Supplementary to “A note on asymptotic distributions in directed exponential random graph models with bi-degree sequences”

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This is a supplementary material that contains proofs of the theorems 2 and 3.

1 Proof of Theorem 2

Note that both $d_i = \sum_{j \neq i} a_{i,j}$ and $b_i = \sum_{i \neq j} a_{i,j}$ are sum of $n - 1$ independent geometric random variables. Also note that $q_n \leq \bar{\alpha}_i^* + \bar{\beta}_i^* \leq Q_n$ and $V = F'(\bar{\theta}^*) \in \mathcal{L}_n(m, M)$, thus we have $(n - 1)Q_n^{-2} \leq v_{i,i} \leq (n - 1)q_n^{-2}$. Therefore, $v_{i,i}^{1/2}(d_i - \mathbb{E}(d_i))$ is asymptotically standard normal if $Q_n/q_n = o(n^{1/2})$. Similarly, $v_{n+j,n+j}^{-1/2}(b_j - \mathbb{E}(b_j))$ is also asymptotically standard normal under the same condition. Similar to Proposition 2 in subsection 5.2 of the main texts, we have:

Proposition 1. *Assume that $A \sim \mathbb{P}_{\theta^*}$ and $Q_n/q_n = o(n^{1/2})$, as $n \rightarrow \infty$, then $\bar{c}^\top S(\mathbf{g} - \mathbb{E}(\mathbf{g}))$ is asymptotically normally distributed with mean zero and variance*

$$\sum_{i=1}^{\infty} \lambda_i^2 + \sum_{i,j=1}^{\infty} \lambda_i \lambda_j H_i H_j + \sum_{i=1}^{\infty} \kappa_i^2 + \sum_{i,j=1}^{\infty} \kappa_i \kappa_j H_{n+i} H_{n+j} - 2 \sum_{i,j=1}^{\infty} \lambda_i \kappa_j H_i H_{n+j},$$

where $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \dots, \lambda_n v_{nn}^{1/2})^\top$, $\bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \dots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top$ and $\bar{c} = (\bar{\lambda}, \bar{\kappa})$.

Before proving Theorem 2, we present two lemmas. The proof of lemma 1 is similar to that of lemma 1 in subsection 5.2 of the main texts and we omit it.

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Lemma 1. Let $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \dots, \lambda_n v_{nn}^{1/2})^\top$, $\bar{\kappa} = (\kappa_1 v_{n+1, n+1}^{1/2}, \dots, \kappa_{n-1} v_{2n-1, 2n-1}^{1/2})^\top$ and $\bar{c} = (\bar{\lambda}, \bar{\kappa})$. Then

$$\text{Var}[\bar{c}^\top W\{\mathbf{g} - \mathbb{E}(\mathbf{g})\}] \leq \frac{c_1 M^3 + 3M^2 m^2}{m^3(n-1)} \times \left\{ \left(\sum_{i=1}^n |\lambda_i| \right)^2 + \left(\sum_{i=1}^{n-1} |\kappa_i| \right)^2 + 2 \left(\sum_{i=1}^n |\lambda_i| \right) \left(\sum_{j=1}^{n-1} |\kappa_j| \right) \right\} \quad (1)$$

where $m = 1/Q_n^2$, $M = 1/q_n^2$ and c_1 given in proposition 1 in subsection 5.1 of the main texts.

The following lemma is due to Theorem 3 in [Yan et al. \(2016\)](#).

Lemma 2. Assume that $\bar{\boldsymbol{\theta}}^*$ satisfies $q_n \leq \bar{\alpha}_i^* + \bar{\beta}_j^* \leq Q_n$ and $A \sim \mathbb{P}_{\bar{\boldsymbol{\theta}}^*}$, where $\mathbb{P}_{\bar{\boldsymbol{\theta}}^*}$ denote the probability distribution (1) of the main texts on A under the parameter $\bar{\boldsymbol{\theta}}^*$. If $Q_n/q_n = o\{(n/\log n)^{1/18}\}$, then as n goes to infinity, with probability approaching one, the MLE $\hat{\boldsymbol{\theta}}$ exist and satisfies

$$\|\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^*\|_\infty = O_p\left(\frac{Q_n^9 (\log n)^{1/2}}{n^{1/2} q_n^9}\right) = o_p(1).$$

Further, if the MLE exists, it is unique.

Proof of Theorem 2. By Lemma 2, with probability approaching one, we have

$$\hat{\rho}_n := \max_{1 \leq i \leq 2n-1} |\hat{\theta}_i - \bar{\theta}_i^*| = O\left(\frac{Q_n^9 (\log n)^{1/2}}{n^{1/2} q_n^9}\right) \quad (2)$$

The following calculations are based on the event (2). Let $\hat{\gamma}_{i,j} = \hat{\alpha}_i + \hat{\beta}_j - \bar{\alpha}_i^* - \bar{\beta}_j^*$. For any $i \neq j$, a direct calculation gives

$$\begin{aligned} \frac{1}{\hat{\alpha}_i + \hat{\beta}_j} - \frac{1}{\bar{\alpha}_i^* + \bar{\beta}_j^*} &= \frac{\bar{\alpha}_i^* - \hat{\alpha}_i + \bar{\beta}_j^* - \hat{\beta}_j}{(\hat{\alpha}_i + \hat{\beta}_j)(\bar{\alpha}_i^* + \bar{\beta}_j^*)} \\ &= \frac{\bar{\alpha}_i^* - \hat{\alpha}_i + \bar{\beta}_j^* - \hat{\beta}_j}{(\bar{\alpha}_i^* + \bar{\beta}_j^*)^2} \left(\frac{\bar{\alpha}_i^* + \bar{\beta}_j^*}{\hat{\alpha}_i + \hat{\beta}_j} - 1 \right) + \frac{\bar{\alpha}_i^* - \hat{\alpha}_i + \bar{\beta}_j^* - \hat{\beta}_j}{(\bar{\alpha}_i^* + \bar{\beta}_j^*)^2} \\ &= h_{i,j} + \frac{\bar{\alpha}_i^* - \hat{\alpha}_i + \bar{\beta}_j^* - \hat{\beta}_j}{(\bar{\alpha}_i^* + \bar{\beta}_j^*)^2}, \end{aligned}$$

where

$$h_{i,j} = \frac{\hat{\gamma}_{i,j}^2}{(\hat{\alpha}_i + \hat{\beta}_j)(\bar{\alpha}_i^* + \bar{\beta}_j^*)^2},$$

and $0 \leq \phi_{i,j} \leq 1$. By the likelihood equations (4) in subsection 2.2 of the main texts, we

have

$$\mathbf{g} - \mathbb{E}(\mathbf{g}) = V(\widehat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^*) + \mathbf{h}, \quad (3)$$

where $\mathbf{h} = (h_1, \dots, h_{2n-1})^\top$ and

$$\begin{aligned} h_i &= \sum_{k=1, k \neq i}^n h_{i,k}, \quad i = 1, \dots, n, \\ h_{n+i} &= \sum_{k=1, k \neq i}^n h_{k,i}, \quad i = 1, \dots, n-1. \end{aligned}$$

Equivalently,

$$\widehat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^* = V^{-1}(\mathbf{g} - \mathbb{E}(\mathbf{g})) + V^{-1}\mathbf{h}$$

By the definition of $\hat{\rho}_n$, we have

$$|h_{i,j}| \leq \frac{4\hat{\rho}_n^2}{q_n^2(q_n - \hat{\rho}_n)}, \quad |h_i| \leq \sum_{j \neq i} |h_{i,j}| \leq \frac{4(n-1)\hat{\rho}_n^2}{q_n^2(q_n - \hat{\rho}_n)}.$$

Note that $(S\mathbf{h})_i = h_i/v_{i,i} + (-1)^{1_{\{i > n\}}} h_{2n}/v_{2n,2n}$, $(n-1)Q_n^{-2} \leq v_{i,i} \leq (n-1)q_n^{-2}$ and $(V^{-1}\mathbf{h})_i = (S\mathbf{h})_i + (W\mathbf{h})_i$. By direct calculations, we have

$$\begin{aligned} |(V^{-1}\mathbf{h})_i| &= |(S\mathbf{h})_i| + |(W\mathbf{h})_i| \leq \frac{|h_i|}{v_{i,i}} + \frac{|h_{2n}|}{v_{2n,2n}} + \{\|W\|_\infty \times [(2n-1) \max_i |h_i|]\} \\ &\leq \frac{8Q_n^2 \hat{\rho}_n^2}{q_n^3} + \frac{8c_1 Q_n^6 \hat{\rho}_n^2}{q_n^7}, \\ &= O\left(\frac{\log(n)Q_n^{24}}{nq_n^{21}}\right) \end{aligned}$$

where c_1 given in proposition 1 in subsection 5.1 of the main texts.

Note that $\bar{c} = (\bar{\lambda}, \bar{\kappa})$, $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \dots, \lambda_n v_{nn}^{1/2})^\top$ and $\bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \dots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top$. Consequently, if $\sum_{i=1}^\infty |\lambda_i| < \infty$, $\sum_{i=1}^\infty |\kappa_i| < \infty$ and $Q_n/q_n = o(n^{1/50}/(\log n)^{1/25})$, then as $n \rightarrow \infty$,

$$\begin{aligned} \left| \sum_{i=1}^{2n-1} \bar{c}_i (V^{-1}\mathbf{h})_i \right| &= \left| \sum_{i=1}^n \lambda_i \sqrt{v_{i,i}} (V^{-1}\mathbf{h})_i + \sum_{i=1}^{n-1} \kappa_i \sqrt{v_{n+i,n+i}} (V^{-1}\mathbf{h})_{n+i} \right| \\ &= O\left(\frac{\log(n)Q_n^{24}}{n^{1/2}q_n^{23}} \left(\sum_{i=1}^n |\lambda_i| + \sum_{i=1}^{n-1} |\kappa_i| \right) \right) \rightarrow 0. \end{aligned} \quad (4)$$

In view of (4) and Lemma 1, if $Q_n/q_n = o(n^{1/50}/(\log n)^{1/25})$ and condition (2) of the main

texts hold, then

$$\begin{aligned}\bar{c}^\top(\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^*) &= \bar{c}^\top S(\mathbf{g} - \mathbb{E}(\mathbf{g})) + \bar{c}^\top W\{\mathbf{g} - \mathbb{E}(\mathbf{g})\} + \left| \sum_i \bar{c}_i (V^{-1}\mathbf{h})_i \right| \\ &= \bar{c}^\top S(\mathbf{g} - \mathbb{E}(\mathbf{g})) + o_p(1).\end{aligned}$$

Theorem 2 is immediately comes from Proposition 1. $\square$

2 Proof of theorem 3

Note that both $d_i = \sum_{j \neq i} a_{i,j}$ and $b_i = \sum_{i \neq j} a_{i,j}$ are sums of $n-1$ independent geometric random variables. Also note that $q_n \leq \bar{\alpha}_i^* + \bar{\beta}_i^* \leq Q_n$ and $V = F'(\bar{\boldsymbol{\theta}}^*) \in \mathcal{L}_n(m, M)$, here $m = \frac{e^{Q_n}}{(e^{Q_n}-1)^2}$ and $M = \frac{e^{q_n}}{(e^{q_n}-1)^2}$, thus we have

$$\begin{aligned}\frac{e^{Q_n}}{(e^{Q_n}-1)^2} &\leq v_{i,j} \leq \frac{e^{q_n}}{(e^{q_n}-1)^2}, i = 1, \dots, n, j = n+1, \dots, 2n, j \neq n+i, \\ \frac{(n-1)e^{Q_n}}{(e^{Q_n}-1)^2} &\leq v_{i,i} \leq \frac{(n-1)e^{q_n}}{(e^{q_n}-1)^2}, i = 1, \dots, 2n.\end{aligned}$$

Note that if $e^{Q_n/2}/q_n = o(n^{1/2})$, $v_{i,i}^{1/2}(d_i - \mathbb{E}(d_i))$ ($i = 1, \dots, n$) is asymptotically standard normal and $v_{n+j,n+j}^{-1/2}(b_j - \mathbb{E}(b_j))$ ($j = 1, \dots, n$) is asymptotically standard normal. Similar to Proposition 2 in subsection 5.2 of the main texts, we have:

Proposition 2. Assume that $e^{Q_n/2}/q_n = o(n^{1/2})$, as $n \rightarrow \infty$, then $\bar{c}^\top S(\mathbf{g} - \mathbb{E}(\mathbf{g}))$ is asymptotically normally distributed with mean zero and variance

$$\sum_{i=1}^{\infty} \lambda_i^2 + \sum_{i,j=1}^{\infty} \lambda_i \lambda_j H_i H_j + \sum_{i=1}^{\infty} \kappa_i^2 + \sum_{i,j=1}^{\infty} \kappa_i \kappa_j H_{n+i} H_{n+j} - 2 \sum_{i,j=1}^{\infty} \lambda_i \kappa_j H_i H_{n+j},$$

where $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \dots, \lambda_n v_{nn}^{1/2})^\top$, $\bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \dots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top$ and $\bar{c} = (\bar{\lambda}, \bar{\kappa})$.

Before proving Theorem 3, we present two lemmas. The proof lemma 3 is similar to that of lemma 1 in subsection 5.2 of the main texts and we omit it. Lemma 4 is due to Theorem 5 in Yan et al. (2016).

Lemma 3. Let $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \dots, \lambda_n v_{nn}^{1/2})^\top$, $\bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \dots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top$ and $\bar{c} = (\bar{\lambda}, \bar{\kappa})$. Then

$$\text{Var}[\bar{c}^\top W\{\mathbf{g} - \mathbb{E}(\mathbf{g})\}] \leq \frac{c_1 M^3 + M^2 m}{m^3(n-1)} \times \left\{ \left(\sum_{i=1}^n |\lambda_i| \right)^2 + \left(\sum_{i=1}^{n-1} |\kappa_i| \right)^2 + 2 \left(\sum_{i=1}^n |\lambda_i| \right) \left(\sum_{j=1}^{n-1} |\kappa_j| \right) \right\} \quad (5)$$

where $m = \frac{e^{Q_n}}{(e^{Q_n}-1)^2}$, $M = \frac{e^{q_n}}{(e^{q_n}-1)^2}$ and c_1 given in proposition 1 in subsection 5.1 of the main texts.

Lemma 4. Assume that $\bar{\boldsymbol{\theta}}^*$ satisfies $q_n \leq \bar{\alpha}_i^* + \bar{\beta}_j^* \leq Q_n$ for all $i \neq j$ and $A \sim \mathbb{P}_{\bar{\boldsymbol{\theta}}^*}$, where $\mathbb{P}_{\bar{\boldsymbol{\theta}}^*}$ denote the probability distribution (1) of the main texts on A under the parameter $\bar{\boldsymbol{\theta}}^*$. If $(1 + q_n^{-11})e^{6Q_n} = o(n^{1/2}/(\log n)^{1/2})$, then as n goes to infinity, with probability approaching one, the MLE $\hat{\boldsymbol{\theta}}$ exist and satisfies

$$\|\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^*\|_\infty = O_p(e^{3Q_n}(1 + \frac{1}{q_n^5})\sqrt{\frac{\log n}{n}}) = o_p(1)$$

Further, if the MLE exists, it is unique.

Proof of Theorem 3. By Lemma 4, with probability approaching one, we have

$$\hat{\rho}_n := \max_{1 \leq i \leq 2n-1} |\hat{\theta}_i - \bar{\theta}_i^*| = O(e^{3Q_n}(1 + \frac{1}{q_n^5})\sqrt{\frac{\log n}{n}}). \quad (6)$$

The following calculations are based on the event (6). Let $\hat{\gamma}_{i,j} = \hat{\alpha}_i + \hat{\beta}_j - \bar{\alpha}_i^* - \bar{\beta}_j^*$. By Taylor's expansion, for any $i \neq j$, we have

$$\frac{1}{e^{\hat{\alpha}_i + \hat{\beta}_j} - 1} - \frac{1}{e^{\bar{\alpha}_i^* + \bar{\beta}_j^*} - 1} = -\frac{e^{\bar{\alpha}_i^* + \bar{\beta}_j^*}}{(e^{\bar{\alpha}_i^* + \bar{\beta}_j^*} - 1)^2} \hat{\gamma}_{i,j} + h_{i,j},$$

where

$$h_{i,j} = -\frac{e^{\bar{\alpha}_i^* + \bar{\beta}_j^* + \eta_{i,j}\hat{\gamma}_{i,j}}(1 + e^{\bar{\alpha}_i^* + \bar{\beta}_j^* + \eta_{i,j}\hat{\gamma}_{i,j}})}{(e^{\bar{\alpha}_i^* + \bar{\beta}_j^* + \eta_{i,j}\hat{\gamma}_{i,j}} - 1)^3},$$

and $0 \leq \eta_{i,j} \leq 1$. It is not difficult to verify that

$$\mathbf{g} - \mathbb{E}(\mathbf{g}) = V(\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^*) + \mathbf{h},$$

where $\mathbf{h} = (h_1, \dots, h_{2n-1})^\top$ and

$$\begin{aligned} h_i &= \sum_{k=1, k \neq i}^n h_{i,k}, \quad i = 1, \dots, n, \\ h_{n+i} &= \sum_{k=1, k \neq i}^n h_{k,i}, \quad i = 1, \dots, n-1. \end{aligned}$$

Equivalently,

$$\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^* = V^{-1}(\mathbf{g} - \mathbb{E}(\mathbf{g})) + V^{-1}\mathbf{h}.$$

Since $\bar{\alpha}_i^* + \bar{\beta}_j^* > 0$, $q_n - \hat{\rho}_n < \alpha_i + \beta_j + \eta_{i,j}\hat{\gamma}_{i,j} \leq Q_n + q_n$ and $\hat{\rho}_n$ is sufficiently small, by

the definition of $\hat{\rho}_n$, we have

$$|h_{i,j}| \leq \frac{e^{(q_n - \hat{\rho}_n)}(1 + e^{(q_n - \hat{\rho}_n)})}{(e^{(q_n - \hat{\rho}_n)} - 1)^3} \hat{\rho}_n^2, \quad |h_i| \leq \sum_{j \neq i} |h_{i,j}| \leq (n-1) \frac{e^{(q_n - \hat{\rho}_n)}(1 + e^{(q_n - \hat{\rho}_n)})}{(e^{(q_n - \hat{\rho}_n)} - 1)^3} \hat{\rho}_n^2.$$

Note that $(S\mathbf{h})_i = h_i/v_{i,i} + (-1)^{1_{\{i > n\}}} h_{2n}/v_{2n,2n}$. By direct calculations, we have

$$\begin{aligned} |(V^{-1}\mathbf{h})_i| &= |(S\mathbf{h})_i| + |(W\mathbf{h})_i| \leq \frac{|h_i|}{v_{i,i}} + \frac{|h_{2n}|}{v_{2n,2n}} + \{\|W\|_\infty \times [(2n-1) \max_i |h_i|]\} \\ &\leq O\left(\frac{\log(n)}{n} \times e^{9Q_n}(1 + q_n^{-15})\right), \end{aligned}$$

Let $\bar{c} = (\bar{\lambda}, \bar{\kappa})$, $\bar{\lambda} = (\lambda_1 v_{11}^{1/2}, \dots, \lambda_n v_{nn}^{1/2})^\top$ and $\bar{\kappa} = (\kappa_1 v_{n+1,n+1}^{1/2}, \dots, \kappa_{n-1} v_{2n-1,2n-1}^{1/2})^\top$. Consequently, if $\sum_{i=1}^\infty |\lambda_i| < \infty$, $\sum_{i=1}^\infty |\kappa_i| < \infty$ and $e^{9Q_n}(1 + q_n^{-15}) = o(n^{1/2}/\log n)$, then as $n \rightarrow \infty$,

$$\begin{aligned} \left| \sum_{i=1}^{2n-1} \bar{c}_i (V^{-1}\mathbf{h})_i \right| &= \left| \sum_{i=1}^n \lambda_i \sqrt{v_{i,i}} (V^{-1}\mathbf{h})_i + \sum_{i=1}^{n-1} \kappa_i \sqrt{v_{n+i,n+i}} (V^{-1}\mathbf{h})_{n+i} \right| \\ &\leq O\left(\frac{\log(n)}{n^{1/2}} \times e^{9Q_n}(1 + q_n^{-15}) \left(\sum_{i=1}^n |\lambda_i| + \sum_{i=1}^{n-1} |\kappa_i| \right) \right) \rightarrow 0 \end{aligned} \tag{7}$$

In view of (7) and Lemmas 3 and 4, if $e^{9Q_n}(1 + q_n^{-15}) = o(n^{1/2}/\log n)$ and condition(2) of the main texts hold, then

$$\begin{aligned} \bar{c}^\top (\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}^*) &= \bar{c}^\top S(\mathbf{g} - \mathbb{E}(\mathbf{g})) + \bar{c}^\top W\{\mathbf{g} - \mathbb{E}(\mathbf{g})\} + \left| \sum_i \bar{c}_i (V^{-1}\mathbf{h})_i \right| \\ &= \bar{c}^\top S(\mathbf{g} - \mathbb{E}(\mathbf{g})) + o_p(1) \end{aligned}$$

Theorem 3 is immediately comes from Proposition 2. □

References

Yan, T., Leng, C., and Zhu, J. (2016). Asymptotics in directed exponential random graph models with an increasing bi-degree sequence. *The Annals of Statistics*, 44(1):31–57.