Supplementary material for the article: Academic performance of students from entrance to graduation via quasi U-statistics: a study at a Brazilian research university

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Appendix A. Analytical results of the MLE for the normal and triangular distribution

For the Normal distribution

Let

$$\mu_g = E[X_i^g], \quad \mu_{g2} = E[(X_i^g)^2], \quad \mu_{g3} = E[(X_i^g)^3] \quad \text{and} \quad \mu_{g4} = E[(X_i^g)^4]$$

be the first, second, third and fourth moments of X_i^g . Then,

$$\begin{aligned} \phi_1(x_1^g) &= E[\phi(X_1^g, X_2^g) \mid X_1^g = x_1^g] = (x_1^g)^2 - 2x_1^g\mu_g + \mu_{g2}, \\ E[\phi_1(X_1^g)] &= E[(X_1^g)^2 - 2X_1^g\mu_g + \mu_{g2}] = 2\sigma_g^2, \\ E(\phi_1^2) &= \mu_{g4} - 4\mu_g\mu_{g3} + 3\mu_{g2}^2. \end{aligned}$$

Therefore,

$$\sqrt{n_g}(\bar{D}_g - 2\sigma^2) \xrightarrow{D} N(0, 4\xi_1), \tag{A.1}$$

where

$$\xi_1 = E[\phi_1(X_1^g) - \theta_g] = \mu_{g4} - 4\mu_g\mu_{g3} + 3\mu_{g2}^2 - 4\sigma_g^4.$$

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For $\bar{D}_{gg'}$, which is a U-statistic of degree (1, 1), let

$$\begin{aligned}\theta_{gg'} &= E(\phi(X_1^g, X_2^{g'})) = \mu_{g2} - 2\mu_g\mu_{g'} + \mu_{g'2}, \\ \phi_{10} &= E[\phi(X_1^g, X_2^{g'}) | X_1^g = x_1^g] = x_1^g - 2x_1^g\mu_{g'} + \mu_{g'2}, \\ \phi_{01} &= E[\phi(X_1^g, X_2^{g'}) | X_2^{g'} = x_2^{g'}] = \mu_{g2} - x_2^{g'}\mu_g + x_2^{g'}.\end{aligned}$$

Then,

$$\xi_{10} = E[\phi_{10}(X_1^g) - \theta_{gg'}^2] = \mu_{g4} - 4\mu_{g3}\mu_{g'} + 2\mu_{g2}\mu_{g'2} - 4\mu_g\mu_{g'}\mu_{g'2} + \mu_{g'2}^2 - \theta_{gg'}^2$$

and

$$\xi_{01} = E[\phi_{01}(X_2^{g'}) - \theta_{gg'}^2] = \mu_{g'4} - 4\mu_{g'3}\mu_g + 2\mu_{g'2}\mu_{g2} - 4\mu_g\mu_{g'}\mu_{g2} + \mu_{g2}^2 - \theta_{gg'}^2.$$

Therefore,

$$Var(\bar{D}_{gg'}) = \frac{1}{n_g}\xi_{10} + \frac{1}{n_{g'}}\xi_{01}. \quad (\text{A.2})$$

As B_n is a linear function of U-statistics ($\bar{D}_g$ and $\bar{D}_{gg'}$), its variance will be dependent on the moments of X_i^g .

For instance, if $G = 2$,

$$Var(B_n) = \frac{n_1^2 n_2^2}{n^2(n-1)^2} 4 \left[\frac{\xi_{10}}{n_1} + \frac{\xi_{01}}{n_2} + \frac{\xi_1}{n_1} + \frac{\xi_1}{n_2} \right] + O\left(\frac{1}{n}\right).$$

For the normal distribution we can apply the property of invariance of maximum likelihood estimators (MLE) to get the estimation of the mean and variance of B_n . Therefore, the MLE estimators of the mean and variance are

$$\begin{aligned}\hat{\mu} &= \bar{X}, \\ \hat{\sigma}^2 &= (n-1)S^2/n.\end{aligned}$$

As $\hat{\sigma}^2$ is biased for σ^2 , we will use S^2 as an estimator of σ^2 , since it is an unbiased estimator for σ^2 . The estimators of the second, third and fourth moments are

$$\begin{aligned}\hat{\mu}_2 &= S^2 + \bar{X}^2, \\ \hat{\mu}_3 &= 3\bar{X}S^2 + \bar{X}^3, \\ \hat{\mu}_4 &= 3S^4 + 6S^2\bar{X}^2 + \bar{X}^4.\end{aligned}$$

For the Triangular distribution

Let Z be a random variable with a triangular distribution limited in $[a, b]$ with mode m , then

$$f(z) = \frac{2(z-a)}{(b-a)(m-a)} I(a \leq z \leq m) + \frac{2(b-z)}{(b-a)(b-m)} I(m \leq z \leq b). \quad (\text{A.3})$$

The k -th moments of Z are

$$\begin{aligned} \mu_k &= c_1 \left(\frac{m^{k+2}}{k+2} - \frac{am^{k+1}}{k+1} - \frac{a^{k+2}}{k+2} + \frac{a^{k+2}}{k+1} \right) \\ &+ c_2 \left(\frac{b^{k+2}}{k+1} - \frac{b^{k+2}}{k+2} - \frac{bm^{k+1}}{k+1} + \frac{m^{k+2}}{k+2} \right), \end{aligned} \quad (\text{A.4})$$

for all $k = 1, 2, \dots$, with $c_1 = \frac{2}{(b-a)(m-a)}$ and $c_2 = \frac{2}{(b-a)(b-m)}$.

If $Z_1, Z_2, \dots, Z_n$ are independent random variables with p.d.f given by (A.3) and $\mathbf{Z} = (Z_{(1)}, Z_{(2)}, \dots, Z_{(n)})$ is the vector of order statistics, the distribution of $\mathbf{Z}$ is given by

$$f(z_{(1)}, z_{(2)}, \dots, z_{(n)}) = \left(\frac{2}{b-a} \right)^n \left\{ \prod_{i=1}^r \frac{z_{(i)} - a}{m-a} \prod_{i=r+1}^n \frac{b - z_{(i)}}{b-m} \right\}, \quad (\text{A.5})$$

where r is such that $Z_{(r)} \leq m < z_{(r+1)}$, $z_{(0)} \equiv a$ and $z_{(n+1)} \equiv b$.

Therefore, for fixed values of a and b , satisfying $a < z_{(1)}$ and $b > z_{(n)}$, we have that

$$\max_{a \leq m \leq b} L(a, m, b \mid \mathbf{z}) = \left(\frac{2}{b-a} \right)^n \{M(a, b, \hat{r}(a, b))\}, \quad (\text{A.6})$$

where

$$\hat{r}(a, b) = \arg \max_{r \in \{1, \dots, n\}} M(a, b, r) \text{ and } M(a, b, r) = \prod_{i=1}^{r-1} \frac{z_{(i)} - a}{z_{(r)} - a} \prod_{i=r+1}^n \frac{b - z_{(i)}}{b - z_{(r)}}.$$

The MLE of m (as a function of a and b) is given by $\hat{m}(a, b) = Z_{(\hat{r}(a, b))}$. Note that the function $\hat{r}(a, b)$ indicates in which order statistic the MLE of m is achieved as a function of the lower and upper limit, a and b , respectively.

From (A.6) we have

$$\max_{S(a, m, b)} [\log\{L(a, m, b; \mathbf{z})\}] = \max_{a < z_{(1)}, b > z_{(n)}} [\log\{n \log 2 + G(a, b)\}], \quad (\text{A.7})$$

where $S(a, b) = \{(a, m, b) \mid a < z_{(1)}, b > z_{(n)}, a \leq m \leq b\}$ and $G(a, b) = \log\{M(a, b, \hat{r}(a, b))\} - n \log(b-a)$.

Note that $G(a, b)$ is defined only for values of $a < z_{(1)}$ and $b > z_{(n)}$. So, the three dimensional problem of maximization of $L(a, m, b; \mathbf{z})$ is reduced to a two dimensional problem of maximization of $G(a, b)$ under $a < z_{(1)}$ and $b > z_{(n)}$. From the likelihood structure, one can see that for all values of m such that $z_{(1)} < m < z_{(n)}$, the likelihood $L(a, m, b; \mathbf{z}) \rightarrow 0$ (and therefore $\log L(a, m, b; \mathbf{z}) \rightarrow \infty$) when $a \uparrow z_{(1)}$ or $b \downarrow z_{(n)}$.

To obtain the MLE for the triangular distribution, Kotz and Dorp (2004) suggests the use of the routines *BSearch* and *ABSearch* combined. The MLE in the triangular case can be obtained using the software *MLE-Estimator* (www.seas.gwu.edu/dorppjr/tab4/publications_book.html).

Appendix B. Algorithm for the quasi U-statistic nonparametric method

Step 1 - Compute the relative gain according to the introduction.

Step 2 - Separate the data set according to the groups of interest.

Step 3 - Compute observed $\bar{D}_g$'s, $\bar{D}_{gg'}$'s and then B_n from the data, call it $B_n(obs)$.

Step 4 - Apply the jackknife re-sampling method to obtain the SE of B_n .

Step 4.1 - Compute the jackknife's replication $B_n(-i)$ by removing from the sample the i -th individual and then apply Step 3 again, for all individuals in the sample, one by one.

Step 4.2 - Compute the $B_n(\cdot) = \sum_{i=1}^n \frac{B_n(-i)}{n}$.

Step 4.3 - The Jackknife Standard Error (SE_J) is then estimated by

$$SE_J = \left[\frac{n-1}{n} \sum_{i=1}^n [B_n(-i) - B_n(\cdot)]^2 \right]^{1/2}.$$

Step 5 - Compute the p -value (one-sided test), (i.e, $P[B_n > B_n(obs)]$), using the normal approximation below for B_n (see details in the main file of the article) and Standard Deviation given by the SE_J calculated in step 4.

$$(M_n V_n^*)^{-1/2} B_n \xrightarrow{\mathcal{D}} N(0, 1) \text{ as } n \rightarrow \infty,$$

with $M_n = O(n^2)$, $V_n^* = U_n^{(2,2)} - U_n^{(3)}$, $U_n^{(2,2)} = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \phi^2(X_i, X_j)$ and $U_n^{(3)} = \frac{1}{n(n-1)(n-2)} \sum_{i \neq j \neq l} \phi(X_i, X_j) \phi(X_i, X_l)$.

References

Kotz, S. and Dorp, J. R. van (2004) *Beyond Beta, other continuous families of distributions with bounded support and applications*. World Scientific Press, Singapore.