Supplement to:

A geometric perspective on counting non-negative integer solutions and combinatorial identities

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Theorem 5.1 is the result of counting the number of nonnegative integer solutions to $x + y + z = n$ for $x < i$, $y < j$, and $z < k$ in two different ways.

Theorem 5.1: For nonnegative integers i, j, k , and n such that $\max\{i, j, k\} \leq n$,

$$\begin{aligned} T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1} + 1 \\ = T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1} \end{aligned}$$

As a challenge, see if you can construct the figures to prove Theorem 5.1 for the following cases:

- 1) $k \leq n - i - j$
- 2) $0 < n - i - j < k$
 - a) $k < \min\{n - i, n - j\}$
 - b) $\min\{n - i, n - j\} \leq k < \max\{n - i, n - j\}$
 - c) $\max\{n - i, n - j\} \leq k$
- 3) $n - i - j \leq 0$
 - a) $k < \min\{n - i, n - j\}$
 - b) $\min\{n - i, n - j\} \leq k < \max\{n - i, n - j\}$
 - c) $\max\{n - i, n - j\} \leq k$.

To aid you in creating your figures we provide n , i , j , and k for each case. For all cases we use $n = 20$. The parameters i , j , and k can be written as (i, j, k) , and we use the following parameters in our figures: Case (1): (9,5,2); Case (2a): (9,6,9); Case (2b): (9,5,13); Case (2c): (9,5,18); Case (3a): (16,10,2); Case (3b): (16,10,7); and Case (3c): (16,10,14). As a reminder, Case (1) is a degenerate case because there are no solutions satisfying the constraints.

Proof. Case (1): See Figure 1.

Figure 1. Case (1): $k \leq n - i - j$.

$$\begin{aligned} & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1} + 1 \\ &= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - T_{n-i-j-k+1} \end{aligned}$$

$$\begin{aligned} & | + \cup \oplus \cup * \cup \otimes | - | * \cup \otimes | - | \oplus \cup \otimes | - | all | \\ & + | \times \cup \otimes \cup * \cup \otimes | + | \odot \cup \otimes \cup \oplus \cup \otimes | - | \otimes \cup \otimes | + 1 \\ &= | all | - | \times \cup \otimes \cup * \cup \otimes | - | \odot \cup \otimes \cup \oplus \cup \otimes | \\ & - | + \cup \oplus \cup * \cup \otimes | + | \otimes \cup \otimes | + | * \cup \otimes | + | \oplus \cup \otimes | - | \otimes | \end{aligned}$$

Case (2a): See Figure 2.

Figure 2. Case (2a): $0 < n - i - j < k$ with $k < \min\{n - i, n - j\}$.

$$\begin{aligned} & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1 \\ &= T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} \end{aligned}$$

$$\begin{aligned} & | \bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus \cup * | - | \times \cup \otimes \cup * | \\ & - | \odot \cup \otimes \cup \oplus | - | + \cup \oplus \cup * | + | * | + | \oplus | + 1 \\ &= | all | - | \times \cup \otimes \cup * | - | \odot \cup \otimes \cup \oplus | - | + \cup \oplus \cup * | + | \otimes | + | * | + | \oplus | \end{aligned}$$

Case (2b): See Figure 3.

Figure 3. Case (2b): $0 < n - i - j < k$ with $\min\{n - i, n - j\} \leq k < \max\{n - i, n - j\}$.

$$\begin{aligned}
 & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1 \\
 & = T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}
 \end{aligned}$$

$$\begin{aligned}
 & |\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus| - |\times \cup \otimes| - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus| + |\oplus| + 1 \\
 & = |\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus| - |\times \cup \otimes| - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus| + |\otimes| + |\oplus|
 \end{aligned}$$

Case (2c): See Figure 4.

Figure 4. Case (2c): $0 < n - i - j < k$ with $\max\{n - i, n - j\} \leq k$.

$$\begin{aligned}
 & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - \mathbb{T}_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1 \\
 & = T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + T_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}
 \end{aligned}$$

$$\begin{aligned}
 & |\bullet \cup \odot \cup \times \cup \otimes \cup +| - |\odot \cup \otimes| - |\times \cup \otimes| - |+| + 1 \\
 & = |\bullet \cup \odot \cup \times \cup \otimes \cup +| - |\times \cup \otimes| - |\odot \cup \otimes| - |+| - |\otimes|
 \end{aligned}$$

Case (3a): See Figure 5.

Figure 5. Case (3a): $n - i - j \leq 0$ with $k < \min\{n - i, n - j\}$.

$$\begin{aligned} & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1 \\ & = T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + \mathbb{T}_{n-i-j+1} + T_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} \end{aligned}$$

$$\begin{aligned} & |\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus \cup * \cup -| - |\times \cup \otimes \cup *| \\ & - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus \cup *| - |-| + |*| + |\oplus| + 1 \\ & = |\bullet \cup \odot \cup \times \cup + \cup \oplus \cup *| - |\times \cup *| - |\odot \cup \oplus| - |+ \cup \oplus \cup *| + |*| + |\oplus| \end{aligned}$$

Case (3b): See Figure 6.

Figure 6. Case (3b): $n - i - j \leq 0$ with $\min\{n - i, n - j\} \leq k < \max\{n - i, n - j\}$.

$$\begin{aligned} & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1 \\ & = T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + \mathbb{T}_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + T_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} \end{aligned}$$

$$\begin{aligned} & |\bullet \cup \odot \cup \times \cup \otimes \cup + \cup \oplus \cup -| - |\times \cup \otimes| - |\odot \cup \otimes \cup \oplus| - |+ \cup \oplus| - |-| + |\oplus| + 1 \\ & = |\bullet \cup \odot \cup \times \cup + \cup \oplus| - |\odot \cup \oplus| - |\times| - |+ \cup \oplus| + |\oplus| \end{aligned}$$

Case (3c): See Figure 7.

Figure 7. Case (3c): $n - i - j \leq 0$ with $\max\{n - i, n - j\} \leq k$.

$$\begin{aligned}
 & T_{i+j+1} - T_{j+1} - T_{i+1} - T_{n+1-k} - T_{i+j-n-2} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1} + 1 \\
 & = T_{n+1} - T_{n-i+1} - T_{n-j+1} - T_{n-k+1} + \mathbb{T}_{n-i-j+1} + \mathbb{T}_{n-i-k+1} + \mathbb{T}_{n-j-k+1} - \mathbb{T}_{n-i-j-k+1}
 \end{aligned}$$

$$\begin{aligned}
 & |\bullet \cup \odot \cup \times \cup \otimes \cup + \cup -| - |\times \cup \otimes| - |\odot \cup \otimes| - |+|-|-| + 1 \\
 & = |\bullet \cup \odot \cup \times \cup +| - |\times|-|\odot|-|+|
 \end{aligned}$$

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