A new perspective on static and low order anti-windup synthesis

Matthew C. Turner ^{*†} and Ian Postlethwaite [‡]

Control and Instrumentation Research Group,
Department of Engineering,
University of Leicester,
Leicester,
LE1 7RH, UK.

Abstract

By viewing the anti-windup problem as a decoupled set of subsystems and relating this configuration to a general static anti-windup set-up, LMI conditions are established which guarantee stability and performance of the resulting closed-loop system. The approach taken, and the mapping used for the performance index, are logical and intuitive - and, it is argued, central to the ‘true’ anti-windup objective. The approach enables one to construct static anti-windup compensators in a systematic and numerically tractable manner. The idea is extended to allow low-order anti-windup compensators to be synthesised, which, while being sub-optimal, can improve transient performance and possess several desired properties (such as low computational overhead and sensible closed-loop pole locations). In addition, low-order anti-windup synthesis is often feasible when the corresponding static synthesis is not.

Keywords: Anti-windup compensation; saturation; linear systems.

1 Introduction

The problem of actuator saturation is well-documented in the literature ¹ and many researchers have sought to address the problems associated with its presence. It has been tackled from various different perspectives which include synthesising controllers which directly account for saturation constraints (see Garcia *et al.* (1999), De Dona *et al.* (2000), Turner *et al.* (2000) or Saberi *et al.* (1996) for example); model predictive control strategies where the control constraints are incorporated into the resulting optimisation procedure; and anti-windup techniques, which this paper considers.

One of the most successful methodologies used to cope with actuator saturation is *anti-windup compensation* or *conditioning* (see Kothare *et al.* (1994) or Astrom & Rundqwist (1989) for example). This well-known technique involves a two-step procedure whereby a controller is first synthesised for a nominal, usually linear,

^{*}Corresponding author. Email: mct6@le.ac.uk. Tel.: +44 (0)116 2522548

[†]Email: ixp@le.ac.uk. Tel.: +44 (0)116 2522547

[‡]Work supported by the UK Engineering and Physical Science Research Council

¹The literature on this subject is now vast and any attempt we make to summarise it would inevitably omit valuable contributions made by many scholars.

system ignoring saturation. An anti-windup compensator is then designed such that, when the control signal undergoes saturation, this compensator becomes active and leads to improved behaviour during this period. This has the advantage that the small signal response of the system is left as intended and that the local performance of the controller is not restricted.

Building upon the success achieved in guaranteeing stability of stable plants subject to actuator saturation, there has been some recent activity aimed at securing performance guarantees for such systems. Although this has always been sought informally, one of the first forays into establishing this notion concretely was made in Mulder *et al.* (2001) (see also Wada & Saeki (1999) for a similar, but less general, treatment), where the $\mathcal{L}_2$ gain was picked as an appropriate induced system norm to measure an anti-windup compensator's performance. Furthermore, Mulder *et al.* (2001) gave existence conditions, in terms of a set of linear matrix inequalities (LMI's), by which anti-windup compensators could be synthesised. Mulder *et al.* (2001) only tackled static anti-windup synthesis: the synthesis of anti-windup compensators which were themselves not dynamic, although their inclusion altered the dynamics of the "effective" controller.

This line of research was continued in more generality in Grimm *et al.* (2001b) and Grimm *et al.* (2003) where it was shown that, for stable plants, there always exists an anti-windup compensator of order greater than or equal to that of the plant which, in addition satisfies a finite $\mathcal{L}_2$ gain constraint. The results of Grimm *et al.* (2001b) and Grimm *et al.* (2003) served to complement known results on the existence of globally stabilising full order anti-windup compensators found in, for example, Teel & Kapoor (1997), Weston & Postlethwaite (2000) and Miyamoto & Vinnicombe (1996). Furthermore, the work in Grimm *et al.* (2001b) gave a characterisation of all optimal linear anti-windup compensators meeting an $\mathcal{L}_2$ performance constraint and a procedure by which to find a certain, possibly reduced order, compensator. Generally this problem was cast as a non-convex optimisation procedure, although in the full-order and static cases the problem is actually convex, which makes the results of Mulder *et al.* (2001) a special case of the general result in Grimm *et al.* (2001b).

Both the works of Mulder *et al.* (2001) and those of Grimm *et al.* (2003) were successful at synthesising compensators which yielded improved performance, particularly when compared to ad hoc methods in existence and when compared to internal model control (IMC), yet in a certain way, they both failed to achieve the true goal of an anti-windup compensator.

The "true goal" of an anti-windup compensator is, of course, subjective, but a notion of this which has been held informally since the methodology's inception is the notion of a *swift return to linear behaviour*. This was really only formalised in Teel & Kapoor (1997) (see also Kapoor *et al.* (1998)), although it was first recognised in the work of Miyamoto & Vinnicombe (1996) and, independently, in Weston & Postlethwaite (1998). In all of these works the performance of an anti-windup compensator was based on it returning the output of the system to the nominal, that is, what it would have been had saturation *not* been present.

In the works of Grimm *et al.* (2003) and Mulder *et al.* (2001), the $\mathcal{L}_2$ gain condition is not directly related to the system's return to linear behaviour; indeed there is no direct way of capturing this into the formulae in those papers. Instead, attention there is focused on the minimisation of a certain closed-loop induced norm, which is deemed appropriate to capture the performance of the system. In our paper, we re-visit the work of Weston & Postlethwaite (1998) (see also Weston & Postlethwaite (2000)) and use the decoupled structure derived in that paper to formulate conditions which encapsulate the return to linear behaviour of a stable system containing an anti-windup compensator. At first we consider purely static anti-windup compensators, which are, from

a practical point of view, most desirable. Then these ideas are extended to the sub-optimal synthesis of “low-order” compensators, which are often feasible for problems for which static compensators are not. It is important to remark that our work assumes that linear behaviour is desirable, and hence a return to linear behaviour marks a return to ‘good’ performance; in most cases this is a valid assumption.

Other work related to these results, and to this paper, can be found in Crawshaw & Vinnicombe (2000) and Kapoor *et al.* (1998), both of which take the return to linear behaviour as a central objective. The work of Crawshaw & Vinnicombe (2000) builds on the work of Miyamoto & Vinnicombe (1996) but uses a novel loop-transformation to convert a small gain condition into, effectively, the less conservative Circle Criterion. Kapoor *et al.* (1998) use an observer-based anti-windup compensator, a special case of static compensator (early work on this subject was contributed by Walgama & Sternby (1990)), and a subspace algorithm to make a certain triplet passive, which ensures recovery of linear behaviour. This latter technique has the advantage of computational simplicity although it does not explicitly tackle an $\mathcal{L}_2$ gain objective, as we propose here.

The notation used in the paper is mainly standard throughout. We denote the $\mathcal{L}_p$ norm of a time-varying vector $y(t) \in \mathbb{R}^{n_y}$ as $\|y\|_p$ and the induced $\mathcal{L}_p$ norm of a possibly nonlinear operator $\mathcal{Y} : \mathbb{Y}_1 \mapsto \mathbb{Y}_2$ from one Lebesgue space to another, as $\|\mathcal{Y}\|_{i,p}$. To avoid notational clutter, we often omit the time variable (t) or Laplace argument (s) of a vector or transfer function if we think that no confusion will arise. Likewise, we often do not distinguish between a system’s transfer function and its associated linear operator in the time domain. $\|y(t)\| = \sqrt{y(t)'y(t)}$ is the Euclidean norm of the vector $y(t)$. The *distance* between a vector $y(t)$ and a compact set, $\mathcal{Y}$, is denoted $dist(y, \mathcal{Y}) := \inf_{w \in \mathcal{Y}} \|y - w\|$. $\mathcal{R}^{i \times j}$ represents the space of all real rational transfer function matrices of dimension $i \times j$; $\mathcal{RH}_\infty$ denotes the set of real rational transfer function matrices, analytic in the closed right-half complex plane and with norm with supremum on the imaginary axis.

2 Problem Formulation

2.1 System Description

We consider the stabilisable, detectable and finite dimensional linear-time-invariant (FDLTI) plant subject to input saturation:

$$G(s) \sim \begin{cases} \dot{x}_p &= A_p x_p + B_p u_m + B_{pd} d \\ y &= C_p x_p + D_p u_m + D_{pd} d \end{cases} \quad (1)$$

with $x_p \in \mathbb{R}^{n_p}$, $u_m \in \mathbb{R}^m$, $y \in \mathbb{R}^q$ and $d \in \mathbb{R}^d$. From this we define the following transfer function matrices to represent the disturbance feedforward and feedback parts of $G(s)$:

$$G_1(s) \sim \left[\begin{array}{c|c} A_p & B_{pd} \\ \hline C_p & D_{pd} \end{array} \right] \quad G_2(s) \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array} \right] \quad (2)$$

As the work here is seeking global results, we are necessarily forced to assume that $G(s)$ is asymptotically stable; that is $G(s) \in \mathcal{RH}^\infty$. This is necessary in the approach we take, as will be clear later.

We assume the following stabilisable, detectable, linear controller has been designed to control the plant $G(s)$,

$$K(s) \sim \begin{cases} \dot{x}_c &= A_c x_c + B_c y + B_{cr} r \\ y_c &= C_c x_c + D_c y + D_{cr} r \end{cases} \quad (3)$$

where $x_c \in \mathbb{R}^{n_c}$ and $r \in \mathbb{R}^{n_r}$. From this we designate the following transfer functions

$$K_1(s) \sim \left[\begin{array}{c|c} A_c & B_{cr} \\ \hline C_c & D_{cr} \end{array} \right] \quad K_2(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array} \right] \quad (4)$$

As is the convention in anti-windup design, we assume that the plant input, u_m , is given by $u_m = \text{sat}(u)$, where

$$\text{sat}(u) = [\text{sat}(u_1), \dots, \text{sat}(u_m)] \quad (5)$$

and $\text{sat}(u_i) = \text{sign}(u_i) \times \min\{|u_i|, \bar{u}_i\}$, $\bar{u}_i > 0 \quad \forall i \in \{1, \dots, m\}$. If there is no saturation present (i.e. $u_m = u$), we call the system in which this is the case the *nominal closed loop system*. Furthermore, we assume that either u_m is available for direct measurement, or that a good estimate of it is available. For convenience we define the compact set $\mathcal{U} \subset \mathbb{R}^m$

$$\mathcal{U} := [-\bar{u}_1, \bar{u}_1] \times \dots \times [-\bar{u}_m, \bar{u}_m] \quad (6)$$

Note that $\forall u \in \mathcal{U}$, $\text{sat}(u) = u$. This definition is similar to that in Teel & Kapoor (1997), but not exactly the same as we only consider asymptotically stable plants. The deadzone function is given by the identity

$$Dz(u) = u - \text{sat}(u) \quad (7)$$

Note that $\forall u \in \mathcal{U}$, $Dz(u) = 0$. Central to our results will be the fact that the deadzone function satisfies the following sector property.

Definition 1 *The decentralised nonlinearity $\mathcal{N} = \text{diag}(\eta_1, \dots, \eta_m)$ is said to belong to Sector $[0, I]$ if the following inequality holds:*

$$\eta_1(u_i)^2 \leq \eta_i(u_i)u_i \leq u_i^2 \quad \forall i \in \{1, \dots, m\} \quad (8)$$

This definition is specialised to the Sector $[0, I]$ case from the more general definition given in Khalil (1996), Definition 10.1 - this less general sector bound is sufficient for our purposes.

We make the following assumption on the nominal closed loop system.

Assumption 1 • *The poles of $\begin{bmatrix} I & -K_2(s) \\ -G_2(s) & I \end{bmatrix}^{-1}$ are in the open left-half complex plane.*

• *The limit $\lim_{s \rightarrow \infty} (I - K_2(s)G_2(s))^{-1}$ exists.* □

The first statement ensures the nominal closed-loop system is stable; the second ensures it is mathematically well-posed and, in state-space terms, is equivalent to the existence of the matrices $\Delta := (I - D_p D_c)^{-1}$ and $\tilde{\Delta} := (I - D_c D_p)^{-1}$

Figure 1: Generic anti-windup scheme

2.2 Anti-windup configuration

We consider the problem of designing an anti-windup compensator, $\Theta(s)$, as depicted in Figure 1. The anti-windup compensator emits two signals, $\theta_1 \in \mathbb{R}^m$ and $\theta_2 \in \mathbb{R}^q$ which enter the control input and the plant output respectively. This is less general than in the work of Grimm *et al.* (2003), where θ_2 is permitted to enter the controller state directly (i.e. $\theta_2 \in \mathbb{R}^{n_c}$). This is because the current paper has been motivated by engineering problems where it may not always be plausible to inject θ_2 directly into the controller state equation². There is also some similarity to the work of Kapoor *et al.* (1998), although only observer-based synthesis is considered there ($\theta_1 = 0$).

Figure 2: Conditioning with $M(s)$

A novel way of representing most anti-windup configurations was introduced in Weston & Postlethwaite (1998), where one interprets the conditioning of controllers in terms of a single transfer function $M(s)$; this is shown in Figure 2. In Weston & Postlethwaite (1998) it was shown that, with all signals labelled identically and noting that $Dz(u) = u - \text{sat}(u)$, Fig 2 can be re-drawn as Fig 3. This configuration reveals an attractive decoupling into nominal linear system, nonlinear loop and disturbance filter. Note that if no saturation occurs, then the nominal linear system is all that is required to determine the system's behaviour. However if saturation occurs, the nonlinear loop and disturbance filter become active. Using this representation, note that the question of nonlinear stability for the complete system is translated into determining whether the nonlinear loop is stable. The dynamics of the disturbance filter determine the manner in which the nominal linear behaviour is affected during and after saturation.

The representation in Figure 3 was analysed in terms of existing schemes in Weston & Postlethwaite (2000)

²For example, in some embedded applications this is frequently the case.

Figure 3: Equivalent representation of conditioning with $M(s)$

and ways of analysing stability and performance were also suggested. In particular, stability and performance synthesis for anti-windup compensators of order greater than or equal to that of the plant can be conveniently performed using an LMI formulation of the Circle Criterion.

Although such a treatment yielded attractive results, from a computational point of view, the method still has deficiencies: normally one would like to keep the dynamics introduced by a conditioning scheme to a minimum. Also the treatment tackled stability of the nonlinear loop and performance of the disturbance filter separately, leading to the likelihood of sub-optimality. The topic of this paper is to address the synthesis issues of static and low-order anti-windup compensators using the Weston & Postlethwaite (1998) representation.

In this formulation, the mapping $\mathcal{T} : u_{lin} \mapsto y_d$ determines the deviation of the conditioned system from the nominal system, so keeping $\|\mathcal{T}\|_{i,p}$ small in some appropriate $\mathcal{L}_p$ norm yields, in some sense, good anti-windup performance. Note that this type of performance *is not* captured explicitly in the formulations of Grimm *et al.* (2001b) and Mulder *et al.* (2001): the performance index in these papers is taken as something which is defined on the system as a whole, which allows Grimm *et al.* (2001b) to conclude that the anti-windup performance is at best, greater than that of the nominal open and closed loop linear systems. Here, we do not have this constraint, and indeed, our performance index is purely defined on the *saturated* system. We now formally define the anti-windup problem we will consider in the remainder of the paper.

Definition

The anti-windup compensator $\Theta(s)$ is said to solve the anti-windup problem if the closed loop system in Figure 3 is well-posed and the following hold:

1. If $\text{dist}(u_{lin}, \mathcal{U}) = 0$, $\forall t \geq 0$, then $y_d = 0$, $\forall t \geq 0$ (assuming zero initial conditions for $\Theta(s)$).
2. If $\text{dist}(u_{lin}, \mathcal{U}) \in \mathcal{L}_p$, then $y_d \in \mathcal{L}_p$ for some integer $p \in [1, \infty)$.

The anti-windup compensator $\Theta(s)$ is said to *solve strongly the anti-windup problem* if, in addition, the following condition is satisfied:

3. The operator $\mathcal{T} : u_{lin} \mapsto y_d$ is well-defined and finite gain $\mathcal{L}_p$ stable for some integer $p \in [1, \infty)$.

Remark 1: In this paper, we only deal with the strong version of the anti-windup problem. The weaker version is included to enable the reader to make comparisons with other techniques in the literature, such as that in Teel & Kapoor (1997), partly on which this definition is based. Note that Condition 3 implies Condition 2. □

Remark 2: We exclude the consideration of $\mathcal{L}_\infty$ performance as $\mathcal{T}$ could be $\mathcal{L}_\infty$ stable yet the output could exhibit a limit cycle. □

Remark 3: It is important to stress that the work in this paper, and most other papers concerning anti-windup compensation, assumes that the model of the plant is reasonably accurate, that is G_2 is known precisely. It is not trivial to incorporate a robustness analysis into anti-windup compensation design, although some steps in this direction are described by (Turner *et al.* 2003).

3 Static anti-windup synthesis

3.1 Representing $M(s)$

In order to analyse linear conditioning in terms of Figure 3, it is necessary to derive an expression for $M(s)$ in terms of Θ . Partitioning Θ as

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \Theta \tilde{u} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} \tilde{u}, \quad \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \in \mathbb{R}^{m+q}, \quad \tilde{u} \in \mathbb{R}^m \quad (9)$$

from Figures 1 and 2, u can be derived, respectively, as

$$u = K_1 r + K_2 y + (K_2 \Theta_2 - \Theta_1) \tilde{u} \quad (10)$$

$$u = K_1 r + K_2 y - [(I - K_2 G_2)M - I] \tilde{u} \quad (11)$$

Obviously, for the two schemes to be equivalent, we must have

$$M = (I - K_2 G_2)^{-1} (-K_2 \Theta_2 + \Theta_3) \quad (12)$$

where $\Theta_3 := \Theta_1 + I$. Note that $M(s)$ is well defined by virtue of Assumption 1.

From Figure 3, note that we must consider $M - I$ for stability of $\mathcal{T}$ and GM for the system's recovery after saturation. We will do this using LMI's, but before that we must derive a state-space representation of these two transfer functions. It is easy to derive a non-minimal realisation but for numerical robustness we desire a "minimal realisation", such as is given in Appendix A³. Thus we assume that we have the following state space representation for the transfer functions $M(s) - I$ and $G_2(s)M(s)$:

$$\begin{bmatrix} M(s) - I \\ G_2(s)M(s) \end{bmatrix} \sim \begin{cases} \dot{\tilde{x}} &= \bar{A}\tilde{x} + (B_0 + \bar{B}\Theta)\tilde{u} \\ u_d &= \bar{C}_1\tilde{x} + (D_{01} + \bar{D}_1\Theta)\tilde{u} \\ y_d &= \bar{C}_2\tilde{x} + (D_{02} + \bar{D}_2\Theta)\tilde{u} \end{cases} \quad (13)$$

where Θ is a static matrix by virtue of Θ_1, Θ_2 being static and the precise form of the state-space matrices is described in Appendix A.

³This realisation is minimal iff (if and only if) the realisations for K_2 and G_2 are minimal; if this is not the case then there will be the same number of extra modes as there are in K_2 and G_2 .

3.2 Stability and Performance Analysis

The aforementioned stability and performance problem can be captured by ensuring that $\mathcal{T} : u_{lin} \mapsto y_d$ is small in some sense. We now state a condition which ensures finite gain $\mathcal{L}_2$ performance of the compensator; furthermore, the result allows the synthesis of an optimal compensator using standard convex optimisation routines.

Theorem 1 *There exists a static anti-windup compensator $\Theta = [\Theta'_1 \quad \Theta'_2] \in \mathbb{R}^{(m+q) \times m}$ which solves strongly the anti-windup problem for $p = 2$ if there exist matrices $Q > 0$, $U = \text{diag}(\mu_1, \dots, \mu_m) > 0$, $L \in \mathbb{R}^{(m+q) \times m}$ and a postive real scalar γ such that the following LMI is satisfied*

$$\begin{bmatrix} Q\bar{A}' + \bar{A}Q & B_0U + \bar{B}L - Q\bar{C}'_1 & 0 & QC'_2 \\ \star & -2U - UD'_{01} - D_{01}U - L'\bar{D}'_1 - \bar{D}_1L & I & UD'_{02} + L'\bar{D}'_2 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0 \quad (14)$$

Furthermore, if this inequality is satisfied, a suitable Θ achieving $\|\mathcal{T}\|_{i,2} < \gamma$ is given by $\Theta = LU^{-1}$. $\square$

Proof: First note that as Θ is linear, the first condition in the definition of the anti-windup problem is guaranteed trivially. Next, as $Dz(\cdot) \in \text{Sector}[0, I]$, we have

$$\tilde{u}_i u_i \geq \tilde{u}_i^2, \quad \forall i \in \{1, \dots, m\} \quad (15)$$

From this it follows that for some matrix $W = \text{diag}(w_1, \dots, w_m) > 0$

$$\tilde{u}'W(u - \tilde{u}) \geq 0 \quad (16)$$

Next assume $\exists v(\bar{x}) = \bar{x}'P\bar{x} > 0$, then if

$$\frac{d}{dt}\bar{x}'P\bar{x} + \|y_d\|^2 - \gamma^2\|u_{lin}\|^2 < 0 \quad (17)$$

it follows that $\|y_d\|_2 < \gamma\|u_{lin}\|_2$ and hence that $\|\mathcal{T}\|_{i,2} < \gamma$; in other words, the anti-windup problem is solved in the $\mathcal{L}_2$ sense. Next, as $\tilde{u}'W(u - \tilde{u}) \geq 0$, if

$$\frac{d}{dt}\bar{x}'P\bar{x} + \|y_d\|^2 - \gamma^2\|u_{lin}\|^2 + 2\tilde{u}'W(u - \tilde{u}) < 0 \quad (18)$$

then inequality (17) will be satisfied and hence $\|\mathcal{T}\|_{i,2} < \gamma$. Thus we consider inequality (18), which after evaluating the derivative with respect to time and substituting for $\dot{\bar{x}}$, y_d and $\bar{u}$, becomes

$$\begin{bmatrix} \bar{x} \\ \tilde{u} \\ u_{lin} \end{bmatrix}' \begin{bmatrix} \bar{A}'P + P\bar{A} + \bar{C}'_2\bar{C}_2 & PB_0 + P\bar{B}\Theta - W\bar{C}'_1 + \bar{C}'_2(D_{02} + \bar{D}_2\Theta) & 0 \\ \star & -2W + (D_{02} + \bar{D}_2\Theta)'(D_{02} + \bar{D}_2\Theta) - W(D_{01} + \bar{D}_1\Theta) - (D_{01} + \bar{D}_1\Theta)'W & W \\ \star & \star & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{u} \\ u_{lin} \end{bmatrix} < 0$$

The LMI in the theorem then follows using standard Schur complement and congruency transformation arguments, as reported, for example, in Mulder *et al.* (2001) (although the details are not the same).

We must now prove that such a compensator guarantees that the system is well-posed. To assist in proving existence, we use the following lemma, based on a result from Grimm *et al.* (2001b) (proof deferred to the appendix C)

Lemma 1 Assume that $-2V - \tilde{D}V - V\tilde{D}' < 0$ for some diagonal positive definite matrix $V > 0$. Also assume that the map $\Pi(w(t)) : \mathbb{R}^l \mapsto \mathbb{R}^{l \times l}$ is unique for all $w(t)$. Then $I + \tilde{D}\Pi(w(t))$ is nonsingular for all matrices $\Pi \in \mathbf{\Pi}$, where

$$\mathbf{\Pi} := \{\Pi(w(t)) : \Pi = \text{diag}(\pi_1(w_1(t)), \dots, \pi_l(w_l(t)), \pi_i(w_i(t)) \in [0, 1]\} \quad (19)$$

□

Note that in order for the anti-windup system to be well-posed, we must have existence and uniqueness of the equations in the nonlinear loop. For this, we must be able to determine u_d uniquely from the expression

$$u_d(t) = \bar{C}_1 \bar{x}(t) + (D_{01} + \bar{D}_1 \Theta) \tilde{u}(t) \quad (20)$$

where $\tilde{u} = Dz(u)$, or, equivalently, for some time varying, diagonal gain $K(u(t)) = \text{diag}(k_1(u_1(t)), \dots, k_m(u_m(t)))$ we can write $\tilde{u}(t) = K(u(t))(u_{lin}(t) - u_d(t))$, where $k_i(u_i(t)) \in [0, 1]$, $\forall u_i(t)$ by virtue of $Dz(\cdot) \in \text{Sector}[0, I]$. Furthermore, by Lipschitz continuity of $Dz(\cdot)$ we know that $K(u(t))$ is unique for each $u(t)$. Hence existence and uniqueness of solutions to (20) are equivalent to studying existence and uniqueness of solutions to

$$u_d(t) = \bar{C}_1 \bar{x}(t) + \tilde{D}K(u(t))u_{lin}(t) - \tilde{D}K(u(t))u_d(t) \quad (21)$$

where we have defined $\tilde{D} := (D_{01} + \bar{D}_1 \Theta)$. A solution (or solutions) exist iff $I + \tilde{D}K(u(t))$ is invertible for all $u(t)$. Note that, in terms of Lemma 1, $K(u(t)) \in \mathbf{\Pi}$, and hence $I + \tilde{D}K(u(t))$ will be invertible for $u(t)$, as the map $K(\cdot) : \mathbb{R}^m \mapsto \mathbb{R}^{m \times m}$ is unique, if $-2V - \tilde{D}V - V\tilde{D}' < 0$ for some positive definite diagonal matrix V : but this will be the case if the LMI (14) is satisfied for $\Theta = LU^{-1}$ (from looking at the 2,2 term). This has proved that solutions exist, to further prove uniqueness is somewhat harder and is therefore deferred to appendix D. □□

If only stability of the anti-windup scheme is required, and finite-gain $\mathcal{L}_2$ performance is not a priority, one can ignore the last two rows and columns of the LMI in Theorem 1 and instead consider the corollary below.

Corollary 1 There exists a static anti-windup compensator $\Theta = [\Theta'_1 \quad \Theta'_2]' \in \mathbb{R}^{(m+q) \times m}$ which solves strongly the anti-windup problem for all $p \in [1, \infty)$ if there exist matrices $Q > 0$, $U = \text{diag}(\mu_1, \dots, \mu_m) > 0$ and $L \in \mathbb{R}^{(m+q) \times m}$ such that the following LMI is satisfied

$$\begin{bmatrix} Q\bar{A}' + \bar{A}Q & B_0U + \bar{B}L - Q\bar{C}'_1 \\ \star & -2U - UD'_{01} - D_{01}U - L'\bar{D}'_1 - D_1L \end{bmatrix} < 0 \quad (22)$$

Furthermore, if this inequality is satisfied, then a suitable Θ achieving $\|\mathcal{T}\|_{i,p} < \gamma$, for some $\gamma > 0$ and all integers $p \in [1, \infty)$, is given by $\Theta = LU^{-1}$. □

Proof: The proof of the LMI is much the same as before, except that we do not consider the performance term $\|y_d\|^2 - \gamma^2 \|u_{lin}\|^2$ in the derivation. To prove that it still satisfies the strong anti-windup problem, we must prove that $\|\mathcal{T}\|_{i,p} < \gamma$, for some $\gamma > 0$. To see this, note that the LMI (22) gives sufficient conditions for the existence of a quadratic Lyapunov function, $v(\bar{x}) = \bar{x}'P\bar{x} > 0$ such that $\dot{v}(\bar{x}) < -\alpha\|\bar{x}\|^2$, when $u_{lin} = 0$. This implies

that the origin of (13) with $\tilde{u} = Dz(u_d)$ - the state equation of $\mathcal{T}$ - is exponentially stable. Now note that the functions

$$f(\bar{x}, u_{lin}) = \bar{A}\bar{x} + (B_0 + \bar{B}\Theta)Dz(\tilde{u}) \quad (23)$$

$$h(\bar{x}, u_{lin}) = \bar{C}_2\bar{x} + (D_{02} + \bar{D}_2\Theta)Dz(\tilde{u}) \quad (24)$$

are globally Lipschitz in both $\bar{x}$ and u_{lin} (refer to (13) for a description of $\tilde{u}$ and note well-posedness of the interconnection). Then Theorem 6.1 in Khalil (1996) can be applied to establish that $\|\mathcal{T}\|_{i,p} < \gamma$ for some $\gamma > 0$ and all $p \in [1, \infty)$, or, in other words, that the anti-windup problem is solved strongly. $\square\square$

Remark 4: One may wonder about the point of Theorem 1 in the presence of Corollary 1 as they both solve strongly the anti-windup problems but with Corollary 1 having less constraints and thus being more tractable numerically. However, Corollary 1 gives no direct way of optimising the size of $\mathcal{T}$, whereas in Theorem 1 this is addressed directly by introducing γ into the LMI, for the case of $p = 2$ at least. On the other hand, for large or ill-conditioned problems, the said numerical superiority of the LMI in Corollary 1 may well make it preferable. $\square$

4 Low-order anti-windup compensator synthesis

Thus far the work has focused on synthesising conditioning schemes of the form $\theta = \Theta\tilde{u}$, with Θ being static. This can often lead to satisfactory performance, but does not use any frequency information in the synthesis of the compensators. In particular, we are normally interested in reacting to saturation at low frequencies i.e. when $W_1(s)\tilde{u}$ is large, $W_1(s)$ being some low-pass filter. For example, it is not desirable, or even physically/computationally possible, to react to saturation caused by a very high frequency sinusoidal disturbance at the controller input, say at 10^{12} radians/s. Moreover it is well documented (see Doyle *et al.* (1987) or Grimm *et al.* (2001a)) that static compensation can be infeasible in situations where dynamic compensation may well be feasible.

In contrast, most dynamic schemes available with some sort of stability guarantees, are of order at least equal to that of the plant (for example the IMC scheme, the $\mathcal{L}_2$ scheme of Teel & Kapoor (1997), the normalised coprime factor scheme of Weston & Postlethwaite (2000)). For many applications this can lead to a prohibitively high computational overhead, both for synthesis and implementation. Hence it is logical to address the synthesis of low-order compensators.

In principle, the results of Grimm *et al.* (2001b) may be used to derive low order compensators achieving a prescribed $\mathcal{L}_2$ gain (although, as stated in the introduction, this $\mathcal{L}_2$ gain is defined on a different system, and is less helpful than the one we consider here), although the resulting optimisation problem is, generally, non-convex, meaning that iterative algorithms for computing fixed-order optimal compensators are prone to becoming stuck at local minima. The approach we propose is definitely sub-optimal, but we have found it to give good results in practice.

4.1 A sub-optimal approach

The proposal retains much of the simplicity of the static approach, except we now let our compensator be described by the dynamic equations

$$\Theta(s) = \begin{bmatrix} \Theta_1(s) \\ \Theta_2(s) \end{bmatrix} \in \mathcal{R}^{(m+q) \times m} \quad (25)$$

where

$$\Theta_1(s) := F_1(s)\tilde{\Theta}_1 \quad (26)$$

$$\Theta_2(s) := F_2(s)\tilde{\Theta}_2 \quad (27)$$

where $F_1(s) \in \mathcal{R}^{m \times m}$ and $F_2(s) \in \mathcal{R}^{q \times q}$ are transfer function matrices and $\tilde{\Theta}_1 \in \mathbb{R}^{m \times m}$ and $\tilde{\Theta}_2 \in \mathbb{R}^{q \times m}$ are constant matrices. Obviously, $\Theta_1(s)$ and $\Theta_2(s)$ could be synthesised in an optimal fashion directly, but this could be computationally expensive; lead to unsuitable compensators (large poles, sensitive to parameter variations - well known from $\mathcal{H}_\infty$ based LMI synthesis); and leave little room for intuition.

Instead we propose that $F_1(s), F_2(s)$ be chosen using intuition and experience, and then $\tilde{\Theta}_1, \tilde{\Theta}_2$ are synthesised in an optimal way, similar to the pure static synthesis described earlier. Obviously the resulting compensator will be sub-optimal in terms of its $\mathcal{L}_2$ gain but simulation results have shown that using relatively simple choices for $F_1(s), F_2(s)$, such as first order low pass filters (which suppress high frequency signals in y_d), very good responses can be obtained. In the scalar case, $F_1(s), F_2(s)$ can be chosen, guided by the graphical Circle Criterion. Typically, the modulus of the poles of $F_1(s), F_2(s)$ should be less than or equal to those of the controller, otherwise implementational difficulties can arise.

4.2 A sub-optimal synthesis routine

As before we obtain that

$$M = (I - K_2 G_2)^{-1}(-K_2 \Theta_2 + \Theta_3) \quad (28)$$

except that this time $\Theta_1(s) = F_1(s)\tilde{\Theta}_1$ and $\Theta_2(s) = F_2(s)\tilde{\Theta}_2$ are dynamic. Also as before this implies that

$$M - I = (I - K_2 G_2)^{-1}(-K_2 \Theta_2 + \Theta_3) - I \quad (29)$$

$$G_2 M = G_2 (I - K_2 G_2)^{-1}(-K_2 \Theta_2 + \Theta_3) \quad (30)$$

A minimal state-space realisation for these expressions can be derived as

$$\begin{bmatrix} M(s) - I \\ G_2(s)M(s) \end{bmatrix} \sim \begin{cases} \dot{\bar{x}} &= \bar{A}\bar{x} + (B_0 + \bar{B}\tilde{\Theta})\tilde{u} \\ u_d &= \bar{C}_1\bar{x} + (D_{01} + \bar{D}_1\tilde{\Theta})\tilde{u} \\ y_d &= \bar{C}_2\bar{x} + (D_{02} + \bar{D}_2\tilde{\Theta})\tilde{u} \end{cases} \quad (31)$$

where the matrices⁴ are described in Appendix B and $\tilde{\Theta}$ is given as $\tilde{\Theta} := [\tilde{\Theta}_1' \quad \tilde{\Theta}_2']'$. Similar to the static case we have the following theorem:

⁴The matrices $\bar{A}$ etc. are different from those in the previous section because they also incorporate the dynamics of $\Theta(s)$ in addition.

Theorem 2 Given $F_1(s), F_2(s)$, where $\deg(F_1(s)) = n_1$ and $\deg(F_2(s)) = n_2$, and $\tilde{\Theta} = [\tilde{\Theta}'_1 \ \tilde{\Theta}'_2]' \in \mathbb{R}^{(m+q) \times m}$, then there exists an $(n_1 + n_2)$ 'th order anti-windup compensator of the form

$$\Theta(s) = \begin{bmatrix} F_1(s)\tilde{\Theta}_1 \\ F_2(s)\tilde{\Theta}_2 \end{bmatrix} \quad (32)$$

which solves strongly the anti-windup problem if there exist matrices $Q > 0$, $U = \text{diag}(\mu_1, \dots, \mu_m) > 0$, $L \in \mathbb{R}^{(m+q) \times m}$ and a positive real scalar γ such that the LMI (14) is satisfied (with the state-space realisation of equation (31)). Furthermore, if inequality (14) is statisfied, a suitable $\tilde{\Theta}$ achieving $\|\mathcal{T}\|_{i,2} < \gamma$ is given by $\tilde{\Theta} = LU^{-1}$. $\square$

Proof: The proof is similar, *mutatis mutandis*, to that of Theorem 1.

Corollary 2 Given $F_1(s), F_2(s)$, where $\deg(F_1(s)) = n_1$ and $\deg(F_2(s)) = n_2$, and $\tilde{\Theta} = [\tilde{\Theta}'_1 \ \tilde{\Theta}'_2]'$, then there exists an $(n_1 + n_2)$ 'th order anti-windup compensator of the form

$$\Theta(s) = \begin{bmatrix} F_1(s)\tilde{\Theta}_1 \\ F_2(s)\tilde{\Theta}_2 \end{bmatrix} \quad (33)$$

which solves the anti-windup problem for all integers $p \in [1, \infty)$ if there exist matrices $Q > 0$, $U = \text{diag}(\mu_1, \dots, \mu_m) > 0$ and $L \in \mathbb{R}^{(m+q) \times m}$ such that the LMI (22) is satisfied (with the state-space realisation of equation (31)). Furthermore, if inequality (22) is satisfied, a suitable $\tilde{\Theta}$ achieving $\|\mathcal{T}\|_{i,p} < \gamma$, for some $\gamma > 0$ and all integers $p \in [1, \infty)$, is given by $\tilde{\Theta} = LU^{-1}$. $\square$

Proof: The proof is similar, *mutatis mutandis*, to that of Corollary 1. $\square\square$

Remark 5: Note that throughout the paper so far, stability and dissipativity have been established through the use of a purely quadratic Lyapunov function, $v(\bar{x}) = \bar{x}'P\bar{x} > 0$. In order to reduce the conservatism inherent in such an approach it would be preferable to use a different type of Lyapunov function; in particular a function of the Lur'e type seems appropriate i.e.

$$v(\bar{x}) = \bar{x}'P\bar{x} + 2 \sum_{i=1}^m \eta_i \int_0^{\bar{u}_i} Dz_i(q) dq \quad (34)$$

Unfortunately, as pointed out in Weston & Postlethwaite (2000), such a choice of Lyapunov function generally does not lead to a convex problem for anti-windup compensator synthesis⁵, and hence the determination of a compensator becomes more difficult.

5 Examples

We demonstrate our results on known examples in the literature and compare our anti-windup solutions with other optimal methods. As most of these techniques require the use of LMI's in the synthesis of compensators we decided to use one LMI solver, the Matlab LMI toolbox (Gahinet *et al.* (1995)), exclusively. This seemed reasonably robust, in a numerical sense, although it did seem to be "cautious" compared to some solvers. We also tried solving the problems using the SDPHA LMI solver of Potre *et al.* (1997). Although this tended to be less cautious than the LMI toolbox, and in many cases led to systems with better time responses, it also had instances of finding problems feasible, which were not strictly feasible, thus leading to erroneous results.

⁵For analysis purposes, more general Lyapunov functions can be used

5.1 Simple dynamic compensator example

We consider the example introduced in Mulder & Kothare (2000) and also considered in Grimm *et al.* (2001a) for which static anti-windup compensation is not feasible (in the sense that the LMI's associated with static synthesis are infeasible). This rules out our static results, but we find that our low-order results are feasible. We compare this with the full-order dynamic compensation results of Grimm *et al.* (2001b) and the well-known IMC solution, which for stable plants $G_2(s)$ always leads to a stable, full-order compensator.

The plant is described by

$$A_p = \begin{bmatrix} -0.2 & -0.2 \\ 1 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{pd} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (35)$$

$$C_p = [-0.4 \quad -0.9], \quad D_p = -0.5, \quad D_{pd} = 0 \quad (36)$$

The saturation limits are fixed at ± 0.5 and the controller is described by

$$A_c = 0, \quad B_c = -B_{cr} = 1 \quad (37)$$

$$C_c = 2, \quad D_c = -D_{cr} = 2 \quad (38)$$

For our low-order synthesis we chose our dynamic parts as first order low-pass filters:

$$F_1(s) = \frac{1000}{s + 1000}, \quad F_2(s) = \frac{0.4}{s + 0.4} \quad (39)$$

These values for $F_1(s)$ and $F_2(s)$ were arrived at after several iterations. After solving the LMI in Theorem 2, the optimal value of $\tilde{\Theta}$ returned was

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 \\ \tilde{\Theta}_2 \end{bmatrix} = \begin{bmatrix} -0.9630 \\ -0.7747 \end{bmatrix}, \quad \|\mathcal{T}\|_{i,2} < \gamma \approx 12.7192 \quad (40)$$

Figure 4 shows the responses of the various anti-windup compensators. Clearly, the low-order compensator of Theorem 2 shows the most desirable response with a fast settling time and little oscillation. The dynamic compensation of Grimm *et al.* (2001b) is also good, but it exhibits a slower return to linear behaviour and a slightly more oscillatory response. The IMC compensation, which had yielded $\|\mathcal{T}\|_{i,2} = \|G_2\|_\infty \approx 10.72$, leads to a stable but slow response and obviously the worst behaviour is observed when no anti-windup compensation is present. No static responses are shown due to infeasibility.

It is worth noting that in this case the “low-order” example is not really low order as the dimension of θ is the same as the number of plant states. Hence the compensator returned by Theorem 2 is of the same order as the full-order dynamic compensator obtained using the technique of Grimm *et al.* (2001b). However, it is important to remark that the Grimm *et al.* (2001b) compensator leads to a markedly worse response than that of Theorem 2; this is because Theorem 2 leads to a direct minimisation of $\|\mathcal{T}\|_{i,2}$, whereas the formulae of Grimm *et al.* (2001b) concentrate on a closed-loop transfer function.

5.2 A more realistic example: missile auto-pilot

In this section, we consider a more realistic application: a missile auto-pilot. This example was introduced into the wider literature in Rodriguez & Cloutier (1994) to demonstrate the effectiveness of an ‘error governor’,

Figure 4: Comparison of different anti-windup schemes for low-order example

Figure 5: Nominal linear response of missile

originally proposed by Kapasouris *et al.* (1988). Romanchuk (1999) also notes the interest of this example. The plant in question is a simplified model of the dynamics of the roll-yaw channels of a bank-to-turn missile. The controller used is of the LQG/LTR type which yields excellent nominal closed-loop response types in both time and frequency domains. The state-space data of these systems are described in Appendix E. In accordance with Rodriguez & Cloutier (1994), the actuator limits are set at ± 8 in both channels.

Figure 5 shows the nominal linear responses of the closed-loop system for a pulse of $r = [4.2 \quad -4.2]'$. Note the

Figure 6: Uncompensated saturated response of missile

excellent decoupling, but observe that the control effort strays outside the set $\mathcal{U} = \text{co}\{(-8, -8), (-8, 8), (8, 8), (8, -8)\}$ for some time, meaning that we can expect a poor response if actuator saturation were to be introduced. This intuition is verified in Figure 6, where it can be observed that the decoupling in the nominal response is destroyed by the introduction of actuator limits. Also notice that even in the first channel, tracking performance is lost due to the so-called infeasibility of the reference ⁶.

Figure 7: Static compensation of Grimm *et al.* – compensated; – – nominal linear

Figure 7 shows the missile response with the static anti-windup compensation of Grimm *et al.* (2001b), which is also equivalent to that described by Mulder *et al.* (2001). The dynamic compensation is not shown for two reasons: it leads to responses almost identical to the static compensation; also, it leads to numerical problems as the compensators it generated had large poles and hence required a very small sample time for simulation (this problem with large poles also occurs with LMI-based $\mathcal{H}_\infty$ synthesis). The optimal static compensator leads to an optimal performance index from the reference, r to the objective $z = r - y$ of $\sup_{r \in \mathcal{L}_2} \frac{\|z\|_2}{\|r\|_2} < \gamma = 2.2937$ (the dynamic compensation led to a slightly smaller $\gamma = 1.7834$) ⁷. From the figure it is obvious that this compensation leads to improved performance compared to the uncompensated system, although the second channel still has some significant differences to the linear response. Note that in both channels there is still some error during the pulse due to the infeasibility of the reference.

Figure 8 shows the missile responses with the IMC based anti-windup compensation. The performance, as measured by the $\mathcal{L}_2$ induced norm of the mapping of $\mathcal{T}$, is $\gamma \approx 376.25$. Although the first channel behaves acceptably, the second channel does not and is scarcely better than the uncompensated response.

Figure 9 shows the missile response with the static anti-windup scheme proposed in this paper. Theorem 1 was used to obtain the optimal Θ as

$$\Theta = \begin{bmatrix} -0.9992 & -0.0039 \\ 0.0173 & -0.6920 \\ -0.0112 & -0.5573 \\ -0.2022 & -0.3408 \end{bmatrix}, \quad \|\mathcal{T}\|_{i,2} < \gamma \approx 378.25 \quad (41)$$

⁶A reference is called infeasible if, in the steady state it leads to a control signal which is permanently above the actuator saturation threshold.

⁷NB - these values of γ are not comparable with those associated with $\|\mathcal{T}\|_{i,2}$

Figure 8: IMC compensation: — compensated; — — nominal linear

Figure 9: Static compensation of Theorem 1: — compensated; — — nominal linear

The response of the first channel is again acceptable and channel 2 exhibits an improved response over the IMC and uncompensated responses. However as with the performance obtained with static scheme of Grimm *et al.* (2001b), there is some significant deviation from the linear response. Somewhat surprisingly, the bound on $\|\mathcal{T}\|_{i,2}$ is actually higher for the optimal static compensator than for the IMC compensator.

Figure 10 shows the missile response with a low-order static compensator. As the designer has several tuning parameters, four or five iterations were performed in order to obtain the best response. For this problem we chose

$$F_1(s) = \text{diag}\left(\frac{2}{s+2}, 1\right) \quad F_2(s) = I_2 \quad (42)$$

and, according to Theorem 2, $\tilde{\Theta}$ was calculated as

$$\tilde{\Theta} = \begin{bmatrix} -0.8843 & 1.1316 \\ 2.6062 & 63.9337 \\ -6.2281 & -304.4056 \\ -116.9253 & -9.4427 \end{bmatrix}, \quad \|\mathcal{T}\|_{i,2} < \gamma = 376.7226 \quad (43)$$

This response is the best exhibited by the anti-windup compensators demonstrated here. Both channels behave well when compared to the linear response, the second being almost identical. There is, of course, some deviation

Figure 10: Dynamic compensation of Theorem 2: — compensated; — — nominal linear

due to the infeasibility of the reference variable which results in a temporary error, but return to linear behaviour is swift after this occurrence. The order of this anti-windup compensator is 1, as $F_2(s)$ and one channel of $F_1(s)$ was chosen to be static, making it more attractive than the IMC and dynamic compensators of Grimm *et al.* (2001b) from a computational point of view. The choice of the compensator dynamics was also found helpful and by sensible selection it is easy to avoid large poles which may result from the synthesis procedure of Grimm *et al.* (2001b).

6 Conclusion

This paper has considered the static anti-windup synthesis problem from a new perspective, and one which, we believe is conceptually appealing and central to the “true” anti-windup problem. We have also proposed, we believe, the first constructive, numerically tractable procedure, for the synthesis of low order anti-windup compensators, for which stability and performance of the overall system are guaranteed. Simple examples have shown the effectiveness of these schemes, although we think the true worth of these results will be greater when demonstrated on more realistic and complex systems. In such cases, it is often imperative to have an intuitive scheme to which one can consult if there are problems with initial designs. For this reason there needs to be further work on the choice of dynamics for the low order scheme. This is the subject of continuing research.

It is still necessary to re-examine some of the more traditional techniques in the light of these new results. Although we think our work on low-order dynamic compensators may help to explain why simple compensators can work in many cases, there is still some way to go before we can justify why some of these techniques sometimes work better than optimal techniques. Some insight to this problem is also found in Mulder & Kothare (2000), where it is shown that quadratic Lyapunov functions are sometimes not always sufficiently general for the anti-windup problem, but, again, more research remains to be conducted.

The authors would like to thank Dr. Guido Herrmann of University of Leicester for pointing out an error in the uniqueness part of the proof of Theorem 1 in an early draft, and for other useful comments.

References

- Astrom, K.J. & Rundqwist, L. (1989), ‘Integrator windup and how to avoid it’, *Proc. American Control Conference* **2**, 1693–1698.
- Crawshaw, S. & Vinnicombe, G. (2000), ‘Anti-windup synthesis for guaranteed $\mathcal{L}_2$ performance’, *Proc. IEEE Conference on Decision and Control*.
- De Dona, J.A., Moheimani, S.O.R. & Goodwin, G.C. (2000), ‘Robust combined PLC/LHG controller with allowed over-saturation of the input signal’, *Proc. American Control Conference* pp. 750–751.
- Doyle, J.C., Enns, D.F. & Smith, R.S. (1987), ‘Control of plants with input saturation nonlinearities’, *Proc. American Control Conference* pp. 2147–2152.
- Gahinet, P., Nemirovski, A., Laub, A.J. & Chilali, M. (1995), *LMI Control Toolbox*, The MathWorks Inc.
- Garcia, G., Tarbouriech, S., Suarez, R. & Alvarez-Ramirez, J. (1999), ‘Nonlinear bounded control for norm-bounded uncertain systems’, *IEEE Transactions on Automatic Control* **44**(8), 1254–1258.
- Grimm, G., Hatfield, J., Postlethwaite, I., Teel, A.R., Turner, M.C. & Zaccarian, L. (2003), ‘Anti-windup for stable linear systems with input saturation: an LMI based synthesis’, *IEEE Transactions on Automatic Control* **48**(9), 1509–1525.
- Grimm, G., Postlethwaite, I., Teel, A.R., Turner, M.C. & Zaccarian, L. (2001a), ‘Case studies using linear matrix inequalities for optimal anti-windup compensator synthesis’, *Proc. European Control Conference*.
- Grimm, G., Postlethwaite, I., Teel, A.R., Turner, M.C. & Zaccarian, L. (2001b), ‘Linear matrix inequalities for full and reduced order anti-windup synthesis’, *Proc. American Control Conference*.
- Kapasouris, P., Athans, M. & Stein, G. (1988), ‘Design of feedback control systems for stable plants with saturating actuators’, *Proc. IEEE Conference on Decision and Control*.
- Kapoor, N., Teel, A.R. & Daoutidis, P. (1998), ‘An anti-windup design for linear systems with input saturation’, *Automatica* **34**(5), 559–574.
- Khalil, H.K. (1996), *Nonlinear Systems*, Prentice Hall, New Jersey.
- Kothare, M.V., Campo, P.J., Morari, M. & Nett, C.N. (1994), ‘A unified framework for the study of anti-windup designs’, *Automatica* **30**(12), 1869–1883.
- Miyamoto, S. & Vinnicombe, G. (1996), ‘Robust control of plants with saturation nonlinearity based on coprime factor representations’, *Proc. IEEE Conference on Decision and Control* pp. 2838–2840.

- Mulder, E.F. & Kothare, M.V. (2000), 'Synthesis of stabilising anti-windup controllers using piecewise quadratic Lyapunov functions', *Proc. American Control Conference* pp. 3239–3243.
- Mulder, E.F., Kothare, M.V. & Morari, M. (2001), 'Multivariable anti-windup controller synthesis using linear matrix inequalities', *Automatica* **37**(9), 1407–1416.
- Potre, F., Sheng, R. & Brixius, N. (1997), 'SDPHA: A code for semidefinite programming', <http://www.math.uiowa.edu/rsheng/SDPHA/sdpha.html>.
- Rodriguez, A.A. & Cloutier, J. R. (1994), 'Control of a bank-to-turn missile with saturating actuators', *Proc. American Control Conference*.
- Romanchuk, B.G. (1999), 'Some comments on anti-windup synthesis using LMI's', *International Journal of Robust and Nonlinear Control* **9**(10), 717–734.
- Saberi, A., Lin, Z. & Teel, A.R. (1996), 'Control of linear systems with saturating actuators', *IEEE Transactions on Automatic Control* **41**(3), 368–378.
- Teel, A.R. & Kapoor, N. (1997), 'The $\mathcal{L}_2$ anti-windup problem: Its definition and solution', *Proceedings of the European Control Conference*.
- Turner, M.C., Herrmann, G. & Postlethwaite, I. (2003), 'Accounting for uncertainty in anti-windup synthesis', *submitted*.
- Turner, M.C., Postlethwaite, I. & Walker, D.J. (2000), 'Nonlinear tracking control for multivariable constrained input linear systems', *International Journal of Control* **73**(12), 1160–1172.
- Wada, N. & Saeki, M. (1999), 'Design of a static anti-windup compensator which guarantees robust stability', *Transactions of the Institute of Systems, Control and Instrumentation Engineers* **12**(11), 664–670.
- Walgama, K.S. & Sternby, J. (1990), 'Inherent observer property in a class of anti-windup compensators', *International Journal of Control* **52**, 705–724.
- Weston, P.F. & Postlethwaite, I (1998), 'Analysis and design of linear conditioning schemes for systems containing saturating actuators', *IFAC Nonlinear Control System Design Symposium*.
- Weston, P.F. & Postlethwaite, I (2000), 'Linear conditioning for systems containing saturating actuators', *Automatica* **36**(9), 1347–1354.

A State-space derivation for static anti-windup compensator

Some tedious algebra yields the state space matrices for a minimal realisation of $[M' - I \quad (G_2 M)']'$ as

$$\bar{x} := \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad \bar{A} := \begin{bmatrix} A_p + B_p \tilde{\Delta} D_c C_p & B_p \tilde{\Delta} C_c \\ B_c \Delta C_p & A_c + B_c \Delta D_p C_c \end{bmatrix}, \quad B_0 := \begin{bmatrix} B_p \tilde{\Delta} \\ B_c \Delta D_p \end{bmatrix}, \quad (44)$$

$$\bar{B} := \begin{bmatrix} B_p \tilde{\Delta} & -B_p \tilde{\Delta} D_c \\ B_c \Delta D_p & -B_c \Delta \end{bmatrix}, \quad \Theta := \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}, \quad (45)$$

$$\bar{C}_1 := [\tilde{\Delta} D_c C_p \quad \tilde{\Delta} C_c], \quad D_{01} := \tilde{\Delta} D_c D_p, \quad \bar{D}_1 := [\tilde{\Delta} \quad -\tilde{\Delta} D_c], \quad (46)$$

$$\bar{C}_2 := [\Delta C_p \quad \Delta D_p C_c], \quad D_{02} := \Delta D_p, \quad \bar{D}_2 := [\Delta D_p \quad -\Delta D_p D_c] \quad (47)$$

B State-space derivation for dynamic anti-windup compensator

As $\Theta_1(s) = F_1(s)\tilde{\Theta}_1$ and $\Theta_2(s) = F_2(s)\tilde{\Theta}_2$, we assign the state-space realisations

$$\Theta_3(s) = \Theta_1(s) + I \sim \begin{cases} \dot{x}_1 &= A_1 x_1 + B_1 \tilde{\Theta}_1 \tilde{u} \\ \tilde{n} &= C_1 x_1 + (D_1 \tilde{\Theta}_1 + I) \tilde{u} \end{cases} \quad (48)$$

$$\Theta_2(s) \sim \begin{cases} \dot{x}_2 &= A_2 x_2 + B_2 \tilde{\Theta}_2 \tilde{u} \\ \tilde{r} &= C_2 x_2 + D_2 \tilde{\Theta}_2 \tilde{u} \end{cases} \quad (49)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$. Some tedious algebra then yields the state-space matrices for a minimal realisation of $[M' - I \quad (G_2 M)']'$ as

$$\bar{A} = \begin{bmatrix} A_p + B_p \tilde{\Delta} D_c C_p & B_p \tilde{\Delta} C_c & B_p \tilde{\Delta} C_1 & -B_p \tilde{\Delta} D_c C_2 \\ B_c \Delta C_p & A_c + B_c \Delta D_p C_c & B_c \Delta D_p C_1 & -B_c \Delta C_2 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \quad (50)$$

$$B_0 = \begin{bmatrix} B_p \tilde{\Delta} \\ B_c \Delta D_p \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_p \tilde{\Delta} D_1 & -B_p \tilde{\Delta} D_c D_2 \\ B_c \Delta D_p D_1 & -B_c \Delta D_2 \\ B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad \tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_1 \\ \tilde{\Theta}_2 \end{bmatrix} \quad (51)$$

$$\bar{C}_1 = [\tilde{\Delta} D_c C_p \quad \tilde{\Delta} C_c \quad \tilde{\Delta} C_1 \quad -\tilde{\Delta} D_c C_2] \quad (52)$$

$$\bar{C}_2 = [\Delta C_p \quad \Delta D_p C_c \quad \Delta D_p C_1 \quad -\Delta D_p D_c C_2] \quad (53)$$

$$D_{01} = [\tilde{\Delta} D_c D_p] \quad \bar{D}_1 = [\tilde{\Delta} D_1 \quad -\tilde{\Delta} D_c D_2] \quad (54)$$

$$D_{02} = [\Delta D_p] \quad \bar{D}_2 = [\Delta D_p D_1 \quad -\Delta D_p D_c D_2] \quad (55)$$

where $\bar{A} \in \mathbb{R}^{n_x \times n_x}$, $B_0 \in \mathbb{R}^{x \times m}$, $\bar{B} \in \mathbb{R}^{x \times (m+q)}$, $\bar{C}_1 \in \mathbb{R}^{m \times n_x}$, $D_{01} \in \mathbb{R}^{m \times m}$, $\bar{D}_1 \in \mathbb{R}^{m \times (m+q)}$, $\bar{C}_2 \in \mathbb{R}^{q \times n_x}$, $D_{02} \in \mathbb{R}^{q \times m}$, $\bar{D}_2 \in \mathbb{R}^{q \times (m+q)}$ and $n_x := n_p + n_c + n_1 + n_2$.

C Proof of Lemma 1

This proof is virtually the same as the lemma in Grimm *et al.* (2001b). The difference here is that we allow the matrix $\Pi(w(t))$ to be varying. To prove the lemma, note that for all $w(t)$ we have that $\tilde{D}\Pi(w(t))w(t)$ is well defined; hence, it follows that the operator $I + \tilde{D}\Pi(w(t))$ is also well defined. Now assume that $I + \tilde{D}\Pi(w(t))$ is singular for some $w(t)$, which implies $\exists w \neq 0$ such that

$$[I + \tilde{D}\Pi(w(t))]w = 0 \quad (56)$$

$$\Rightarrow w'\Pi(w(t))V[I + \tilde{D}\Pi(w(t))]w = 0 \quad (57)$$

for some diagonal positive definite $V > 0$. Define $\bar{w} := \Pi(w(t))w$ and note that we have

$$\bar{w}'Vw + \frac{1}{2}\bar{w}V\tilde{D}\bar{w} + \frac{1}{2}\bar{w}'\tilde{D}'V\bar{w} = 0 \quad (58)$$

As V is diagonal and as $\pi(w_i(t)) \in [0, 1]$ it then follows that

$$\bar{w}'Vw = \sum_{i=0}^l \pi_i(w_i(t))v_iw_i^2 \geq \sum_{i=0}^l \pi_i^2(w_i(t))v_iw_i^2 = \bar{w}'V\bar{w} \quad (59)$$

Using this in equation (58) we have that

$$-\bar{w}'V\bar{w} - \frac{1}{2}\bar{w}V\tilde{D}\bar{w} - \frac{1}{2}\bar{w}'\tilde{D}'V\bar{w} \geq 0 \quad (60)$$

but as we have assumed $-2V - \tilde{D}'V - V\tilde{D} < 0$ we have a contradiction, showing that $I + \tilde{D}\Pi(w(t))$ is nonsingular for all $\Pi \in \mathbf{\Pi}$.

D Proof of uniqueness of solutions

The following Lemma will be needed in establishing uniqueness of solutions.

Lemma 2 *Let $\mathbf{\Pi}$ be defined as in Lemma 1. If $I + \tilde{D}K(\cdot)$ is invertible for all $K(\cdot) \in \mathbf{\Pi}$, then*

- $1 + d_{ii}K_i(\cdot)$ is invertible $\forall i$ and $\forall K(\cdot) \in \mathbf{\Pi}$, where d_{ii} is the ii 'th element of $\tilde{D}$.
- Any $k \times k$ ($k \leq m$) minor of $I + \tilde{D}K(\cdot)$ is invertible $\forall K(\cdot) \in \mathbf{\Pi}$.

Proof:

- Assume that for $i \in \{2, 3, \dots, m\}$ we have that $K_i(\cdot) = 0$. Then we have

$$I + \tilde{D}K(\cdot) = \begin{bmatrix} 1 + d_{11}K_1(\cdot) & 0 \\ 0 & I_{(m-1) \times (m-1)} \end{bmatrix} \quad (61)$$

For this to be invertible we therefore must have $1 + d_{ii}K_i(\cdot)$ invertible. A similar argument can be followed for $i \neq 1$.

- Assume that

$$K_i \neq 0 \quad i \in \{1, \dots, k\} \quad (62)$$

$$K_i = 0 \quad i \in \{k+1, \dots, m\} \quad (63)$$

Then, denoting $[M]_{k \times k}$ as the $k \times k$ 'th submatrix of M , we have

$$I + \bar{D}K(\cdot) = \begin{bmatrix} I_{k \times k} + [\bar{D}K(\cdot)]_{k \times k} & 0 \\ 0 & I_{(m-k) \times (m-k)} \end{bmatrix} \quad (64)$$

Obviously for this matrix to be invertible we must have that

$$I_{k \times k} + [\bar{D}K(\cdot)]_{k \times k} = [I + \bar{D}K(\cdot)]_{k \times k} \quad (65)$$

is invertible too, which proves the result when the last $m - k$ elements of $K(\cdot)$ are zero. More generally, after a suitable co-ordinate change we can write

$$I + \bar{D}K(\cdot) = V^{-1} \begin{bmatrix} I_{k \times k} + [\bar{D}K(\cdot)]_{k \times k} & 0 \\ 0 & I_{(m-k) \times (m-k)} \end{bmatrix} V \quad (66)$$

and thus the same argument can be used to show nonsingularity. $\square\square$.

To prove well-posedness we need to prove that the equation

$$u_d = \bar{C}_1 \bar{x} + \tilde{D}Dz(u_{lin} - u_d) \quad (67)$$

has a unique solution, u_d . Without loss of generality, but for considerably more simplicity, we set $u_{lin} = 0$. So we actually consider

$$u_d = \bar{C}_1 \bar{x} - \tilde{D}Dz(u_d) \quad (68)$$

or, after re-arranging,

$$u_d + \tilde{D}Dz(u_d) = \bar{C}_1 \bar{x} =: z \quad (69)$$

From the main proof of Theorem 1, we can set $Dz(u_d) = K(u_d)u_d$, where

$$K(u_d) = \text{diag} \{k_1(u_{d,1}), \dots, k_m(u_{d,m})\}, \quad k_i \in [0, 1] \quad (70)$$

By Lipschitz continuity of $Dz(\cdot)$ we know that $K(\cdot)$ is also Lipschitz continuous. Also, by Lemma 1, we know that the matrix

$$(I + \tilde{D}K(u_d))^{-1} \quad (71)$$

exists for all u_d . To prove that a unique solution exists we use a contradiction argument.

D.1 Proof by contradiction

Assume there exist two solutions to (69), that is $u_d = u_a$ and $u_d = u_b$. Assume further that, after any necessary co-ordinate change, we can isolate the zero components of u_a as

$$u_a = \begin{bmatrix} u_{a1} \\ u_{a2} \end{bmatrix} = \begin{bmatrix} u_{a1} \\ 0 \end{bmatrix} \quad (72)$$

that is $0 \neq u_{a1} \in \mathbb{R}^k$, $(k \leq m)$ and $0 = u_{a2} \in \mathbb{R}^{m-k}$. Now as $u_{a1} \neq 0$, it follows that we can write u_{b1} as

$$u_{b1} = \Sigma_1 u_{a1}, \quad \Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_k) \quad (73)$$

where u_b has been partitioned as

$$u_b = \begin{bmatrix} u_{b1} \\ u_{b2} \end{bmatrix}, \quad u_{b1} \in \mathbb{R}^k, \quad u_{b2} \in \mathbb{R}^{m-k} \quad (74)$$

Partitioning (69) compatibly with u_a and u_b we get

$$\begin{bmatrix} u_{a1} + \tilde{D}_{11}K_1(u_{a1})u_{a1} \\ \tilde{D}_{21}K_1(u_{a1})u_{a1} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (75)$$

and also

$$\begin{bmatrix} u_{b1} + \tilde{D}_{11}K_1(u_{b1})u_{b1} + \tilde{D}_{12}K_2(u_{b2})u_{b2} \\ u_{b2} + \tilde{D}_{21}K_1(u_{b1})u_{b1} + \tilde{D}_{22}K_2(u_{b2})u_{b2} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (76)$$

So, after re-arranging we must have,

$$[I + \tilde{D}_{11}K_1(u_{a1})]u_{a1} = [I + \tilde{D}_{11}K_1(u_{b1})]u_{b1} + \tilde{D}_{12}K_2(u_{b2})u_{b2} \quad (77)$$

$$\tilde{D}_{21}K_1(u_{a1})u_{a1} = [I + \tilde{D}_{22}K_2(u_{b2})]u_{b2} + \tilde{D}_{21}K_1(u_{b1})u_{b1} \quad (78)$$

From equation (78) we have that

$$[I + \tilde{D}_{22}K_2(u_{b2})]u_{b2} = \tilde{D}_{21}[K_1(u_{a1})u_{a1} - K_1(u_{b1})u_{b1}] \quad (79)$$

Next, because $K(\cdot) \in \mathbf{\Pi}$ and, from Lemma 2, we have that each $k \times k$ minor of $[I + \tilde{D}K(\cdot)]$ is invertible, it follows that

$$u_{b2} = [I + \tilde{D}_{22}K_2(u_{b2})]^{-1} \tilde{D}_{21}[K_1(u_{a1})u_{a1} - K_1(u_{b1})u_{b1}] \quad (80)$$

Using this expression in equation (77) and defining

$$X(u_{b2}) := \tilde{D}_{12}K_2(u_{b2})[I + \tilde{D}_{22}K_2(u_{b2})]^{-1}\tilde{D}_{21} \quad (81)$$

we obtain

$$[I + \tilde{D}_{11}K_1(u_{a1})]u_{a1} = [I + \tilde{D}_{11}K_1(u_{b1})]u_{b1} + X(u_{b2})[K_1(u_{a1})u_{a1} - K_1(u_{b1})u_{b1}] \quad (82)$$

Re-arranging this yields

$$[I + \tilde{D}_{11}K_1(u_{a1}) - X(u_{b2})K_1(u_{a1})]u_{a1} = [I + \tilde{D}_{11}K_1(u_{b1}) - X(u_{b2})K_1(u_{b1})]u_{b1} \quad (83)$$

However, by construction we have that $u_{b1} = \Sigma_1 u_{a1}$ so we have

$$[I + \tilde{D}_{11}K_1(u_{a1}) - X(u_{b2})K_1(u_{a1})]u_{a1} = [I + \tilde{D}_{11}K_1(\Sigma_1 u_{a1}) - X(u_{b2})K_1(\Sigma_1 u_{a1})]\Sigma_1 u_{a1} \quad (84)$$

Therefore we must have

$$I + \tilde{D}_{11}K_1(u_{a1}) - X(u_{b2})K_1(u_{a1}) = \Sigma_1 + \tilde{D}_{11}K_1(\Sigma_1 u_{a1})\Sigma_1 - X(u_{b2})K_1(\Sigma_1 u_{a1})\Sigma_1 \quad (85)$$

We shall now show that a necessary and sufficient condition for this to hold is that $\Sigma_1 = I$.

Sufficiency: It is easy to see that setting $\Sigma_1 = I$ is sufficient for equation (85) to hold.

Necessity: Equation (85) can equivalently be written as

$$I + (\tilde{D}_{11} - X(u_{b2}))K_1(u_{a1}) = \Sigma_1 + (\tilde{D}_{11} - X(u_{b2}))K_1(\Sigma u_{a1})\Sigma_1 \quad (86)$$

A necessary condition for this to hold is that

$$1 + (d_{ii} - x_{ii}(u_{b2}))k_{1,i}(u_{a1,i}) = \sigma_i + (d_{ii} - x_{ii}(u_{b2}))k_{1,i}(\sigma_i u_{a1,i})\sigma_i \quad \forall i \quad (87)$$

Defining $\epsilon_{ii} := d_{ii} - x_{ii}(u_{b2})$, a necessary condition for equation (87) to hold is that

$$|1 + \epsilon_{ii}k_{1,i}(u_{a1,i})| = |\sigma_i| |1 + \epsilon_{ii}k_{1,i}(\sigma_i u_{a1,i})| \quad \forall i \quad (88)$$

So assume that $\sigma_i \neq 1$ for some i (this is necessary for $u_a \neq u_b$), and at first we assume that $0 \leq \sigma_i < 1$ (this is equivalent to $u_{a,i} > u_{b,i}$, and will be removed later) which, as $k_i(\cdot)$ is monotonically nondecreasing, implies that

$$|\sigma_i u_{a1,i}| < |u_{a1,i}| \quad (89)$$

$$|k_{1,i}(\sigma_i u_{a1,i})| \leq |k_{1,i}(u_{a1,i})| \quad (90)$$

Now we shall consider three cases and show that for $0 \leq \sigma_i < 1$, equation (88) is contradicted in all cases.

- Assume $\epsilon_{ii} > 0$

We have that

$$|1 + \epsilon_{ii} k_i(u_{1a,i})| = 1 + |\epsilon_{ii}| |k_{1,i}(u_{1a,i})| \quad (91)$$

$$\geq 1 + |\epsilon_{ii}| |k_{1,i}(\sigma_i u_{1a,i})| \quad (92)$$

$$> \sigma_i (1 + |\epsilon_{ii}| |k_{1,i}(\sigma_i u_{1a,i})|) \quad (93)$$

This contradicts equation (88)

- Assume $-1 < \epsilon_{ii} < 0$

We have

$$|1 + \epsilon_{ii} k_{1,i}(u_{a1,i})| = \begin{cases} 1 & |u_{a1,i}| < \bar{u}_i \\ |1 + \epsilon_{ii}(\frac{|u_{a1,i}| - \bar{u}_i}{|u_{a1,i}|})| & |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (94)$$

$$= \begin{cases} 1 & |u_{a1,i}| < \bar{u}_i \\ 1 - |\epsilon_{ii}| + \frac{|\epsilon_{ii}| \bar{u}_i}{|u_{a1,i}|} & |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (95)$$

$$> \begin{cases} \sigma_i & |u_{a1,i}| < \bar{u}_i \\ \sigma_i (1 - |\epsilon_{ii}|) + \frac{|\epsilon_{ii}| \bar{u}_i}{|u_{a1,i}|} & |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (96)$$

$$= \begin{cases} \sigma_i & |u_{a1,i}| < \bar{u}_i \\ \sigma_i - \sigma_i |\epsilon_{ii}| (1 - \frac{\bar{u}_i}{\sigma_i |u_{a1,i}|}) & |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (97)$$

$$= \begin{cases} \sigma_i & |u_{a1,i}| < \bar{u}_i \\ \sigma_i - \sigma_i |\epsilon_{ii}| (\frac{\sigma_i |u_{a1,i}| - \bar{u}_i}{\sigma_i |u_{a1,i}|}) & |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (98)$$

$$= \begin{cases} \sigma_i & |u_{a1,i}| < \bar{u}_i \\ \sigma_i |1 - |\epsilon_{ii}| (\frac{\sigma_i |u_{a1,i}| - \bar{u}_i}{\sigma_i |u_{a1,i}|})| & |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (99)$$

$$> \begin{cases} \sigma_i & |u_{a1,i}| < \bar{u}_i \\ \sigma_i & \bar{u}_i \leq |u_{a1,i}| < \bar{u}_i \sigma_i^{-1} \\ \sigma_i |1 - |\epsilon_{ii}| (\frac{\sigma_i |u_{a1,i}| - \bar{u}_i}{\sigma_i |u_{a1,i}|})| & |u_{a1,i}| \geq \bar{u}_i \sigma_i^{-1} \end{cases} \quad (100)$$

$$= \begin{cases} \sigma_i & \sigma_i |u_{a1,i}| < \bar{u}_i \\ \sigma_i |1 - |\epsilon_{ii}| (\frac{\sigma_i |u_{a1,i}| - \bar{u}_i}{\sigma_i |u_{a1,i}|})| & \sigma_i |u_{a1,i}| \geq \bar{u}_i \end{cases} \quad (101)$$

$$= \sigma_i |1 - |\epsilon_{ii}| k_i(\sigma_i u_{a1,i})| \quad (102)$$

This contradicts equation (88). Note that there is some quite extensive algebra between equation (99) and inequality (100).

- Assume $\epsilon_{ii} \leq -1$

We have that

$$|1 + \epsilon_{ii} k_{1,i}(u_{a1,i})| > \sigma_i |1 + \epsilon_{ii} k_i(u_{a1,i})| \quad (103)$$

$$= \sigma_i ||\epsilon_{ii}k_i(u_{a1,i})| - 1| \quad (104)$$

$$\geq \sigma_i ||\epsilon_{ii}k_i(\sigma_i u_{a1,i})| - 1| \quad (105)$$

$$= \sigma_i |1 + \epsilon_{ii}k_i(\sigma_i u_{a1,i})| \quad (106)$$

which contradicts (88).

In other words we have proved that for equation (87) to hold, we cannot have $\sigma_i < 1$. Now we remove the assumption that $\sigma_i < 1$: if $\sigma_i > 1$, we have that $u_{a1,i} < u_{b1,i}$ and the same argument as above can be applied.

We have proved that for equation (87) to hold we must have $|\sigma_i| = 1$. To prove that the solution is unique, we must prove that equation (87) holds only with $\sigma_i = 1$ (and not -1). To see this note that a necessary condition for equation (87) to hold is that

$$\text{sign}\{1 + \epsilon_{ii}k_{1,i}(u_{a1,i})\} = \text{sign}\{\sigma_i\} \text{sign}\{1 + \epsilon_{ii}k_{1,i}(\sigma_i u_{1a,i})\} \quad (107)$$

As $\text{sign}\{1 + \epsilon_{ii}k_{1,i}(\sigma_i u_{1a,i})\} = \text{sign}\{1 + \epsilon_{ii}k_{1,i}(u_{1a,i})\} \forall u_{a1,i}$ (because $k_{1,i}(\cdot) \in [0, 1]$), it follows that for equation (107) to hold, we must have $\text{sign}(\sigma_i) = 1$. Or in other words we must have $\sigma_i = 1$ for a unique solution. In other words we have proved the $u_{a1} = u_{b1}$. To prove that $u_{b2} = u_{a2}$ consider equation (79) and note that as we now have that $u_{a1} = u_{b1}$, this equation becomes

$$[I + \tilde{D}_{22}K_2(u_{b2})]u_{b2} = 0 \quad (108)$$

From Lemma 2 we know that $[I + \tilde{D}_{22}K_2(u_{b2})]$ has full rank $\forall K_2(\cdot) \in \mathbf{\Pi}$, which implies $u_{b2} = 0$. Therefore we have $u_{b2} = u_{a2} = 0$. Thus $u_a = u_b$ which implies the solutions are unique. $\square\square$.

E Missile dynamics

The simplified roll-yaw dynamics of the missile are given by

$$A_p = \begin{bmatrix} -0.818 & -0.999 & 0.349 \\ 80.29 & -0.579 & 0.009 \\ -2734 & 0.05621 & -2.10 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.147 & 0.012 \\ -194.4 & 37.61 \\ -2716 & -1093 \end{bmatrix}, \quad B_{pd} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (109)$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_p = D_{pd} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (110)$$

The LQG/LTR controller used is described by

$$A_c = \begin{bmatrix} A_{c1} & B_{c1} \\ 0 & 0 \end{bmatrix}, \quad B_c = -B_{cr} = -\begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C_c = [C_{c1} \quad 0], \quad D_{cr} = D_c = 0 \quad (111)$$

where

$$A_{c1} = \begin{bmatrix} -0.29 & -107.8 & 6.67 & -2.58 & -0.4 \\ 107.68 & -97.81 & 63.95 & -4.52 & -5.35 \\ -6.72 & 64.82 & -54.19 & -40.79 & 5.11 \\ 3.21 & 2.1 & 29.56 & -631.15 & 429.89 \\ 0.36 & -3.39 & 3.09 & -460.03 & -0.74 \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} 2.28 & 0.48 \\ -40.75 & 2.13 \\ 18.47 & -0.22 \\ -2.07 & -44.68 \\ -0.98 & -1.18 \end{bmatrix} \quad (112)$$

$$C_{c1} = \begin{bmatrix} 0.86 & 8.54 & -1.71 & 43.91 & 1.12 \\ 2.17 & 39.91 & -18.39 & -8.51 & 1.03 \end{bmatrix} \quad (113)$$