Online Supplement to A Stochastic Inventory Model with Price Quotation

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Proof of Lemma 1. Part (a): We proceed as follows: for $j \geq 1$,

$$\begin{aligned}
 \alpha \sum_{k=0}^{j-1} m_{\alpha}(k) \sum_{l=j-k}^{\infty} \psi(l) &= \alpha \sum_{k=0}^{j-1} m_{\alpha}(k) (1 - \Psi(j-k-1)) \\
 &= \alpha \sum_{k=0}^{j-1} m_{\alpha}(k) - \alpha \Psi(j-1) - \sum_{k=0}^{j-1} \sum_{n=1}^{\infty} \alpha^{n+1} \psi^{(n)}(k) \Psi(j-k-1) \\
 &= \alpha M_{\alpha}(j) - \alpha \Psi(j-1) - \sum_{n=1}^{\infty} \alpha^{n+1} \sum_{k=0}^{j-1} \psi^{(n)}(k) \Psi(j-k-1) \\
 &= \alpha M_{\alpha}(j) - \alpha \Psi(j-1) - \sum_{n=1}^{\infty} \alpha^{n+1} \Psi^{(n+1)}(j-1) \\
 &= \alpha M_{\alpha}(j) - \sum_{n=1}^{\infty} \alpha^n \Psi^{(n)}(j-1) = \alpha M_{\alpha}(j) - M_{\alpha}(j) + 1.
 \end{aligned}$$

We used $M_{\alpha}(j) = 1 + \overline{M}_{\alpha}(j-1) = 1 + \sum_{n=1}^{\infty} \alpha^n \Psi^{(n)}(j-1)$ in the last equality.

Part (b): We observe that, for $j \geq 1$,

$$\begin{aligned}
 &\sum_{k=0}^{j-1} m(k) \sum_{l=j-k}^{\infty} (l+k) \psi(l) - \mu M(j) \\
 &= \sum_{k=0}^{j-1} m(k) \left\{ \sum_{l=j-k}^{\infty} (l+k) \psi(l) - \sum_{l=0}^{\infty} l \psi(l) \right\} = \sum_{k=0}^{j-1} m(k) \left\{ k \sum_{l=j-k}^{\infty} \psi(l) - \sum_{l=0}^{j-k-1} l \psi(l) \right\}.
 \end{aligned}$$

It is easy to see that $\sum_{l=0}^{j-k-1} l \psi(l) = (j-k) \Psi(j-k-1) - \sum_{l=0}^{j-k-1} \Psi(l)$, and $\sum_{k=0}^{j-1} k m(k) = j M(j) - \sum_{k=1}^j M(k)$. Then, straightforward calculations yield

$$j M(j) - \sum_{k=1}^j M(k) - j \sum_{k=0}^{j-1} m(k) \Psi(j-k-1) + \sum_{k=0}^{j-1} m(k) \sum_{l=0}^{j-k-1} \Psi(l).$$

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Now the following results are useful.

$$\begin{aligned}
\sum_{k=0}^{j-1} m(k) \Psi(j-k-1) &= \Psi(j-1) + \sum_{k=0}^{j-1} \sum_{n=1}^{\infty} \psi^{(n)}(k) \Psi(j-1-k) \\
&= \Psi(j-1) + \sum_{n=1}^{\infty} \Psi^{(n+1)}(j-1) = M(j) - 1.
\end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned}
\sum_{k=0}^{j-1} m(k) \sum_{l=0}^{j-k-1} \Psi(l) &= \sum_{p=0}^{j-1} \sum_{k+l=p} m(k) \Psi(l) \\
&= \sum_{p=0}^{j-1} (M(p+1) - 1) = \sum_{p=1}^j M(p) - j
\end{aligned} \tag{12}$$

where we used (11) in the second equality. Plugging (11) and (12) into the original expression, then we obtain zero.

Part (c): From direct computations,

$$\begin{aligned}
\alpha \sum_{k=0}^j \psi(k) m_{\alpha}(j-k) &= \alpha \psi(j) + \alpha \sum_{k=0}^j \psi(k) \bar{m}_{\alpha}(j-k) \\
&= \alpha \psi(j) + \alpha \sum_{n=1}^{\infty} \alpha^n \sum_{k=0}^j \psi(k) \psi^{(n)}(j-k) \\
&= \alpha \psi(j) + \sum_{n=1}^{\infty} \alpha^{n+1} \psi^{(n+1)}(j) = \bar{m}_{\alpha}(j).
\end{aligned}$$

Part (d): Using $M_{\alpha}(0) = 0$ and $M_{\alpha}(k) = 1 + \bar{M}_{\alpha}(k-1)$ for $k \geq 1$, we have

$$\begin{aligned}
\alpha \sum_{k=0}^j M_{\alpha}(k) \psi(j-k) &= \alpha \sum_{k=1}^j \left\{ 1 + \sum_{n=1}^{\infty} \alpha^n \Psi^{(n)}(k-1) \right\} \psi(j-k) \\
&= \alpha \sum_{k=0}^{j-1} \psi(k) + \sum_{n=1}^{\infty} \alpha^{n+1} \sum_{k=1}^j \Psi^{(n)}(k-1) \psi(j-k) \\
&= \alpha \Psi(j-1) + \sum_{n=1}^{\infty} \alpha^{n+1} \Psi^{(n+1)}(j-1) = M_{\alpha}(j) - 1.
\end{aligned}$$

□

Proof of Lemma 2. Let $\mathcal{S}$ be the set of states that are reachable when the system starts from S_1 or S_2 . Then, from the system dynamics and the positive average demand, it is clear that the system with state space $\mathcal{S}$ is irreducible and positive recurrent. Therefore, there exists a stationary distribution $P(\cdot)$ from a standard theory of Markov chains. We can set $P(j) = 0$ if $j \notin \mathcal{S}$.

As a next step, we observe that $P(\cdot)$ satisfies the following relationships. For $j > r$, the state j is reachable from the previous period, say $t - 1$, when (1) $x_{t-1} \geq j$ and the demand size was $x_{t-1} - j$, and (2) $x_{t-1} \leq r$, the lower price p_2 was quoted, and the demand size was $S_2 - j$. This can be written as

$$P(j) = \gamma \phi_2 \psi(S_2 - j) + \sum_{k=j}^{S_2} P(k) \psi(k - j), \quad \gamma := \sum_{k=-\infty}^r P(k).$$

Note that γ is the stationary probability that there is a quotation request in a period. This equation is that of (RE-3), and thus has a solution (with $g(j) = \gamma \phi_2 \psi(S_2 - j)$ and $q = S_2$)

$$P(j) = \sum_{k=0}^{S_2-j} \gamma \phi_2 \psi(S_2 - j - k) m(k) = \gamma \phi_2 \bar{m}(S_2 - j)$$

where the second equality is from (c) of Lemma 1. Moreover,

$$1 = \sum_{j=-\infty}^{S_2} P(j) = \gamma + \sum_{j=r+1}^{S_2} P(j) = \gamma \left(1 + \phi_2 \sum_{j=r+1}^{S_2} \bar{m}(S_2 - j) \right) = \gamma (1 + \phi_2 \bar{M}(S_2 - r - 1)).$$

By definition, $1 + \phi_2 \bar{M}(S_2 - r - 1) = \phi_1 + \phi_2 M(S_2 - r)$, from which part (a) is immediate.

Part (b) concerns with the case $S_1 < j \leq r$ which does not exist if $S_1 = r$. Similarly as above, let us consider each of possible scenarios. The state j is reachable if (1) $x_{t-1} > r$ and the demand size was $x_{t-1} - j$, (2) $x_{t-1} \leq r$, the lower price p_2 was quoted, and the demand size was $S_2 - j$, and lastly (3) $j \leq x_{t-1} \leq r$, the higher price p_1 was quoted (hence no order), and the demand size was $x_{t-1} - j$. Mathematically, it is

$$P(j) = \sum_{k=r+1}^{S_2} P(k) \psi(k - j) + \gamma \phi_2 \psi(S_2 - j) + \phi_1 \sum_{k=j}^r P(k) \psi(k - j).$$

Let us denote the first two terms by $g(j)$ which is simplified to $\gamma \phi_2 \sum_{k=r+1}^{S_2} \psi(k - j) m(S_2 - k)$. Then, again the above equation is of (RE-3) and thus $P(j) = \sum_{k=0}^{r-j} m_{\phi_1}(k) g(j + k)$, which is exactly the formula in (b). $\square$

Proof of (5). We first compute

$$\begin{aligned} \sum_{k=0}^{S_2-r-1} \psi(k) f(S_2 - k) &= \sum_{k=0}^{S_2-r-1} \psi(k) \sum_{l=0}^{S_2-k-r-1} L(S_2 - k - l) m(l) \\ &= \sum_{p=0}^{S_2-r-1} \sum_{k+l=p} L(S_2 - p) \psi(k) m(l) \\ &= \sum_{p=0}^{S_2-r-1} L(S_2 - p) \bar{m}(p) \end{aligned}$$

$$= \sum_{p=0}^{S_2-r-1} L(S_2-p)m(p) - L(S_2).$$

The third equality uses part (c) of Lemma 1. Let us write the first term in the last expression by Π . Then, it is immediate that $\mathcal{C}_2(j) = \phi_2(p_2(S_2-j) + \Pi)$. Plugging this into $\mathbb{E}C$ and regrouping terms, we can see that

$$\begin{aligned} \mathbb{E}C &= \phi_1 L(S_1) + \phi_2 \Pi + \sum_{j=-\infty}^{S_1} \tilde{P}(j) \phi_1 p_1(S_1-j) + \sum_{j=-\infty}^r \tilde{P}(j) \phi_2 p_2(S_2-j) \\ &\quad + \sum_{j=S_1+1}^r \tilde{P}(j) \phi_1 (L(j) - L(S_1)) + Q, \end{aligned}$$

and the third, fourth terms can be re-written as

$$\sum_{j=-\infty}^{S_1} \tilde{P}(j) \phi_1 (p_1 - \lambda)(S_1-j) + \sum_{j=-\infty}^r \tilde{P}(j) \phi_2 (p_2 - \lambda)(S_2-j) + \Upsilon$$

with $\Upsilon = \gamma^{-1} \lambda \left(\phi_1 \sum_{j=-\infty}^{S_1} P(j)(S_1-j) + \phi_2 \sum_{j=-\infty}^r P(j)(S_2-j) \right)$. Note that the expression in the parenthesis is the long-run average of order quantities per period. In the stationary system, this quantity must be equal to the expected demand per period; otherwise, the system accumulates infinite inventory or infinite back-orders. Therefore, $\Upsilon = \gamma^{-1} \lambda \mu$. We also note that $\phi_1(p_1 - \lambda) = \phi_1 \phi_2(p_1 - p_2) = -\phi_2(p_2 - \lambda)$ from $\lambda = \phi_1 p_1 + \phi_2 p_2$. Putting these altogether and using $\sum_{j=-\infty}^r \tilde{P}(j) = 1$, it is not difficult to see that

$$\mathbb{E}C = \phi_1 L(S_1) + \phi_2 \Pi + \phi_1 \phi_2 (p_2 - p_1)(S_2 - S_1) + \gamma^{-1} \lambda \mu + \gamma^{-1} \phi_1 \sum_{j=S_1+1}^r P(j) (\bar{L}(j) - \bar{L}(S_1)) + Q$$

where $\bar{L}(j) - \bar{L}(S_1) = L(j) - L(S_1) + \phi_2(p_1 - p_2)(j - S_1)$ from the definition of $L(\cdot)$. As a last step, we use part (b) of Lemma 2. For the summation, it is equal to

$$\begin{aligned} &\sum_{j=S_1}^r P(j) (\bar{L}(j) - \bar{L}(S_1)) \\ &= \sum_{j=S_1}^r \sum_{k=0}^{r-j} m_{\phi_1}(k) \left(\gamma \phi_2 \sum_{l=r+1}^{S_2} m(S_2-l) \psi(l-j-k) \right) (\bar{L}(j) - \bar{L}(S_1)) \\ &= \gamma \phi_2 \sum_{j=S_1}^r \sum_{k'=j}^r \sum_{l=r+1}^{S_2} m_{\phi_1}(k' - j) m(S_2-l) \psi(l-k') (\bar{L}(j) - \bar{L}(S_1)) \\ &= \gamma \phi_2 \sum_{l=r+1}^{S_2} \sum_{k'=S_1}^r \sum_{j=S_1}^{k'} m_{\phi_1}(k' - j) m(S_2-l) \psi(l-k') (\bar{L}(j) - \bar{L}(S_1)). \end{aligned}$$

Adding this term to $\mathbb{E}C$, the desired result is obtained. $\square$

Proof of Lemma 3. First, consider the case $r < S_1 \leq S_2$. Let (r^u, S_1^u, S_2^u) be an optimal (r, S_1, S_2) policy with the largest S_2 , i.e., $S_2^u = \max\{S_2^* : c_1(r^*, S_1^*, S_2^*) = \min_{\{r < S_1 \leq S_2\}} c_1(r, S_1, S_2)\}$. Suppose that $S_2^u < y_2^*$. We, then, claim that $c_1(r^u + 1, S_1^u + 1, S_2^u + 1) \leq c_1(r^u, S_1^u, S_2^u)$. Indeed, straightforward computations using (4) lead us to

$$\begin{aligned} & (c_1(r^u + 1, S_1^u + 1, S_2^u + 1) - c_1(r^u, S_1^u, S_2^u)) (\phi_1 M(S_1^u - r^u) + \phi_2 M(S_2^u - r^u)) \\ &= \phi_1 \sum_{j=0}^{S_1^u - r^u - 1} m(j) \Delta L(S_1^u - j) + \phi_2 \sum_{j=0}^{S_2^u - r^u - 1} m(j) \Delta L(S_2^u - j). \end{aligned}$$

Now, due to the assumption that $S_1^u \leq S_2^u < y_2^*$, we have $S_i^u - j < y_2^*$ for all $0 \leq j \leq S_i^u - r^u - 1$ and for $i = 1, 2$. Since we have $\Delta L(y) \leq 0$ for $y < y_2^*$, the right side of the inequality is non-positive, and thus the claim is proved. However, this is a contradiction to the maximality of S_2^u .

Let us turn to the second case, $S_1 \leq r < S_2$. Similarly as above, we define (r^u, S_1^u, S_2^u) where minimum of $c_2(\cdot)$ is taken over the set $\{S_1 \leq r < S_2\}$. Straightforwardly again, we obtain

$$\begin{aligned} & (c_2(r^u + 1, S_1^u + 1, S_2^u + 1) - c_2(r^u, S_1^u, S_2^u)) (\phi_1 + \phi_2 M(S_2^u - r^u)) \\ &= \phi_1 \Delta L(S_1^u) \left(1 - \phi_2 \sum_{j=r^u+1}^{S_2^u} m(S_2^u - j) \sum_{k=S_1^u}^{r^u} \psi(j - k) \sum_{l=S_1^u}^k m_{\phi_1}(k - l) \right) \\ &+ \phi_2 \sum_{j=0}^{S_2^u - r^u - 1} m(j) \Delta L(S_2^u - j) \\ &+ \phi_1 \phi_2 \sum_{j=r^u+1}^{S_2^u} m(S_2^u - j) \sum_{k=S_1^u}^{r^u} \psi(j - k) \sum_{l=S_1^u}^k m_{\phi_1}(k - l) \Delta L(l). \end{aligned}$$

The following fact is proved below: for any $S_1 \leq r < S_2$,

$$1 - \phi_2 \sum_{j=r+1}^{S_2} m(S_2 - j) \sum_{k=S_1}^r \psi(j - k) \sum_{l=S_1}^k m_{\phi_1}(k - l) > 0. \quad (13)$$

Again using $\Delta L(y) \leq 0$ for $y < y_2^*$, if $S_2^u < y_2^*$, then $(r^u + 1, S_1^u + 1, S_2^u + 1)$ would be also optimal and this is a contradiction to the maximality of S_2^u . Hence, we have $S_2^u \geq y_2^*$ for this case as well. $\square$

Proof of (13). For any $S_1 \leq r < S_2$, we have

$$\begin{aligned} & 1 - \phi_2 \sum_{j=r+1}^{S_2} m(S_2 - j) \sum_{k=S_1}^r \psi(j - k) \sum_{l=S_1}^k m_{\phi_1}(k - l) \\ &= 1 - \phi_2 \sum_{j=r+1}^{S_2} m(S_2 - j) \sum_{k=S_1}^r \psi(j - k) M_{\phi_1}(k - S_1 + 1) \end{aligned}$$

by definition of $M_{\phi_1}(\cdot)$. If we set $t = S_2 - S_1 + 1$, $n = S_2 - r$, $p = S_2 - j$, and $q = k - S_1 + 1$, then the above expression becomes (using $i = t - q$ and $M_{\phi_1}(0) = 0$)

$$\begin{aligned}
& 1 - \phi_2 \sum_{p=0}^{n-1} m(p) \sum_{q=1}^{t-n} \psi(t-p-q) M_{\phi_1}(q) \\
&= 1 - \phi_2 \sum_{p=0}^{n-1} m(p) \left\{ \sum_{q=0}^{t-p} \psi(t-p-q) M_{\phi_1}(q) - \sum_{q=t-n+1}^{t-p} \psi(t-p-q) M_{\phi_1}(q) \right\} \\
&= 1 - \phi_2 \sum_{p=0}^{n-1} m(p) \sum_{q=0}^{t-p} \psi(t-p-q) M_{\phi_1}(q) + \phi_2 \sum_{p=0}^{n-1} \sum_{i=p}^{n-1} m(p) \psi(i-p) M_{\phi_1}(t-i) \\
&= 1 - \phi_2 \sum_{p=0}^{n-1} m(p) (M_{\phi_1}(t-p) - 1) / \phi_1 + \phi_2 \sum_{i=0}^{n-1} \sum_{p=0}^i m(p) \psi(i-p) M_{\phi_1}(t-i) \\
&= 1 - \phi_2 \sum_{p=0}^{n-1} m(p) (M_{\phi_1}(t-p) - 1) / \phi_1 + \phi_2 \sum_{i=0}^{n-1} m(i) M_{\phi_1}(t-i) - \phi_2 M_{\phi_1}(t)
\end{aligned}$$

where we used part (d) of Lemma 1 in the third equality, and part (c) of the same lemma together with definition of $m(\cdot)$ in the last equality. Now, rearranging terms, we get

$$\begin{aligned}
& 1 - \phi_2 M_{\phi_1}(t) + \frac{\phi_2}{\phi_1} \sum_{p=0}^{n-1} m(p) + \phi_2 \sum_{p=0}^{n-1} m(p) \left(1 - \frac{1}{\phi_1} \right) M_{\phi_1}(t-p) \\
&= 1 - \phi_2 M_{\phi_1}(t) + \frac{\phi_2}{\phi_1} \sum_{p=0}^{n-1} m(p) (1 - \phi_2 M_{\phi_1}(t-p)).
\end{aligned}$$

Moreover, recall that $M_{\phi_1}(\cdot)$ satisfies, for $x \geq 1$,

$$M_{\phi_1}(x) = 1 + \sum_{j=1}^{\infty} \phi_1^j \Psi^{(j)}(x-1) \leq 1 + \sum_{j=1}^{\infty} \phi_1^j = \frac{1}{\phi_2}.$$

Since $M_{\phi_1}(0) = 0$, the above expression is clearly non-negative. Actually, it is necessarily strictly positive. If it were zero, then, for instance, $\Psi^{(j)}(t) = \mathbb{P}(\xi_1 + \dots + \xi_j \leq t) = 1$ for all $j \geq 1$ where the ξ_i denote i.i.d. demand random variables. However, this conclusion is absurd because ξ_i is assumed to have positive mean, and thus $\sum_{i=1}^j \xi_i$ diverges to infinity almost surely. $\square$

Proof of Lemma 4. We prove part (a) only as part (b) can be treated in a similar fashion. As in the proof of Lemma 3, we write (r^l, S_1^l, S_2^l) for an optimal policy with the smallest r , i.e., $r^l = \min\{r^* : c_1(r^*, S_1^*, S_2^*) = \min_{\{r < S_1 \leq S_2\}} c_1(r, S_1, S_2)\}$. Clearly, such a finite r^l exists; otherwise, we would never replenish the inventory and this is absurd. Then, from (4),

$$\left(c_1(r^l, S_1^l, S_2^l) - c_1(r^l - 1, S_1^l - 1, S_2^l - 1) \right) \left(\phi_1 M(S_1^l - r^l) + \phi_2 M(S_2^l - r^l) \right)$$

$$= \phi_1 \sum_{j=0}^{S_1^l - r^l - 1} m(j) \Delta L(S_1^l - j - 1) + \phi_2 \sum_{j=0}^{S_2^l - r^l - 1} m(j) \Delta L(S_2^l - j - 1).$$

We note that $\Delta L(y) \geq 0$ for all $y \geq y_1^*$ by the definition of y_1^* . Suppose that $r^l \geq y_1^*$. Then, $\Delta L(S_1^l - j - 1) \geq 0$ for any $j = 0, \dots, S_2^l - r^l - 1$. This implies that $(r^l - 1, S_1^l - 1, S_2^l - 1)$ is also optimal and this is a contradiction to the definition of r^l . Therefore, $r^l > y_1^*$. $\square$

Proof of Lemma 5. For part (a), we simply note that

$$\begin{aligned} \Delta L(y) &= L(y+1) - L(y) = -b + (b+h)\Psi(y) \uparrow h, \\ \Delta^2 L(y) &= \Delta L(y+1) - \Delta L(y) = (b+h)\psi(y+1) \geq 0. \end{aligned}$$

Since $\Delta \bar{L}(y) = \Delta L(y) + \phi_2(p_1 - p_2)$ and $\Delta^2 \bar{L}(y) = \Delta^2 L(y)$, it is clear that $\bar{L}(\cdot)$ is convex, and that $\Delta \bar{L}(y) = -b + \phi_2(p_1 - p_2)$ for any $y < 0$ and $\Delta \bar{L}(y) > 0$ for all sufficiently large y .

Suppose now that $\bar{y}^* > y_1^*$. Then, $\bar{L}(\bar{y}^* - 1) > \bar{L}(\bar{y}^*)$ by definition of $\bar{y}^*$. However, $\Delta \bar{L}(\bar{y}^* - 1) = \Delta L(\bar{y}^* - 1) + \phi_2(p_1 - p_2) \geq \phi_2(p_1 - p_2) > 0$. Since this is contradictory, we conclude that $\bar{y}^* \leq y_1^*$. As for the second statement of part (a), observe that if $b > \phi_2(p_1 - p_2)$, then $L(\cdot)$ is strictly decreasing in $(-\infty, 0)$ and thus $\bar{y}^*$ is finite. If $b \leq \phi_2(p_1 - p_2)$, then $\bar{L}(\cdot)$ is nondecreasing, leading to $\bar{y}^* = -\infty$.

A proof of part (b) is based on straightforward calculations, but more involved. First, by a change of variable, we have

$$\hat{L}(y) = \phi_2 \sum_{k=0}^{y-\bar{y}^*} m_{\phi_1}(k) (\bar{L}(y-k) - \bar{L}(\bar{y}^*)) - \phi_2(p_1 - p_2)y + \bar{L}(\bar{y}^*),$$

which in turn yields

$$\Delta \hat{L}(y) = \phi_2 \sum_{k=0}^{y-\bar{y}^*} m_{\phi_1}(k) \Delta \bar{L}(y-k) - \phi_2(p_1 - p_2). \quad (14)$$

Using this formula, we arrive at

$$\Delta^2 \hat{L}(y) = \phi_2 m_{\phi_1}(y+1 - \bar{y}^*) \Delta \bar{L}(\bar{y}^*) + \phi_2 \sum_{k=0}^{y-\bar{y}^*} m_{\phi_1}(k) \Delta^2 L(y-k) \geq 0,$$

which means that $\hat{L}(\cdot)$ is convex. Here, we used that $\bar{L}(\cdot)$ is minimized at $\bar{y}^*$ and that $L(\cdot)$ is convex. On the other hand, from (14),

$$\begin{aligned} \Delta \hat{L}(y) &= \phi_2 \sum_{k=0}^{y-\bar{y}^*} m_{\phi_1}(k) \Delta L(y-k) + \phi_2^2(p_1 - p_2) M_{\phi_1}(y+1 - \bar{y}^*) - \phi_2(p_1 - p_2) \\ &\leq \phi_2 M_{\phi_1}(y+1 - \bar{y}^*) \Delta L(y) + \phi_2^2(p_1 - p_2) M_{\phi_1}(y+1 - \bar{y}^*) - \phi_2(p_1 - p_2) \\ &= \Delta \bar{L}(y) (\phi_2 M_{\phi_1}(y+1 - \bar{y}^*) - 1) + \Delta L(y), \end{aligned}$$

where we used $\Delta L(\cdot)$ is nondecreasing in the inequality. Since $M_{\phi_1}(\cdot) \leq \sum_{j=0}^{\infty} \phi_1^j = 1/\phi_2$ and $\Delta \bar{L}(y) \geq 0$ for $y \geq \bar{y}^*$, we conclude that $\Delta \hat{L}(y) \leq \Delta L(y)$ for any $y \geq \bar{y}^*$.

To show $\hat{y}^*$ is finite and not smaller than y_2^* , we assume that $\hat{y}^* < y_2^*$. Then, by definition of $\hat{y}^*$, $\Delta \hat{L}(\hat{y}^*) > 0$. From the above result, this implies that $\Delta L(y_2^* - 1) \geq \Delta L(\hat{y}^*) \geq \Delta \hat{L}(\hat{y}^*) > 0$. This is then a contradiction to the definition of y_2^* . Hence, $\hat{y}^* \geq y_2^*$. We still need to prove that $\hat{y}^*$ is finite. To this end, consider (14). It was re-written as follows above:

$$\Delta \hat{L}(y) = \phi_2 \sum_{k=0}^{y-\bar{y}^*} m_{\phi_1}(k) \Delta L(y-k) - \phi_2(p_1 - p_2) (1 - \phi_2 M_{\phi_1}(y+1 - \bar{y}^*)).$$

The second term is easily seen to converge to zero as y increases. On the other hand, the first term is greater than or equal to

$$\phi_2 m_{\phi_1}(0) \Delta L(y) + \phi_2 \sum_{k=y-y_1^*}^{y-\bar{y}^*} m_{\phi_1}(k) \Delta L(y-k),$$

using the nonnegativity of $\Delta L(y)$ for $y \geq y_1^*$. It is clear that $\lim_{k \rightarrow \infty} m_{\phi_1}(k) = 0$. Thus, $\lim_{y \rightarrow \infty} \Delta \hat{L}(y) \geq \phi_2 m_{\phi_1}(0)h = \phi_2 h / (1 - \phi_1 \psi(0)) > 0$. Hence, $\hat{L}(y)$ is strictly increasing for all sufficiently large y , and therefore $\hat{y}^*$ is finite. $\square$

Proof of Lemma 6. For the first equality, we note that

$$\begin{aligned} & (\alpha \cdot c_1(r, S_1, S_2) + (1 - \alpha)L(r)) (\phi_1 M(S_1 - r + 1) + \phi_2 M(S_2 - r + 1)) \\ &= \phi_1 \sum_{j=0}^{S_1-r-1} m(j) L(S_1 - j) + \phi_2 \sum_{j=0}^{S_2-r-1} m(j) L(S_2 - j) \\ & \quad + \phi_1 \phi_2 (p_2 - p_1)(S_2 - S_1) + (\phi_1 m(S_1 - r) + \phi_2 m(S_2 - r)) L(r) \\ &= \phi_1 \sum_{j=0}^{S_1-r} m(j) L(S_1 - j) + \phi_2 \sum_{j=0}^{S_2-r} m(j) L(S_2 - j) + \phi_1 \phi_2 (p_2 - p_1)(S_2 - S_1) \\ &= c_1(r-1, S_1, S_2) (\phi_1 M(S_1 - r + 1) + \phi_2 M(S_2 - r + 1)). \end{aligned}$$

Since $M(x) > 0$ for $x \geq 1$ and non-decreasing in x , it is obvious that $\alpha \in (0, 1]$. For the second inequality, we proceed as follows:

$$\begin{aligned} & (\beta c_2(S_1, S_1, S_2) + (1 - \beta)L(S_1)) (\phi_1 M(1) + \phi_2 M(S_2 - S_1 + 1)) \\ &= \phi_1 L(S_1) + \phi_2 \sum_{j=0}^{S_2-S_1-1} m(j) L(S_2 - j) + Q + \phi_1 \phi_2 (p_2 - p_1)(S_2 - S_1) \\ & \quad + (\phi_1 \bar{m}(0) + \phi_2 m(S_2 - S_1)) L(S_1) \\ &= \phi_1 m(0) L(S_1) + \phi_2 \sum_{j=0}^{S_2-S_1} m(j) L(S_2 - j) + Q + \phi_1 \phi_2 (p_2 - p_1)(S_2 - S_1) \\ &= c_1(S_1 - 1, S_1, S_2) (\phi_1 M(1) + \phi_2 M(S_2 - S_1 + 1)). \end{aligned}$$

It is also clear that $\beta \in (0, 1]$. $\square$

Proof of Lemma 7. It is obvious that $\gamma \in (0, 1]$. Now, let us consider the case: $r + 1 < S_2$. Recall that we have $S_1 = \bar{y}^* \leq r < S_2$. To simplify notation, we set

$$F(r, S_1, S_2) = \sum_{j=r+1}^{S_2} m(S_2 - j) \sum_{k=S_1}^r \psi(j - k) \sum_{l=S_1}^k m_{\phi_1}(k - l) (\bar{L}(l) - \bar{L}(S_1)).$$

Then, from (6),

$$\begin{aligned} & c_2(r, S_1, S_2)(\phi_1 + \phi_2 M(S_2 - r)) - c_2(r + 1, S_1, S_2)(\phi_1 + \phi_2 M(S_2 - r - 1)) \\ &= \phi_2 m(S_2 - r - 1) L(r + 1) + \phi_1 \phi_2 F(r, S_1, S_2) - \phi_1 \phi_2 F(r + 1, S_1, S_2) \\ &= \phi_2 m(S_2 - r - 1) L(r + 1) \\ &\quad - \phi_1 \phi_2 \sum_{j=r+1}^{S_2} m(S_2 - j) \psi(j - r - 1) \sum_{l=S_1}^{r+1} m_{\phi_1}(r + 1 - l) (\bar{L}(l) - \bar{L}(S_1)) \quad (I) \\ &\quad + \phi_1 \phi_2 m(S_2 - r - 1) \sum_{k=S_1}^{r+1} \psi(r + 1 - k) \sum_{l=S_1}^k m_{\phi_1}(k - l) (\bar{L}(l) - \bar{L}(S_1)). \quad (II) \end{aligned}$$

As for (I), note that

$$\sum_{j=r+1}^{S_2} m(S_2 - j) \psi(j - r - 1) = \sum_{k=0}^{S_2 - r - 1} m(S_2 - r - 1 - k) \psi(k) = \bar{m}(S_2 - r - 1)$$

where we used Lemma 1, part (c) in the second equality. Since $r + 1 < S_2$, the right side is just $m(S_2 - r - 1)$. For (II),

$$\begin{aligned} & \phi_1 \sum_{k=S_1}^{r+1} \psi(r + 1 - k) \sum_{l=S_1}^k m_{\phi_1}(k - l) (\bar{L}(l) - \bar{L}(S_1)) \\ &= \sum_{l=S_1}^{r+1} \left\{ \phi_1 \sum_{k=l}^{r+1} \psi(r + 1 - k) m_{\phi_1}(k - l) \right\} (\bar{L}(l) - \bar{L}(S_1)) \\ &= \sum_{l=S_1}^{r+1} \bar{m}_{\phi_1}(r + 1 - l) (\bar{L}(l) - \bar{L}(S_1)) \\ &= \sum_{l=S_1}^{r+1} m_{\phi_1}(r + 1 - l) (\bar{L}(l) - \bar{L}(S_1)) - (\bar{L}(r + 1) - \bar{L}(S_1)). \end{aligned}$$

Plugging these results into the original formula, we get $\phi_2 m(S_2 - r - 1) \hat{L}(r + 1)$. Now, it is a simple algebra to get the desired expression.

Let us turn our attention to part (b). When $r + 1 = S_2$, we have $M(1) = m(0)$ by definition and thus

$$\begin{aligned} & c_2(r, S_1, S_2) (\phi_1 + \phi_2 m(0)) \\ &= \phi_1 L(S_1) + \phi_2 m(0) L(S_2) + Q + \phi_1 \phi_2 (p_2 - p_1) (S_2 - S_1) \end{aligned}$$

$$+\phi_1\phi_2m(0)\sum_{k=S_1}^r\psi(r+1-k)\sum_{l=S_1}^km_{\phi_1}(k-l)(\bar{L}(l)-\bar{L}(S_1)). \quad (III)$$

The last term (III) can be simplified by observing that

$$\begin{aligned} & \phi_1\phi_2m(0)\sum_{k=S_1}^{r+1}\psi(r+1-k)\sum_{l=S_1}^km_{\phi_1}(k-l)(\bar{L}(l)-\bar{L}(S_1)) \\ &= \phi_1\phi_2m(0)\sum_{l=S_1}^{r+1}\sum_{k=l}^{r+1}\psi(r+1-k)m_{\phi_1}(k-l)(\bar{L}(l)-\bar{L}(S_1)) \\ &= \phi_2m(0)\sum_{l=S_1}^{r+1}\bar{m}_{\phi_1}(r+1-l)(\bar{L}(l)-\bar{L}(S_1)) \\ &= \phi_2m(0)\sum_{l=S_1}^{r+1}m_{\phi_1}(r+1-l)(\bar{L}(l)-\bar{L}(S_1)) - \phi_2m(0)(\bar{L}(S_2)-\bar{L}(S_1)) \\ &= m(0)\left\{\widehat{L}(S_2)-L(S_2)+\bar{L}(S_2)-\bar{L}(S_1)\right\} - \phi_2m(0)(\bar{L}(S_2)-\bar{L}(S_1)) \end{aligned}$$

where we used part (c) of Lemma 1 in the second equality and the definition of $\widehat{L}(\cdot)$ in the last equality. Similarly, we have

$$\begin{aligned} & \phi_1\phi_2m(0)\psi(0)\sum_{l=S_1}^{r+1}m_{\phi_1}(r+1-l)(\bar{L}(l)-\bar{L}(S_1)) \\ &= \phi_1m(0)\psi(0)\left\{\widehat{L}(S_2)-L(S_2)+\bar{L}(S_2)-\bar{L}(S_1)\right\}. \end{aligned}$$

Also, $m(0)(1-\psi(0)) = 1$ by definition, and

$$m(0)(1-\phi_1\psi(0)) = m(0)(\phi_1(1-\psi(0)) + \phi_2) = \phi_1 + \phi_2m(0).$$

Then, regrouping terms, we get

$$\begin{aligned} & c_2(r, S_1, S_2)(\phi_1 + \phi_2m(0)) \\ &= \phi_1L(S_1) + Q + \phi_1\phi_2(p_2 - p_1)(S_2 - S_1) + m(0)(1 - \phi_1\psi(0))\widehat{L}(S_2) \\ &\quad - \phi_1m(0)(1 - \psi(0))\{L(S_2) - \bar{L}(S_2) + \bar{L}(S_1)\} \\ &= Q + \phi_1\phi_2(p_2 - p_1)(S_2 - S_1) + (\phi_1 + \phi_2m(0))\widehat{L}(S_2) \\ &\quad + \phi_1\{L(S_1) - L(S_2) + \bar{L}(S_2) - \bar{L}(S_1)\} \\ &= Q + (\phi_1 + \phi_2m(0))\widehat{L}(S_2). \end{aligned}$$

Now, the result is immediate because $\gamma = \phi_1/(\phi_1 + \phi_2m(0))$. □

Proof of Proposition 7. First note that we can set an optimal $S_1^o = r$ for fixed r and S_2 thanks to Proposition 4. Let us write $c_2(r)$ for $c_2(r, r, S_2)$ to simplify notation. A proof consists of several steps.

Step 1: We claim that there exists $r < \bar{y}^*$ such that $c_2(r) \leq L(r)$. We further claim that $c_2(\tilde{r}) \leq L(\tilde{r})$ for any $\tilde{r} \leq r$ and this leads to $c_2(\tilde{r}) \leq c_2(\tilde{r} - 1)$. To show the first part, we observe from (6) that

$$\begin{aligned} & (c_2(r) - L(r)) (\phi_1 + \phi_2 M(S_2 - r)) \\ &= \phi_2 \sum_{j=0}^{S_2-r-1} m(j) (L(S_2 - j) - L(r)) + Q + \phi_1 \phi_2 (p_2 - p_1)(S_2 - r). \end{aligned}$$

Since $L(y)$ is nonincreasing for $y \leq y_1^*$ and $\lim_{y \rightarrow -\infty} L(y) = \infty$, it is possible to choose $r < \bar{y}^*$ such that $L(S_2 - j) \leq L(r)$ for any $j = 0, \dots, S_2 - r - 1$. Taking a smaller r if necessary, we can make $Q + \phi_1 \phi_2 (p_2 - p_1)(S_2 - r) \leq 0$. This proves the first claim. Let us fix such r . It is then enough to show $c_2(r - 1) \leq L(r - 1)$ for the second part, and this is based on the following computations: using $M(S_2 - r + 1) \geq M(S_2 - r)$, $c_2(r) \leq L(r)$, and the above equality,

$$\begin{aligned} & \left\{ (c_2(r) - L(r)) - (c_2(r - 1) - L(r - 1)) \right\} (\phi_1 + \phi_2 M(S_2 - r + 1)) \\ & \geq \phi_2 \sum_{j=0}^{S_2-r} m(j) (L(r - 1) - L(r)) - \phi_1 \phi_2 (p_2 - p_1), \end{aligned}$$

which is nonnegative because $L(y)$ is nonincreasing for $y \leq y_1^*$ and $p_1 > p_2$. Now, for the last part, note the following relationship which is proved below:

$$c_2(\tilde{r} - 1) = \delta \cdot c_2(\tilde{r}) + (1 - \delta)L(\tilde{r}) + \Pi \quad (15)$$

where $\delta = (\phi_1 + \phi_2 M(S_2 - \tilde{r})) / (\phi_1 + \phi_2 M(S_2 - \tilde{r} + 1))$ and $\Pi = -\phi_1 \Delta \bar{L}(\tilde{r} - 1) / (\phi_1 + \phi_2 M(S_2 - \tilde{r} + 1))$. Indeed,

$$\begin{aligned} & c_2(r - 1) (\phi_1 + \phi_2 M(S_2 - r + 1)) - c_2(r) (\phi_1 + \phi_2 M(S_2 - r)) \\ &= \phi_1 L(r - 1) + \phi_2 \sum_{j=0}^{S_2-r} m(j) L(S_2 - j) + Q + \phi_1 \phi_2 (p_2 - p_1)(S_2 - r + 1) \\ & \quad - \left\{ \phi_1 L(r) + \phi_2 \sum_{j=0}^{S_2-r-1} m(j) L(S_2 - j) + Q + \phi_1 \phi_2 (p_2 - p_1)(S_2 - r) \right\} \\ &= -\phi_1 \Delta L(r - 1) + \phi_2 m(S_2 - r) L(r) + \phi_1 \phi_2 (p_2 - p_1) \\ &= -\phi_1 \Delta \bar{L}(r - 1) + \phi_2 m(S_2 - r) L(r). \end{aligned}$$

Then, we have, for $\tilde{r} \leq r$ such that $c_2(\tilde{r}) \leq L(\tilde{r})$,

$$c_2(\tilde{r} - 1) - c_2(\tilde{r}) = (1 - \delta)(L(\tilde{r}) - c_2(\tilde{r})) + \Pi \geq \Pi$$

where we used that $\delta \in (0, 1]$ and the assumption that $c_2(\tilde{r}) \leq L(\tilde{r})$. Now, since $\tilde{r} < \bar{y}^*$, $\Delta \bar{L}(\tilde{r} - 1) \leq 0$ and thus $\Pi \geq 0$. Therefore, we conclude that $c_2(\tilde{r}) \leq c_2(\tilde{r} - 1)$.

Step 2: The previous step implies that $r^l = \max \{r < \bar{y}^* : c_2(r) \leq L(r)\}$ is well defined and that $c_2(r^l) \leq c_2(r)$ for any $r \leq r^l$. Therefore, an optimal policy r^* with $r^* \geq r^l$ exists for $S_1^o = r^*$ and for given

S_2 . In this step, we suppose that $S_2 > \bar{y}^*$. If $c_2(\bar{y}^*) \leq L(\bar{y}^*)$, then the policy $(\bar{y}^*, \bar{y}^*, S_2) \in W_1$ is at least as good as any policy in W_2 . Indeed, nothing prevents us from extending the previous arguments in Step 1 to the region $\{r \leq \bar{y}^*\}$. If $c_2(\bar{y}^*) > L(\bar{y}^*)$, then we claim that $c_2(r) \geq c_1(r, r+1, S_2)$ for $r = r^l, \dots, \bar{y}^* - 1$, which implies that we do not need to search W_2 for an optimal policy in this case. By the assumption and the definition of r^l , we have $c_2(r+1) > L(r+1)$ for any r in the set $\{r^l, \dots, \bar{y}^* - 1\}$. Now, part (b) of Lemma 6 tells us

$$c_1(r, r+1, S_2) = \beta \cdot c_2(r+1) + (1-\beta)L(r+1), \quad \beta = \frac{\phi_1 + \phi_2 M(S_2 - r - 1)}{\phi_1 M(1) + \phi_2 M(S_2 - r)}.$$

It is clear that $\beta \leq \delta$ in (15) with $\tilde{r} = r+1$. Observe

$$\begin{aligned} c_1(r, r+1, S_2) &= \beta (c_2(r+1) - L(r+1)) + L(r+1) \\ &\leq \delta (c_2(r+1) - L(r+1)) + L(r+1) \\ &= c_2(r) - \Pi \end{aligned}$$

where we utilized (15) and $\Pi = -\phi_1 \Delta \bar{L}(r) / (\phi_1 + \phi_2 M(S_2 - r))$. Since $r+1 \leq \bar{y}^*$, $\Pi \geq 0$ and thus we conclude that $c_1(r, r+1, S_2) \leq c_2(r)$.

Step 3: The last step deals with the remaining case: $S_2 = \bar{y}^*$. Recall that we have $c_2(r^l) \leq c_2(r^l - 1) \leq \dots$ with $r^l < S_2$. Consider $r = r^l, \dots, S_2 - 1$. If $r+1 < S_2$, then Lemma 6 is applicable and we have $c_2(r+1) > L(r+1)$. Hence, the arguments in Step 2 imply that $c_1(r, r+1, S_2) \leq c_2(r)$. Finally, we consider the case $r = S_2 - 1$. We will show that the same conclusion holds by direct computations. By the constraint $S_2 \geq y_2^*$, the assumption $S_2 = \bar{y}^*$ means that $\bar{y}^* = y_1^* = y_2^*$. Note that

$$\begin{aligned} &(c_2(r) - c_1(r, r+1, r+1))m(0)(\phi_1 + \phi_2 m(0)) \\ &= m(0) \{ \phi_1 L(r) + \phi_2 m(0)L(r+1) + Q + \phi_1 \phi_2 (p_2 - p_1) \} - (\phi_1 + \phi_2 m(0)) (m(0)L(r+1) + Q) \\ &= \phi_1 m(0)L(r) - \phi_1 m(0)L(r+1) + \phi_1 Q(m(0) - 1) + \phi_1 \phi_2 (p_2 - p_1)m(0) \\ &= \phi_1 m(0) \{ L(r) - L(r+1) + \phi_2 (p_2 - p_1) \} + \phi_1 Q(m(0) - 1) \\ &= -\phi_1 m(0) \Delta \bar{L}(r) + \phi_1 Q \bar{m}(0). \end{aligned}$$

And the last expression is nonnegative because $r < \bar{y}^*$. The proof is now complete. $\square$

Proof of Proposition 8. We consider two cases separately: (i) $c_1^* \leq c_2^*$ (ii) $c_1^* > c_2^*$.

Step 1: Choose any $c \geq c_1^*$ and denote an optimal policy by $(r^*, S_1^*, S_2^*) \in V$. Let us define an auxiliary function $G(r, y) = Q + \sum_{j=0}^{y-r-1} m(j)L(y-j)$ for $y > r$. Then, from (4), it is immediate to see that

$$c_1(r, S_1, S_2) \left(\phi_1 M(S_1 - r) + \phi_2 M(S_2 - r) \right) = \phi_1 G(r, S_1) + \phi_2 G(r, S_2) + \phi_1 \phi_2 (p_1 - p_2)(S_1 - S_2),$$

which is simplified to $c_1(r, y, y)M(y-r) = G(r, y)$ when $S_1 = S_2 = y > r$. It can be shown that

$$G(r, y) = L(y) + Q(1 - \Psi(y-r-1)) + \sum_{j=0}^{y-r-1} \psi(j)M(y-r-j)c_1(r, y-j, y-j). \quad (16)$$

This equation is shown below. Now, thanks to (16) and $c \geq c_1^*$,

$$\begin{aligned}
& c_1^* \left(\phi_1 M(S_1^* - r^*) + \phi_2 M(S_2^* - r^*) \right) - \phi_1 \phi_2 (p_2 - p_1) (S_2^* - S_1^*) \\
& \geq \sum_{i=1}^2 \phi_i \left\{ L(S_i^*) + Q(1 - \Psi(S_i^* - r^* - 1)) + \sum_{j=0}^{S_i^* - r^* - 1} \psi(j) M(S_i^* - r^* - j) c_1^* \right\} \\
& = \sum_{i=1}^2 \phi_i \left\{ L(S_i^*) + Q(1 - \Psi(S_i^* - r^* - 1)) + (M(S_i^* - r^*) - 1) c_1^* \right\}.
\end{aligned}$$

In the equality, the definition $M(0) = 0$ and part (d) of Lemma 1 are used. This leads us to

$$\begin{aligned}
c_1^* & \geq \sum_{i=1}^2 \phi_i \left\{ L(S_i^*) + Q(1 - \Psi(S_i^* - r^* - 1)) \right\} + \phi_1 \phi_2 (p_2 - p_1) (S_2^* - S_1^*) \\
& \geq \phi_1 L(S_1^*) + \phi_2 L(S_2^*) + \phi_1 \phi_2 (p_2 - p_1) (S_2^* - S_1^*).
\end{aligned}$$

However, the last expression is equal to $\phi_1 \bar{L}(S_1^*) + \phi_2 \tilde{L}(S_2^*)$ which is clearly greater than or equal to $\phi_1 \bar{L}(\bar{y}^*) + \phi_2 \tilde{L}(S_2^*)$. Therefore, $\tilde{L}(S_2^*) \leq \phi_2^{-1} (c_1^* - \phi_1 \bar{L}(\bar{y}^*))$ and this implies that $S_2^* \leq \mathcal{U}(c_1^*)$.

Step 2: An optimal policy (r^*, S_1^*, S_2^*) is found to be in W in this case. Proposition 4 implies that we can set $S_1^* = r^* \wedge \bar{y}^*$. Then, in (5), note that the last term is nonnegative, and therefore,

$$\begin{aligned}
& c_2^* \left(\phi_1 + \phi_2 M(S_2^* - r^*) \right) - \phi_1 \phi_2 (p_2 - p_1) (S_2^* - S_1^*) \\
& \geq \phi_1 L(S_1^*) + \phi_2 \sum_{j=0}^{S_2^* - r^* - 1} m(j) L(S_2^* - j) + Q = \phi_1 (L(S_1^*) + Q) + \phi_2 G(r^*, S_2^*).
\end{aligned}$$

As before, from (16), we have $G(r^*, S_2^*) \geq L(S_2^*) + Q(1 - \Psi(S_2^* - r^* - 1)) + (M(S_2^* - r^*) - 1) c_1^*$. Plugging this into the above expression and rearranging terms, it is not difficult to see that

$$\begin{aligned}
& c_2^* \left(\phi_1 + \phi_2 M(S_2^* - r^*) \right) \\
& \geq \phi_1 (\bar{L}(S_1^*) + Q) + \phi_2 \left\{ \tilde{L}(S_2^*) + Q(1 - \Psi(S_2^* - r^* - 1)) + (M(S_2^* - r^*) - 1) c_1^* \right\}.
\end{aligned}$$

Applying the assumption $c_2^* < c_1^*$ and collecting terms with c_2^* , we get

$$\begin{aligned}
c_2^* & \geq \phi_1 (\bar{L}(S_1^*) + Q) + \phi_2 \left\{ \tilde{L}(S_2^*) + Q(1 - \Psi(S_2^* - r^* - 1)) \right\} \\
& \geq \phi_1 \bar{L}(S_1^*) + \phi_2 \tilde{L}(S_2^*) \geq \phi_1 \bar{L}(\bar{y}^*) + \phi_2 \tilde{L}(S_2^*).
\end{aligned}$$

The same argument as in Case (i) yields the desired result.

Step 3: Lastly, we show (16). The equation is equivalent to the recursive expression:

$$G(r, y) = L(y) + Q \sum_{j=y-r}^{\infty} \psi(j) + \sum_{j=0}^{y-r-1} \psi(j) G(r, y-j).$$

Indeed, for the right hand side, we have

$$\begin{aligned}
& L(y) + Q \sum_{j=y-r}^{\infty} \psi(j) + \sum_{j=0}^{y-r-1} \psi(j) G(r, y-j) \\
&= L(y) + Q \sum_{j=y-r}^{\infty} \psi(j) + \sum_{j=0}^{y-r-1} \psi(j) \left\{ Q + \sum_{k=0}^{y-r-1-j} m(k) L(y-j-k) \right\} \\
&= L(y) + Q + \sum_{j=0}^{y-r-1} \sum_{k=0}^{y-r-1-j} \psi(j) m(k) L(y-j-k) \\
&= L(y) + Q + \sum_{i=0}^{y-r-1} \sum_{j+k=p} \psi(j) m(k) L(y-i) \\
&= L(y) + Q + \sum_{i=0}^{y-r-1} \overline{m}(i) L(y-i) = Q + \sum_{i=0}^{y-r-1} m(i) L(y-i)
\end{aligned}$$

where we used part (c) of Lemma 1 in the fourth equality. The last expression is $G(r, y)$. □