), \rho_{j^*}(k-1))$, where

$$j^* = \arg \min_{1 \leq j \leq M} |y_s(k) - \mu_j(k-1)|. \quad (33)$$

This can be realized in the following two steps.

- 1) Create a temporary $(M+1)$ -order model by adding the newly created $(M+1)$ -th Gaussian kernel based on $y_s(k)$ to the estimate (32) according to

$$\hat{p}^{(M+1)}(y_s; \boldsymbol{\lambda}_{M+1}(k), \boldsymbol{\mu}_{M+1}(k), \boldsymbol{\rho}_{M+1}(k)) = \frac{M}{M+1} \sum_{j=1}^M \lambda_j(k-1) G(y_s; \mu_j(k-1), \rho_j(k-1)) + \frac{1}{M+1} G(y_s; y_s(k), \rho_0). \quad (34)$$

Clearly, we set $\lambda_{M+1}(k) = \frac{1}{M+1}$, and for $1 \leq j \leq M$

$$\lambda_j(k) = \frac{M\lambda_j(k-1)}{M+1}, \quad (35)$$

to satisfy the constraints $\lambda_j(k) > 0$ for $1 \leq j \leq M+1$ and $\boldsymbol{\lambda}_{M+1}^T(k) \mathbf{1}_{M+1} = 1$, while using $\mu_{M+1}(k) = y_s(k)$, $\rho_{M+1}(k) = \rho_0$, and for $1 \leq j \leq M$

$$\mu_j(k) = \mu_j(k-1), \quad (36)$$

$$\rho_j(k) = \rho_j(k-1). \quad (37)$$

- 2) Merge the j^* -th, where j^* is determined in (33), and $(M+1)$ -th mixtures in the temporary estimate (34) into the single new j^* -th mixture, so that

$$\lambda_{j^*}(k) G(y_s; \mu_{j^*}(k), \rho_{j^*}(k)) \approx \frac{M\lambda_{j^*}(k-1)}{M+1} G(y_s; \mu_{j^*}(k-1), \rho_{j^*}(k-1)) + \frac{1}{M+1} G(y_s; y_s(k), \rho_0). \quad (38)$$

Thus, the new j^* -th weight $\lambda_{j^*}(k)$ is given by

$$\lambda_{j^*}(k) = \frac{M\lambda_{j^*}(k-1)}{M+1} + \frac{1}{M+1}, \quad (39)$$

while the new j^* -th mean and standard deviation, $\mu_{j^*}(k)$ and $\rho_{j^*}(k)$, are updated by matching the mean and standard deviation of the two mixtures with the new single Gaussian. As is shown in Appendix A, we have the following adaptation rules

$$\mu_{j^*}(k) = \frac{M\lambda_{j^*}(k-1)}{M\lambda_{j^*}(k-1) + 1} \mu_{j^*}(k-1) + \frac{1}{M\lambda_{j^*}(k-1) + 1} y_s(k), \quad (40)$$

$$\rho_{j^*}^2(k) = \frac{M\lambda_{j^*}(k-1)(\rho_{j^*}^2(k-1) + \mu_{j^*}^2(k-1))}{M\lambda_{j^*}(k-1) + 1} + \frac{\rho_0^2 + y_s^2(k)}{M\lambda_{j^*}(k-1) + 1} - \mu_{j^*}^2(k). \quad (41)$$

The PDF of the signed decision variable at sample time k can thus be approximated by

$$\hat{p}_y(y_s, k) = \hat{p}^{(M)}(y_s; \boldsymbol{\lambda}_M(k), \boldsymbol{\mu}_M(k), \boldsymbol{\rho}_M(k)) = \sum_{j=1}^M \lambda_j(k) G(y_s; \mu_j(k), \rho_j(k)), \quad (42)$$

in which the j^* -th weight, mean and standard deviation are given in (39), (40) and (41), respectively, while the j -th weights, means and standard deviations, where $1 \leq j \leq M$ and $j \neq j^*$, are given in (35), (36) and (37), respectively. Note that only $\mu_{j^*}(k)$ and $\rho_{j^*}^2(k)$ contain the new information provided by $y_s(k)$. This on-line estimator is a much more accurate estimate of the true PDF $p_y(y_s)$ than the one-sample estimate of (25), at the cost of a modest increase in computational requirements, as the number of mixtures M is a very small number.

B. Adaptive MBER Equalizer

Based on the estimated PDF of the signed decision variable at sample k , as is given in (42), the error probability of the equalizer can be expressed as

$$\begin{aligned}\hat{P}_E(\mathbf{w}, k) &= \int_{-\infty}^0 \hat{p}_y(y_s, k) dy_s \\ &= \int_{-\infty}^0 \left(\sum_{j=1, j \neq j^*}^M \lambda_j(k) G(y_s; \mu_j(k), \rho_j(k)) + \lambda_{j^*}(k) G(y_s; \mu_{j^*}(k), \rho_{j^*}(k)) \right) dy_s \\ &= \hat{R}_E(\mathbf{w}, k) + \sum_{j=1, j \neq j^*}^M \lambda_j(k) \int_{-\infty}^0 G(y_s; \mu_j(k), \rho_j(k)) dy_s,\end{aligned}\quad (43)$$

where

$$\hat{R}_E(\mathbf{w}, k) = \frac{\lambda_{j^*}(k)}{\sqrt{2\pi}\rho_{j^*}(k)} \int_{-\infty}^0 e^{-\frac{(y_s - \mu_{j^*}(k))^2}{2\rho_{j^*}^2(k)}} dy_s = \frac{\lambda_{j^*}(k)}{\sqrt{2\pi}} \int_{g_{j^*}(\mathbf{w}, k)}^{\infty} e^{-\frac{y^2}{2}} dy, \quad (44)$$

with

$$g_{j^*}(\mathbf{w}, k) = \frac{\mu_{j^*}(k)}{\rho_{j^*}(k)}. \quad (45)$$

In (43) only $\mu_{j^*}(k)$ and $\rho_{j^*}(k)$ depend on the current equalizer's parameter vector $\mathbf{w}$, since only $\mu_{j^*}(k)$ and $\rho_{j^*}(k)$ depends on $y_s(k) = b_0(k-d)y(k) = b_0(k-d)f(\mathbf{r}(k); \mathbf{w})$. Therefore, we have the “instantaneous” gradient of $\hat{P}_E(\mathbf{w}, k)$ given by

$$\nabla \hat{P}_E(\mathbf{w}, k) = \nabla \hat{R}_E(\mathbf{w}, k) = -\frac{\lambda_{j^*}(k)}{\sqrt{2\pi}} e^{-\frac{\mu_{j^*}^2(k)}{2\rho_{j^*}^2(k)}} \frac{\partial g_{j^*}(\mathbf{w}, k)}{\partial \mathbf{w}}, \quad (46)$$

where

$$\frac{\partial g_{j^*}(\mathbf{w}, k)}{\partial \mathbf{w}} = \frac{(\rho_{j^*}^2(k) + \mu_{j^*}^2(k)) \operatorname{sgn}(b_0(k-d)) - \mu_{j^*}(k)y(k)}{\rho_{j^*}^3(k)(M\lambda_{j^*}(k-1) + 1)} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}, \quad (47)$$

the derivation of which is given in Appendix B.

At sample k , given the equalizer's signed decision variable $y_s(k) = b_0(k-d)f(\mathbf{r}(k); \mathbf{w}(k-1))$, the on-line parameter updating for the adaptive MBER equalizer based on "stochastic" gradient descent is therefore expressed by

$$\begin{aligned} \mathbf{w}(k) &= \mathbf{w}(k-1) - \eta \nabla \hat{P}_E(\mathbf{w}(k-1), k) \\ &= \mathbf{w}(k-1) + \frac{\eta \lambda_{j^*}(k)}{\sqrt{2\pi}} e^{-\frac{\mu_{j^*}^2(k)}{2\rho_{j^*}^2(k)}} \frac{\partial g_{j^*}(\mathbf{w}(k-1), k)}{\partial \mathbf{w}}, \end{aligned} \quad (48)$$

where $\eta > 0$ is a small step size.

C. OMG-MBER Algorithm Summary

It is well known that the mixture of Gaussians with a small M is capable of accurately estimating an arbitrary PDF [36], [37], [44], [45]. Therefore, the number of mixtures M in the on-line kernel density estimator (28) can be chosen as a very small number, and typically $M = 4$ to 6 is sufficient for most applications. The initial means $\mu_j(0)$ for $1 \leq j \leq M$ of the PDF estimator can be simply set to some small positive and random values. Our extensive simulation experience suggests that the choice of the initial means $\mu_j(0)$ for $1 \leq j \leq M$ are not critical at all to the performance of the adaptive equalizer. This is because the propose OMG PDF estimator has an excellent adaptation capability and is capable of adapting to the underlying true data distribution from different choices of $\mu_j(0)$. By the same reason, the choice of the initial kernel width ρ_0 does not have a major impact on the achievable performance.

The nonlinear equalizer we employed is the RBF equalizer (9). The number of the RBF nodes n_c is problem dependent. Specifically, n_c is related to the size N_r of the state set $\mathbf{R}$. With n_c set to N_r , the RBF equalizer has the potential of attaining the full optimal performance of the Bayesian equalizer (7). However, this is rarely achievable as N_r is typically huge. In general, there exists a trade-off between achievable performance and affordable cost in choosing n_c . A large n_c has the potential of achieving a better performance at the cost of higher complexity and slower adaptation process. In practice, we often choose a very small n_c , compared with the problem size N_r . We are now ready to summarize the proposed OMG-MBER RBF equalizer.

Initialization ($k \leq 0$):

- 1) Given the equalizer order m and the RBF equalizer size n_c , initialize the RBF equalizer, namely, set the initial equalizer's parameter vector $\mathbf{w}(0)$.

- 2) Given the number of mixtures M and the initial kernel width ρ_0 , initialize the mixture of Gaussians based PDF estimator (31), namely, set the initial means $\mu_j(0)$, $1 \leq j \leq M$, of the M Gaussians.

Adaptation ($k \geq 1$):

- 1) Given $\mathbf{w}(k-1)$ and the new data $\{\mathbf{r}(k), b_0(k-d)\}$, calculate the RBF equalizer output $y(k) = f(\mathbf{r}(k); \mathbf{w}(k-1))$ using (9) and compute the signed decision variable $y_s(k) = b_0(k-d)y(k)$.
- 2) Update the mixture of Gaussians based PDF estimator from (32) into (42). Specifically,
 - Find the index j^* according to (33).
 - Update the j^* -th weight, mean and standard deviation according to (39), (40) and (41), respectively.
 - Update the j -th weights, means and standard deviations for $1 \leq j \leq M$ and $j \neq j^*$ according to (35), (36) and (37), respectively.
- 3) Given the step size η , update the RBF equalizer's parameter vector from $\mathbf{w}(k-1)$ to $\mathbf{w}(k)$. Specifically,
 - Calculate $\frac{\partial f(\mathbf{r}(k); \mathbf{w}(k-1))}{\partial \mathbf{w}}$ according to (10).
 - Calculate $\frac{\partial g_{j^*}(\mathbf{w}(k-1), k)}{\partial \mathbf{w}}$ according to (47).
 - Update the RBF equalizer's parameter vector according to (48).

Because the initial choice of the RBF equalizer's parameter vector $\mathbf{w}(0)$ has some influence on the convergence speed of the on-line adaptive procedure, we choose the initial centers $\mathbf{c}_j(0)$, $1 \leq j \leq n_c$, from the data $\{\mathbf{r}(k), b_0(k-d), k \leq 0\}$ according to the following heuristic rules. Half of the initial centers are chosen from the data points $\mathbf{r}(k)$ that corresponds to the desired signal $b_0(k-d) = +1$, while the other half of the initial centers are chosen from the data points $\mathbf{r}(k)$ which has the desired signal value $b_0(k-d) = -1$. Moreover, the chosen initial centers must have a minimum distance apart. Specifically, for $1 \leq i, j \leq n_c$ and $i \neq j$,

$$\|\mathbf{c}_j(0) - \mathbf{c}_i(0)\| > \frac{\zeta_0}{n_c}, \quad (49)$$

where the preset positive parameter ζ_0 controls the distribution of the initial RBF nodes. The initial weights $\alpha_j(0)$ are set either to $+\alpha_0$ or to $-\alpha_0$, depending on whether $\mathbf{c}_j(0)$ are related to $b_0(k-d) = +1$ or $b_0(k-d) = -1$, where $\alpha_0 > 0$ is a preset parameter. All the initial variances $\sigma_j^2(0)$, $1 \leq j \leq n_c$, are set to the same value of $\frac{\sigma_0^2}{n_c^2}$, where the preset positive value σ_0^2 controls

the influencing field of the initial RBF nodes. This RBF equalizer initialization procedure is similar to the one given in [33].

The values of ζ_0 , α_0 and σ_0 as well as the step size η are carefully chosen based on trial-and-error to achieve a best possible performance.

IV. SIMULATION STUDY

Two examples were used to compare the proposed OMG-MBER RBF equalizer with the LBER RBF equalizer given in [33] and the conventional linear MMSE equalizer based on the LMS algorithm. For the both examples, the number of mixtures was set to $M = 6$ with the initial kernel parameters $\rho_0 = 1$ and $\mu_j(0)$, $1 \leq j \leq M$, randomly chosen from $[0.5, 1.5]$ for the proposed OMG PDF estimator, while the step size of the OMG-MBER RBF equalizer was set to $\eta = 0.3$.

Example One. In the first experiment, the transfer functions of the channel and co-channel were given as $A_0(z) = 0.5 + 1.0z^{-1}$ and $A_1(z) = \lambda(1.0 + 0.5z^{-1})$, respectively, where λ was set to yield the signal to interference ratio (SIR) of 12 dB. The equalizer order and decision delay were set to $m = 2$ and $d = 1$, respectively, and therefore the size of the state set $\mathbf{R}$ was $N_r = 64$. For both the OMG-MBER and LBER based RBF equalizers, we had $n_c = 4$. For the OMG-MBER RBF equalizer, the initialization was carried out with $\zeta_0 = 8$, $\sigma_0 = 6$ and $\alpha_0 = 10$. For the LBER RBF equalizer, the same initialization as in [33] was used and moreover the LBER algorithmic parameters were set to $\eta_k = 0.5k^{-0.25}$ and $\rho^2 = 20\sigma_n^2$, the same as in [33].

Fig. 1. Comparison of decision boundaries (thin dotted: linear MMSE, thick dotted: LBER RBF, thin solid: proposed OMG-MBER RBF, and thick solid: optimal Bayesian) for *Example 1* given SNR= 15 dB.

Fig. 2. Comparison of learning curves for the linear MMSE equalizer based on the LMS, the LBER RBF equalizer and the OMG-MBER RBF equalizer, in terms of BER performance and averaged over 100 runs for *Example 1* given SNR= 15 dB.

Fig. 1 compares the decision boundaries of the linear MMSE equalizer, the LBER RBF equalizer and the proposed OMG-MBER RBF equalizer using the optimal Bayesian equalizer (7) as the benchmark, under the signal to noise ratio (SNR) condition of 15 dB. It can be clearly seen from Fig. 1 that the linear MMSE equalizer could only realize a linear decision boundary, while the LBER and OMG-MBER RBF equalizers with only 4 RBF nodes were capable of approximating well the optimal nonlinear decision boundary of the 64-nodes Bayesian equalizer. The learning curves of the four-center LBER and OMG-MBER RBF equalizers as well as the

Fig. 3. BER performance comparison of the linear MMSE equalizer based on the LMS, the LBER RBF equalizer and the OMG-MBER RBF equalizer as well as the optimal Bayesian equalizer for *Example 1*.

two-tap linear MMSE equalizer based on the LMS algorithm are compared in Fig. 2 given SNR= 15 dB, where the BER results were averaged over 100 runs. As expected, the linear MMSE equalizer had the worst BER performance. From Fig. 2, it can be seen that the OMG-MBER RBF equalizer considerably outperformed the LBER RBF equalizer both in convergence speed and achievable BER performance. The achievable BER performance of the three adaptive equalizers are depicted in Fig. 3 together with the BER of the optimal Bayesian equalizer. The results of Fig. 3 again confirm that the proposed OMG-MBER RBF equalizer had the superior performance over the LBER RBF equalizer.

Example Two. The transfer functions of the channel and co-channel were specified by $A_0(z) = 0.3482 + 0.8704z^{-1} + 0.3482z^{-2}$ and $A_1(z) = \lambda(0.6 + 0.8z^{-1})$, respectively, where the value of λ was chosen to yield the SIR= 20 dB. The equalizer order was set to $m = 4$ and the decision delay $d = 1$. The number of the signal states in $\mathbf{R}$ was $N_r = 2048$. We used $n_c = 16$ for both the LBER and OMG-MBER RBF equalizers. For the initialization of the OMG-MBER RBF equalizer, we used $\zeta_0 = 12$, $\sigma_0 = 16$ and $\alpha_0 = 10$. For the LBER RBF equalizer, $\eta_k = 0.3k^{-0.25}$ and $\rho^2 = 10\sigma_n^2$ as well as the same initialization as given in [33] were used.

The learning curves of the three adaptive equalizers are shown in Fig. 4 given the SNR= 15 dB, where the results were averaged over 100 runs. As expected, the 4-tap linear MMSE equalizer trained by the LMS algorithm had the fastest convergence speed but its achievable BER performance was the worst. Fig. 5 depicts the BER performance of the three adaptive equalizers tested. From both Figs. 4 and 5, it can be clearly seen that the proposed OMG-

Fig. 4. Comparison of learning curves for the linear MMSE equalizer based on the LMS, the LBER RBF equalizer and the OMG-MBER RBF equalizer, in terms of BER performance and averaged over 100 runs for *Example 2* given SNR= 15 dB.

Fig. 5. BER performance comparison of the linear MMSE equalizer based on the LMS, the LBER RBF equalizer and the OMG-MBER RBF equalizer for *Example 2*.

MBER RBF equalizer significantly outperformed the LBER RBF equalizer, in terms of both convergence speed and achievable BER. It is worth noting that for this example the optimal full Bayesian solution would require 2048 nodes. Thus, this optimal Bayesian equalizer becomes impractical when the channel and interfering co-channel both have high orders.

V. CONCLUSIONS

The performance of an adaptive MBER nonlinear equalizer critically depends on the on-line PDF estimator used for estimating the signed decision variable's distribution. In this contribution, a new on-line small-size Gaussian mixture based PDF estimator has been proposed to aid an adaptive MBER nonlinear equalizer. The resulting on-line MBER nonlinear equalizer, referred to as the OMG-MBER nonlinear equalizer, has been introduced under the generic setting of intersymbol interference channel plus co-channel interference. Compared with the existing LBER nonlinear equalizer, which relies on a one-sample PDF estimator, our on-line mixture of Gaussians based PDF estimator is capable of providing a much more accurate estimate of the signed decision variable's PDF while maintaining a very low computational requirements for on-line PDF estimation. Consequently, our proposed OMG-MBER nonlinear equalizer significantly outperforms the LBER nonlinear equalizer, in terms of both convergence speed and achievable BER performance.

APPENDIX

A. Merging Two Gaussians as One

Consider merging a mixture of two Gaussians

$$\widehat{p}^{(2)}(y_s; \boldsymbol{\lambda}_2, \boldsymbol{\mu}_2, \boldsymbol{\rho}_2) = \sum_{j=1}^2 \lambda_j G(y_s; \mu_j, \rho_j), \quad (50)$$

into one mixture by matching the resultant mean and variance. The mean μ of the two mixtures is given by

$$\mu = \int y_s \widehat{p}^{(2)}(y_s; \boldsymbol{\lambda}_2, \boldsymbol{\mu}_2, \boldsymbol{\rho}_2) dy_s = \sum_{j=1}^2 \lambda_j \int y_s G(y_s; \mu_j, \rho_j) dy_s = \sum_{j=1}^2 \lambda_j \mu_j, \quad (51)$$

while the variance ρ^2 of the two mixtures is

$$\begin{aligned} \rho^2 &= \int (y_s - \mu)^2 \widehat{p}^{(2)}(y_s; \boldsymbol{\lambda}_2, \boldsymbol{\mu}_2, \boldsymbol{\rho}_2) dy_s = \int y_s^2 \widehat{p}^{(2)}(y_s; \boldsymbol{\lambda}_2, \boldsymbol{\mu}_2, \boldsymbol{\rho}_2) dy_s - \mu^2 \\ &= \sum_{j=1}^2 \lambda_j \int y_s^2 G(y_s; \mu_j, \rho_j) dy_s - \mu^2 = \sum_{j=1}^2 \lambda_j (\rho_j^2 + \mu_j^2) - \mu^2. \end{aligned} \quad (52)$$

B. The Derivation of $\frac{\partial g_{j^}(\mathbf{w}, k)}{\partial \mathbf{w}}$*

Noting $y_s(k) = \text{sgn}(b_0(k-d))f(\mathbf{r}(k); \mathbf{w})$, (40) becomes

$$\mu_{j^*}(k) = \frac{M\lambda_{j^*}(k-1)}{M\lambda_{j^*}(k-1) + 1} \mu_{j^*}(k-1) + \frac{\text{sgn}(b_0(k-d))}{M\lambda_{j^*}(k-1) + 1} f(\mathbf{r}(k); \mathbf{w}). \quad (53)$$

The partial derivative of $\mu_{j^*}(k)$ with respect to the equalizer's parameter vector $\mathbf{w}$, denoted as $\mu'_{j^*}(k)$, is then given by

$$\mu'_{j^*}(k) = \frac{\partial \mu_{j^*}(k)}{\partial \mathbf{w}} = \frac{\text{sgn}(b_0(k-d))}{M\lambda_{j^*}(k-1) + 1} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}. \quad (54)$$

From (41), we also have

$$\begin{aligned} \rho_{j^*}^2(k) &= \frac{M\lambda_{j^*}(k-1)(\rho_{j^*}^2(k-1) + \mu_{j^*}^2(k-1))}{M\lambda_{j^*}(k-1) + 1} + \frac{\rho_0^2 + (\text{sgn}(b_0(k-d))f(\mathbf{r}(k); \mathbf{w}))^2}{M\lambda_{j^*}(k-1) + 1} - \mu_{j^*}^2(k) \\ &= \frac{M\lambda_{j^*}(k-1)(\rho_{j^*}^2(k-1) + \mu_{j^*}^2(k-1))}{M\lambda_{j^*}(k-1) + 1} + \frac{\rho_0^2 + f^2(\mathbf{r}(k); \mathbf{w})}{M\lambda_{j^*}(k-1) + 1} - \mu_{j^*}^2(k). \end{aligned} \quad (55)$$

Thus the partial derivation of $\rho_{j^*}^2(k)$ with respect to the equalizer's parameter vector $\mathbf{w}$ is

$$\frac{\partial \rho_{j^*}^2(k)}{\partial \mathbf{w}} = \frac{2y(k)}{M\lambda_{j^*}(k-1) + 1} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}} - 2\mu_{j^*}(k) \frac{\partial \mu_{j^*}(k)}{\partial \mathbf{w}}. \quad (56)$$

Substituting (54) into (56) yields

$$\begin{aligned}\frac{\partial \rho_{j^*}^2(k)}{\partial \mathbf{w}} &= \frac{2y(k)}{M\lambda_{j^*}(k-1)+1} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}} - \frac{2\mu_{j^*}(k)\text{sgn}(b_0(k-d))}{M\lambda_{j^*}(k-1)+1} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}} \\ &= \frac{2(y(k) - \mu_{j^*}(k)\text{sgn}(b_0(k-d)))}{M\lambda_{j^*}(k-1)+1} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}.\end{aligned}\quad (57)$$

Denoting $\rho'_{j^*}(k) = \frac{\partial \rho_{j^*}(k)}{\partial \mathbf{w}}$, then

$$\frac{\partial \rho_{j^*}^2(k)}{\partial \mathbf{w}} = 2\rho_{j^*}(k)\rho'_{j^*}(k). \quad (58)$$

From (57) and (58), we have

$$\rho'_{j^*}(k) = \frac{y(k) - \mu_{j^*}(k)\text{sgn}(b_0(k-d))}{\rho_{j^*}(k)(M\lambda_{j^*}(k-1)+1)} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}. \quad (59)$$

Finally by making use of (59) and (54), we have

$$\begin{aligned}\frac{\partial g_{j^*}(\mathbf{w}, k)}{\partial \mathbf{w}} &= \frac{\mu'_{j^*}(k)\rho_{j^*}(k) - \mu_{j^*}(k)\rho'_{j^*}(k)}{\rho_{j^*}^2(k)} \\ &= \frac{(\rho_{j^*}^2(k) + \mu_{j^*}^2(k))\text{sgn}(b_0(k-d)) - \mu_{j^*}(k)y(k)}{\rho_{j^*}^3(k)(M\lambda_{j^*}(k-1)+1)} \frac{\partial f(\mathbf{r}(k); \mathbf{w})}{\partial \mathbf{w}}.\end{aligned}\quad (60)$$

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