Online Supplement to “A self-similar map of rhythmic components”

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This Online Supplement provides examples and potential applications of the rhythmic building blocks and generative operations described in the main article.

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1. Examples

The use of rhythmic encoding will be applied here, first to a piece of music from the Romantic period, then to a rhythmic figure that widely occurs in music of the past century, and finally to a song where derivation of that figure forms part of the musical surface.

1.1. *Brahms, Symphony No. 4, First Movement, measures 1–16*

The opening theme of the first movement of Brahms, Symphony No. 4 has rhythmic patterns that are parsed very efficiently into rhythmic components mapped to the Sierpinski gasket. The attack pattern of the entire first eight measures of the first violin part can be represented by a single ternary address, 111010. See Figure 1. Measures 9–12 can be represented by the address 11011, measures 13–14 by the address 1101, and measures 15–16 by the address 1110, respectively. The smaller number of digits in the address reflects the relatively local scope of the shorter passages. Specifying addresses on an effectively smaller Sierpinski gasket allows comparison of internal structure without imposing an overall scheme that would be implied by locating all passages, short or long, on the same gasket.

The rhythms represented by these encodings all happen to be strictly d -rhythms, that is, rhythmic configurations that result solely from application of d -operations e and u . This is apparent from the fact that no 2 appears in any of the addresses. The simple derivation and high degree of parallelism are reflected by the large number of 1s and the single, or at most two, 0s in each address.

The rhythm that repeats throughout measures 9–12 of the first violin part is shown in Figure 2 mapped to its address, 11011, on the Sierpinski gasket, drawn to eighth note resolution. The highest point marked (in dark gray) on the gasket is the location given by the address. The locations branching out downward are those for addresses whose

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Figure 1. Brahms, Symphony No. 4, First Movement, First Violin.

Figure 2. Representation of the first violin rhythm in measures 9–12 of Brahms, Symphony No. 4, First Movement, encoded by ternary integer 11011.

associated rhythmic configurations are subsumed by the higher one; that is, locations higher on the gasket are elaborations of those below whenever the higher location has an address identical to that of the lower location except for one or more 0s or 2s in the lower location's address have been replaced by 1s in the higher location's address. Eventually, on the bottom row, are addresses corresponding to single time-displacements.

The accompanying wind and horn parts exhibit a regular, syncopated rhythm, with attacks on the second and fourth beat of measures 1–14. The first eight measures, each of which has the rhythm *ooxooxo* at eighth note resolution, can be represented by address 111120. Again, the 1s denote parallelism at every metrical level above the quarter note level, the 2 reflects a syncopation at the quarter note level, with the final 0 indicating no elaboration or syncopation below the quarter note level.

The melody, largely composed of uneven note durations, is very much in phase with meter, whereas the accompaniment has even note durations, but is out of phase with meter. These two types of congruence between rhythm and meter complement each other

to produce a consistent, but moderate, controlled rhythmic tension.

Although this analysis of the passage might occur straightforwardly to any human with sufficient musical training, this degree of formalization makes it more likely that an automated process might also arrive at such an analysis.

1.2. *The tresillo, or kinka, rhythm*

A different kind of syncopation occurs in the *tresillo*, as it is known in Cuban music. This 3+3+2 rhythm has gained widespread use over the past century, particularly as an element of popular music (Toussaint 2005, 49). In traditional West African music it is known as the *kinka* rhythm.

Within binary meter, attacks that are three time-steps apart create syncopation because the expected attack on the next stronger beat is not present. However unlike the Brahms example, this syncopation is not a displacement of rhythm that is in phase with binary meter. Rather, this rhythm unfolds at a different speed than the pulse at any metrical level of binary meter.

This rhythm is an example of a *maximally even rhythm*, defined by Godfried T. Toussaint as one whose “attack points are distributed in time as evenly as possible” (Toussaint 2013, 123). The almost-even distribution in this case consists of time-points that are multiples of 3. By the unique factorization theorem, no multiple of 3 can be a power of 2, and so this rhythm (as well as with the *bossa nova clave* described below) is by nature non-congruent with binary meter.

Figure 3. Agawu's proposed derivation of the *kinka* rhythm.

Kofu Agawu theorizes that the *kinka* rhythm, shown in the third line of Figure 3, may be derived from the more basic rhythms shown in the first two lines (Agawu 2006, 33). The first two rhythms are in fact highly congruent with binary meter; they would in fact each be encoded by a single ternary address, 110 and 010 respectively. Seeing the syncopated second attack as an addition to a rhythm with such congruence to binary meter recasts the non-binary syncopation of the *tresillo* as a drive toward the upbeat that does belong to a binary congruent rhythm. The lack of any 1s in an address indicates that the single attack represented by that address cannot be combined with any other attacks into a rhythmic configuration represented by a single address.

The *tresillo* is the first half of a number of variations on the *clave* pattern, another influential Afro-Cuban rhythm. One such variation is the *bossa nova clave*, and this rhythm is highly non-congruent with binary meter. When repeated, it consists of five attacks spaced three sixteenth's apart, starting on the second eighth of the second bar. In the sense described in Section 2.2, this rhythm is highly complex, as it achieves no compression whatsoever when encoded as mapped rhythmic components; five components

Figure 4. Bossa nova pattern.

are required in order to account for five attacks, as shown in Figure 4.

The two attacks in the second bar each contain a stronger beat between themselves and the downbeat, in violation of Komar’s constraint in Section 4.1 of the main article, because they precede that downbeat by multiples of 3, rather than powers of 2. In general, not even a single pair of attacks here can result from application of any d -operation. An encoding of this rhythm as described here therefore shifts attention away from schematic predictability toward other explanations for the rhythm’s character, such as the approximate evenness described above, or dynamic expectancy based on repeated exposures.

1.3. “*Superstition*” by Stevie Wonder

Another musical example is taken from Stevie Wonder’s song “Superstition” on his 1972 album *Talking Book*. The drum part unambiguously establishes a 4/4 meter before the song’s signature clavinet riff, which occurs twice before the vocal begins. Here we consider the clavinet note riff as a one-bar loop, reducing ornamental variations to a basic pattern consisting in part of unsyncopated eighth notes that run uninterrupted from an eighth note upbeat through the beginning of the third beat. Compositional aspects of this song have been analyzed at length, but the sole focus here will be the rhythm of that particular figure (Hughes 2003, 140–177).

Figure 5. Clavinet pattern from “Superstition.”

If the beginning of the third beat was followed by two more unsyncopated eighths then the rhythm would simply consist entirely of unbroken eighth notes. Instead, the two “missing” eighths between beat three and the second eighth of beat four are replaced by a single attack on the third sixteenth of bar three, splitting the time between the beginning of beat three and the second eighth of beat four into two intervals of three

sixteenths each (instead of three intervals of one eighth each).

The second half of the bar forms the *kinka* rhythm discussed in Section 1.2, while the first half forms an unsyncopated even rhythm. If the second half is heard as a rhythmic elaboration of the first, then the song explicitly turns simple rhythmic derivation into a musical foreground element. Note also that whereas the *kinka* rhythm typically appears as the first half of a rhythm where the second half is highly syncopated, in this case the *kinka* figure appears in the second half of the rhythm following an entirely unsyncopated first half. This repetitive derivation of one foreground element from another places rhythmic evolution right onto the musical surface.

The perfect symmetry of regular unsyncopated eighth notes is thereby broken into a pattern that repeats after one bar, where the “background” sixth and seventh eighth notes have been replaced by a single attack that falls on the sixteenth note between them. Represented as ternary addresses this rhythm and its derivation can be captured by three integers: 2001, 1010, and 0202.

The address 2110 represents the four regular eighth notes that terminate on the beginning of beat three. The fact that these notes occupy only one half the time span of the riff is immediately seen by the 2 in the first ternary place, which indicates that the pattern only contains attacks within the first half note of the bar (being the second half note backward in time from the downbeat, hence the 2). A 1 in place of that 2 would simply indicate a full bar of unsyncopated eighths.

The address 1010 represents the downbeat and the third beat each with an eighth note pickup. The third beat and its pickup are also represented by address 2110 above, but the point here is that again a single ternary digit, the 0 in the second ternary place of 1010, is the only departure from a pure background rhythm of straight eighths that would be indicated if that 0 was replaced by a 1.

In summary, the pure background rhythm of straight eighths represented by 1110 (with the 0 indicating that only the first of each sixteenth note pair is attacked) is replaced by two addresses that carve away the two eighths immediately following the beginning of the third beat. They do this by operating on different metrical levels. Address 2110 omits any attacks during the time-span of the second half note, and 1010 omits attacks during the time-spans of the first and third quarter notes. In their union, it is only during the time-span of the third quarter that attacks are excluded.

Finally the single syncopated sixteenth note in beat three is represented by address 0202. The 2 in the ternary place corresponding to the sixteenth note metrical level combined with the 2 in the place for the quarter note metrical level indicates that the attack has been displaced back in time one quarter plus one sixteenth from the subsequent downbeat. This corresponds to the second attack in the 3+3+2 pattern that characterizes *kinka* rhythms.

2. Potential applications

Parsing of surface rhythms as defined here might provide useful inputs to musical analyses and processes where low-level elements of musical expectation play a role. Some conceivable applications are briefly sketched here, though these are not meant to be exhaustive or representative of the overall range of possibilities.

2.1. *Higher-level musical analysis*

Lerdahl and Jackendoff, in *GTTM*, allude to a potential role for low-level rhythmic data in higher-level analyses.

[The theory] has not included the rhythmic patterns at the musical surface: its patterns of attack points, durations, and rests; its patterns of pitch-rhythmic repetitions and parallelisms. These features are an input to the theory, not part of it. (Lerdahl and Jackendoff 1983, 285)

GTTM describes grouping structure as “units organized hierarchically,” as opposed to meter, which consists of “beats organized hierarchically” (Lerdahl and Jackendoff 1983, 25). Degrees of congruence between rhythm and meter are not claimed here to reveal gestalts such as those described by *GTTM*. But such congruence might have potential for marshalling data¹ for consideration of preference rules such as those for parallelism in grouping and time-span structures (Lerdahl and Jackendoff 1983, 75 and 164).

2.2. *Measures of rhythmic complexity*

Godfried Toussaint defined a rhythmic simplicity measure called *metricity*: “the sum of all the metrical accents of the beats present in a rhythm.” From that he derived an inversely proportional measure called *metric complexity* (Toussaint 2002, 162–163). This complexity measure was refined and used in the context of algorithmic composition by George Sioros and Carlos Guedes (Sioros and Guedes 2011, 381).

In that spirit, rhythmic simplicity might be measured by the sort of congruence between meter and rhythm described here, leading to a complexity measure that is inversely proportional to the degree of compression achieved by the encoding proposed in this paper. In other words, the number of derived rhythmic components required to represent actual rhythms. By this measure the first violin part in Section 1.1 and the *bossa nova* pattern in Section 1.2 provide examples of low and high complexity, respectively. This might correspondingly aid algorithmic composition as speculated in Section 2.3.

2.2.1. *Syncopation properties*

It is tricky to conclude too much about syncopation based on an address alone, because the associated rhythmic configuration may overlap with other configurations, which may include attacks that fulfill one or more anticipations in the first. But considered in isolation, the presence of 2s in an address signals syncopation.

In fact the address can be used to directly compute the syncopation index defined by Fitch and Rosenfeld (2007, 57) based on work by Longuet-Higgins and Lee (1984, 430–431). The following algorithm tallies syncopation operations possibly nested within elaborations to calculate a syncopation score without examining the attack pattern itself.

In this algorithm, ternary places are numbered from low to high counting from 0.

¹In a different direction, marshalling of data as described above forms an aspect of the *Multiresolution Principle*, which means “Music is a patterning of sound in the time-frequency plane. Analyze music by looking for repetition of patterns of time-frequency structures over multiple time-scales, and multiple levels of resolution in the time-frequency plane” (Walker 2008, 196). See also Cheng, Hart, and Walker (2009) and Don et al. (2010) for more on wavelets and rhythm, although wavelets go in a direction different from the ideas of the present paper.

Data: *address*
Result: *score*
if *address* contains no 1s **then**
 | set *score* to 0;
else
 | set *numLevels* to the number of ternary places in *address*;
 | set *lowestSyncopatedLevel* to the lowest ternary place in *address* containing 2;
 | set *score* to *numLevels* - *lowestSyncopatedLevel*;
 | set *n* to *lowestSyncopatedLevel* + 1;
 while *n* < *numLevels* **do**
 | set *p* to the *i*th ternary place in *address*;
 if *p* = 0 or *p* = 2 **then**
 | set *n* to *i* - *lowestSyncopatedLevel*;
 | set *m* to the number of ternary places in *address* higher than *n* that
 | contain 1;
 | set *score* to *score* + *n* * 2^m ;
 end
 end
end

Algorithm 1: How to compute the syncopation index of an address.

2.3. Algorithmic composition

A typical aspect of music composition is the creation of localized musical patterns that are used repeatedly, possibly in altered form, within larger musical contexts. This process is made explicit by software-based music sequencers, with which a short passage can be copied to other locations where it might be edited, copied again, and so on.

Recent proposals for algorithmic variation have included manipulation of abstract musical objects such as grouping structures and time-span reductions (Hamanaka, Hirata, and Tojo 2008). As noted in Section 2.1 the parsing described here might provide low-level input useful in consideration of relevant preference rules, for instance the following respective Time-Span Reduction and Prolongational Reduction rules of *GTTM*.

TSRPR 4 (Parallelism) If two or more time-spans can be construed as motivically and/or rhythmically parallel, preferably assign them parallel heads. (Lerdahl and Jackendoff 1983, 164)

PRPR 5 (Parallelism) Prefer a prolongation reduction in which parallel passages receive parallel analyses. (Lerdahl and Jackendoff 1983, 226)

On the other hand, one might aim rather to explore serendipitous emergence of higher-level musical structure while retaining organizational characteristics closer to the musical surface, by analyzing and generating low-level structures. A process for varying and hybridizing music via a hidden Markov model, with observable rhythms and hidden states that consist of intersections of derived rhythmic components, might form the topic of a subsequent paper.

2.3.1. Fractal music composition

The self-similarity inherent in this approach might invite comparisons with fractal music composition, which may employ scaling, tiling, and automata to form explicitly self-

similar musical surfaces (Johnson 2006).

However, this paper makes no particular focus on such musical surfaces and makes no assertion that noteworthy musical structures generally exhibit any particular degree of self-similarity on an individual basis. Rather, self-similarity characterizes an underlying set of possible rhythmic configurations, some subset of which may be used to dissect a given musical surface.

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