Supplemental material for "A varying-coefficient expectile model for estimating Value at Risk"

Shangyu Xie, Yong Zhou and Alan T.K. Wan

This supplementary file contains the detailed proof of Lemma A.1 and Lemma A.2.

Throughout the supplementary, we use M to represent a (generic) constant, which can take on different values at different places. Now, write $\psi(t|\mathbf{x}, u) = E[Q_\theta(Y - \eta(U, \mathbf{X}) + t)|\mathbf{X} = \mathbf{x}, U = u]$, and denote $\partial\psi(t|\mathbf{x}, u)/\partial t$ and $\partial^2\psi(t|\mathbf{x}, u)/\partial t^2$ by $\psi'(t|\mathbf{x}, u)$ and $\psi''(t|\mathbf{x}, u)$ respectively.

Proof of Lemma A.1: Recall that

$$G_T(\xi) \equiv G_T(\xi; \theta, \mathbf{X}, U, u_0) = \sum_{t=1}^T \{Q_\theta(Y_t^* - \xi^T \mathbf{Z}_t^* / \sqrt{Th}) - Q_\theta(Y_t^*)\} K_t, \quad (\text{A.1})$$

where $Y_t^* = Y_t - \beta_0^T \mathbf{Z}_t = Y_t - \bar{\eta}(u_0, U_t, \mathbf{X}_t)$. Write

$$G_T(\xi) = E(G_T(\xi)|\mathbf{X}, U) - \frac{1}{\sqrt{Th}} \left(\sum_{t=1}^T Q'_\theta(Y_t^*) \mathbf{Z}_t^* K_t - E(Q'_\theta(Y_t^*)|\mathbf{X}_t, U_t) \mathbf{Z}_t^* K_t \right)^T \xi + R_T^*(\xi). \quad (\text{A.2})$$

Let us show that

$$E(G_T(\xi)|\mathbf{X}, U) = -\frac{1}{\sqrt{Th}} E(Q'_\theta(Y_t^*)|\mathbf{X}_t, U_t) \mathbf{Z}_t^{*T} K_t \xi + \frac{1}{2} \xi^T D(\theta, u_0) \xi + o_p(1).$$

Now, from (A.1) and by the definition of $\psi(t|\mathbf{x}, u)$, we have

$$\begin{aligned} & E(G_T(\xi)|\mathbf{X}, U) \\ &= \sum_{t=1}^T \left[\psi(\eta(U_t, \mathbf{X}_t) - \bar{\eta}(u_0, U_t, \mathbf{X}_t) - \frac{\xi^T \mathbf{Z}_t^*}{\sqrt{Th}} | \mathbf{X}_t, U_t) - \psi(\eta(U_t, \mathbf{X}_t) - \bar{\eta}(u_0, U_t, \mathbf{X}_t) | \mathbf{X}_t, U_t) \right] K_t \\ &= -\sum_{t=1}^T \psi'(\eta(U_t, \mathbf{X}_t) - \bar{\eta}(u_0, \mathbf{X}_t, U_t) | \mathbf{X}_t, U_t) \frac{\xi^T \mathbf{Z}_t^*}{\sqrt{Th}} K_t \\ &\quad + \frac{1}{2} \sum_{t=1}^T \psi''(\eta(U_t, \mathbf{X}_t) - \bar{\eta}(u_0, \mathbf{X}_t, U_t) | \mathbf{X}_t, U_t) \left(\frac{\xi^T \mathbf{Z}_t^*}{\sqrt{Th}} \right)^2 K_t (1 + o_p(1)) \\ &= -\frac{1}{\sqrt{Th}} \sum_{t=1}^T E(Q'_\theta(Y_t^*)|\mathbf{X}_t, U_t) \mathbf{Z}_t^{*T} K_t \xi + \frac{1}{2Th} \xi^T \left(\sum_{t=1}^T K_t \psi''(0|\mathbf{X}_t, U_t) \mathbf{Z}_t^* \mathbf{Z}_t^{*T} \right) \xi (1 + o_p(1)). \end{aligned} \quad (\text{A.3})$$

Using Lemma B.2 (see below) we can show that $R_T^*(\xi) = o_p(1)$. Taking this together with (A.2), (A.3) and Lemma B.3 (see below) yields

$$G_T(\xi) = \frac{1}{2} \xi^T D(\theta, u_0) \xi - \frac{1}{\sqrt{Th}} W_T^T \xi + R_T(\xi), \quad (\text{A.4})$$

where $R_T(\xi) = o_p(1)$ for each fixed ξ , and

$$W_T = \sum_{t=1}^T Q'_\theta(Y_t^*) K_t \mathbf{Z}_t^*.$$

Next let us show that

$$\sup_{\xi \in \mathcal{K}} |R_T(\xi)| = o_p(1)$$

for any compact set $\mathcal{K}$ of ξ .

It is readily seen that $W_T/\sqrt{Th}$ has a bounded second moment and is thus stochastically bounded. As the convex function $G_T(\xi)$ converges in probability to the convex function $(1/2)\xi^T D(w, u_0)\xi - W_T^T \xi/\sqrt{Th}$, it follows from the convexity lemma (Pollard, 1991) that for any compact set $\mathcal{K}$,

$$\sup_{\xi \in \mathcal{K}} |R_T(\xi)| = o_p(1).$$

This completes the proof of Lemma A.1.

Proof of Lemma A.2. Let us first show that

$$E\{\mathbf{Z}_1^* Q'_\theta(Y_1^*) K_1\} = \frac{h^3}{2} f_U(u_0) \begin{pmatrix} a''(u_0)\mu_2 \\ a''(u_0)\mu_3 \end{pmatrix} \otimes \Gamma(\theta, u_0)\{1 + o(1)\}$$

and

$$E\{\mathbf{Z}_1^* Q'_\theta(Y_1^*) K_1\}^{\otimes 2} = h f_U(u_0) \begin{pmatrix} \nu_0 & \nu_1 \\ \nu_1 & \nu_2 \end{pmatrix} \otimes A(\theta, u_0)\{1 + o(1)\}$$

as $T \rightarrow \infty$.

Note that $E\{\mathbf{Z}_1^* Q'_\theta(Y_1 - \eta(U_1, \mathbf{X}_1)) K_1\} = 0$, and a Taylor seires expansion $\eta(u, \mathbf{X}_t) = \bar{\eta}(u_0, U_t, \mathbf{X}_t) + \frac{1}{2} \sum_{j=1}^p a_j''(u_0) X_{tj} (U_t - u_0)^2 + o(h^2)$, we have

$$\begin{aligned} & E\{\mathbf{Z}_1^* Q'_\theta(Y_1^*) K_1\} \\ &= E\left[\mathbf{Z}_1^* Q''_\theta(Y_1 - \eta(U_1, \mathbf{X}_1)) K_1 \mathbf{X}_1^T \left\{\frac{1}{2} \mathbf{a}''(u_0)(U_1 - u_0)^2\right\}\right] \{1 + o(1)\} \\ &= E\left[\left(1, \frac{U_1 - u_0}{h}\right)^T K\left(\frac{U_1 - u_0}{h}\right) \frac{1}{2} \mathbf{a}''(u_0)(U_1 - u_0)^2 \right. \\ &\quad \left. \{2(1 - \theta)I(Y_1 - \eta(U_1, \mathbf{X}_1) \leq 0) + 2\theta I(Y_1 - \eta(U_1, \mathbf{X}_1) > 0)\} \otimes \mathbf{X}_1^T \mathbf{X}_1\right] \{1 + o(1)\} \\ &= \frac{h^3}{2} f_U(u_0) \begin{pmatrix} \mathbf{a}''(u_0)\mu_2 \\ \mathbf{a}''(u_0)\mu_3 \end{pmatrix} \otimes \Gamma(\theta, u_0)\{1 + o(1)\} \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} & E\{\mathbf{Z}_1^* Q'_\theta(Y_1^*) K_1\}^{\otimes 2} \\ &= E\left\{\begin{pmatrix} 1 & \frac{U_1 - u_0}{h} \\ \frac{U_1 - u_0}{h} & (\frac{U_1 - u_0}{h})^2 \end{pmatrix} K^2\left(\frac{U_1 - u_0}{h}\right) Q_\theta'^2(Y_1^*) \otimes \mathbf{X}_1 \mathbf{X}_1^T\right\} \\ &= h f_U(u_0) \begin{pmatrix} \nu_0 & \nu_1 \\ \nu_1 & \nu_2 \end{pmatrix} \otimes A(\theta, u_0)\{1 + o(1)\} \\ &= h \Sigma(\theta, u_0)\{1 + o(1)\}. \end{aligned} \tag{A.6}$$

Our proof of the asymptotic normality of W_T is based on the Cramér-Wold method. Now, for any unit vector $d \in R^{2p}$,

$$\begin{aligned} S_T &\equiv \frac{d^T}{\sqrt{Th}}(W_T - EW_T) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \frac{d^T}{\sqrt{h}} \left\{ \mathbf{Z}_t^* Q'_\theta(Y_t^*) K_t - E\{\mathbf{Z}_t^* Q'_\theta(Y_t^*) K_t\} \right\} \\ &\triangleq \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} W_{T,t}^*. \end{aligned}$$

Let us partition $\{1, 2, \dots, T\}$ into $2k+1$ subsets with large blocks of size $r = r_T$ and small blocks of size $s = s_T$. Set

$$k = \left\lfloor \frac{T}{r_T + s_T} \right\rfloor. \quad (\text{A.7})$$

Similar to Lemma B.1, and combining (A.5) and (A.6), we obtain

$$\text{Var}(W_{T,0}^*) = d^T \Sigma(\theta, u_0) d(1 + o(1)) \equiv \Omega(\theta, u_0)(1 + o(1)) \quad (\text{A.8})$$

and

$$\sum_{l=0}^{T-1} |\text{cov}(W_{T,0}^*, W_{T,l}^*)| = o(1). \quad (\text{A.9})$$

Now, define the random variables

$$\begin{aligned} \eta_j &= \sum_{i=j(r+s)}^{j(r+s)+r-1} W_{T,i}^*, \\ \varsigma_j &= \sum_{i=j(r+s)+r}^{(j+1)(r+s)} W_{T,i}^*, \end{aligned}$$

and

$$\zeta_k = \sum_{i=k(r+s)}^{T-1} W_{T,i}^*,$$

for $0 \leq j \leq k-1$, and denote

$$\begin{aligned} S_T &= \frac{1}{\sqrt{T}} \left\{ \sum_{j=0}^{k-1} \eta_j + \sum_{j=0}^{k-1} \varsigma_j + \zeta_k \right\} \\ &\equiv \frac{1}{\sqrt{T}} \{S_{T,1} + S_{T,2} + S_{T,3}\}. \end{aligned}$$

Now, from Theorem 18.4.1 of Ibragimov and Linnik (1971), to prove $S_T \xrightarrow{\mathcal{L}} N(0, \Omega(\theta, u_0))$, it suffices to show that

$$\frac{1}{T} E[S_{T,2}]^2 \rightarrow 0, \quad \frac{1}{T} E[S_{T,3}]^2 \rightarrow 0, \quad (\text{A.10})$$

$$\left| E[\exp(itS_{T,1})] - \prod_{j=0}^{k-1} E[\exp(it\eta_j)] \right| \rightarrow 0, \quad (\text{A.11})$$

$$\frac{1}{T} \sum_{j=0}^{k-1} E(\eta_j^2) \rightarrow \Omega(w, u_0) = d^T \Sigma d \quad (\text{A.12})$$

and

$$\frac{1}{T} \sum_{j=0}^{k-1} E \left[\eta_j^2 I\{|\eta_j| \geq \varepsilon \Sigma(w, u_0) \sqrt{T}\} \right] \rightarrow 0, \quad (\text{A.13})$$

for every $\varepsilon > 0$. Equation (A.10) implies that $S_{T,2}$ and $S_{T,3}$ are asymptotically negligible in probability, (A.11) states that the summands η_j in $S_{T,1}$ are asymptotically independent, and (A.12) and (A.13) are the standard Lindeberg-Feller conditions for the asymptotic normality of $S_{T,1}$ under the independent setup.

Now, to prove (A.10), we consider large block sizes. Condition A.3 implies that there is a sequence of positive constants $\gamma_T \rightarrow \infty$ such that

$$\gamma_T s_T = o(\sqrt{Th_T})$$

and

$$(Th_T^{-1})^{1/2} \alpha(s_T) \rightarrow 0. \quad (\text{A.14})$$

Define the large-block size r_T by $r_T = \lfloor (Th_T)^{1/2} / \gamma_T \rfloor$ and the small-block size s_T . Then it can be easily shown from (A.14) that as $T \rightarrow \infty$,

$$s_T / r_T \rightarrow 0, \quad r_T / n \rightarrow 0, \quad r_T (Th_T)^{-1/2} \rightarrow 0 \quad (\text{A.15})$$

and

$$(T / r_T) \alpha(s_T) \rightarrow 0. \quad (\text{A.16})$$

By the properties of stationary, (A.8) and (A.9), we have

$$\begin{aligned} E[S_{T,2}]^2 &= \sum_{j=0}^{k-1} \text{var}(\varsigma_j) + 2 \sum_{0 \leq t < l \leq k-1} \text{cov}(\varsigma_t, \varsigma_l) \\ &= ks_T [\Omega(\theta, u_0) + o(1)] + O\left(T \sum_{j=0}^{T-1} |\text{cov}(W_{T,0}^*, W_{T,j}^*)|\right) \\ &= O(ks_T) + o(T). \end{aligned}$$

Hence by (A.7) and (A.15), we have

$$\frac{1}{T} E[S_{T,2}^2] = O(ks_T T^{-1}) + o(1) = o(1).$$

Similarly, we can show that

$$\begin{aligned} &\frac{1}{T} E[S_{T,3}]^2 \\ &\leq \frac{1}{T} (T - k(s_T + r_T)) \text{var}(W_{T,0}^*) + 2 \sum_{j=1}^{T-1} |\text{cov}(W_{T,0}^*, W_{T,j}^*)| \\ &\leq \frac{r_T + s_T}{T} \Omega(\theta, u_0) + o(1) = o(1). \end{aligned}$$

To prove (A.11), we use Lemma 1.1 of Volkonskii and Rozanov (1959) to obtain

$$\left| E[\exp(itS_{T,1})] - \Pi_{j=0}^{k-1} E[\exp(it\eta_j)] \right| \leq 16(T/r_T) \alpha(s_T) \rightarrow 0.$$

Also, (A.12) follows by using (A.8), (A.9) and (A.15).

Finally, by the definition of $W_{T,i}^*$, we have

$$\frac{1}{\sqrt{T}} \max_{0 \leq j \leq p-1} |\eta_j| \leq Mr_T / \sqrt{Th_T} \rightarrow 0$$

and

$$\max_{0 \leq j \leq p-1} P(|\eta_j| \geq \varepsilon \Omega(\theta, u_0)^{1/2} \sqrt{T}) \rightarrow 0$$

as $T \rightarrow \infty$.

Here we establish the asymptotic normality of S_T , such as

$$S_T \xrightarrow{\mathcal{L}} N(0, \Omega(\theta, u_0)).$$

This completes the proof of Lemma A.2.

Appendix B

This appendix provides the proofs of some of the results used for proving Lemma A.1 and Lemma A.2.

Lemma B.1 *For any $x, y \in \mathbb{R}$, and $w \in (0, 1)$,*

$$|Q_\theta(x+y) - Q_\theta(x) - Q'_\theta(x)y| \leq 4y^2,$$

and

$$|Q'_\theta(x+y) - Q'_\theta(x) - Q''_\theta(x)y| \leq 4|y|,$$

where $Q'_\theta(x) = d^2\{Q_\theta(x)\}/dx^2$ for $x \neq 0$, and $Q''_\theta(0) = 0$.

Proof: See Yao and Tong (1996).

Lemma B.2 *For $\xi \in \mathbb{R}^{2p}$, $R_T^*(\xi) \rightarrow 0$ in probability.*

Proof: From the expression of $R_T^*(\xi)$ in (A.2), we have

$$R_T^*(\xi) = \sum_{t=1}^T \{V_t - E(V_t | \mathbf{X}_t, U_t)\},$$

where

$$V_t \equiv V_t(\xi; \theta, x, u_0) = \left\{ Q_\theta(Y_t^* - \mathbf{Z}_t^{*T} \xi / \sqrt{Th}) - Q_\theta(Y_t^*) - Q'_\theta(Y_t^*) \mathbf{Z}_t^{*T} \xi / \sqrt{Th} \right\} K_t.$$

Note that $ER_T^*(\xi) = 0$. From Lemma B.1,

$$EV_1^2 \leq 16E(\mathbf{Z}_1^{*T} \xi / \sqrt{Th})^4 K_1^2 = O\left(\frac{1}{T^2 h}\right).$$

Then we have

$$ER_T^{*2} \leq TEV_1^2 + 2 \sum_{l=1}^{T-1} (T-l) \text{Cov}(V_1, V_{1+l}),$$

with $TEV_1^2 = O((Th)^{-1})$. Let $d_T \rightarrow \infty$ be a sequence of positive integers such that $d_T h_T \rightarrow 0$. Define

$$J_1 = \sum_{l=1}^{d_T-1} T |\text{Cov}(V_1, V_{1+l})|$$

and

$$J_2 = \sum_{l=d_T}^{T-1} T |Cov(V_1, V_{1+l})|.$$

By Davydov's Lemma (see Hall and Heydte, 1980, Corollary A.2), we have

$$|Cov(V_1, V_{1+l})| \leq c[\alpha(l)]^{1-2/\delta} [E|V_1|^\delta]^{2/\delta}.$$

By Lemma B.1 and assumption A.2, there exists a constant M such that

$$E|V_1|^\delta \leq 4^\delta E\left(\frac{\mathbf{Z}^{*T}\xi}{\sqrt{Th}}\right)^{2\delta} K_1^\delta \leq M \frac{h}{(Th)^\delta} = MT^{-\delta} h^{1-\delta}.$$

Then we have

$$\begin{aligned} J_2 &\leq TcM^{2/\delta} T^{-2} h^{\frac{2}{\delta}-2} \sum_{l=d_T}^{\infty} \alpha(l)^{1-2/\delta} \\ &\leq cM^{2/\delta} T^{-1} h^{\frac{2}{\delta}-2} d_T^{-a} \sum_{l=d_T}^{\infty} l^a \alpha(l)^{1-2/\delta} \\ &= o\left(\frac{1}{Th}\right), \end{aligned}$$

Let us choose d_T such that $h^{1-2/\delta} d_T^a = C$ for $\delta > 2$, to satisfy the requirement that $d_T h \rightarrow 0$. Now, by the Cauchy-Schwartz inequality,

$$|Cov(V_1, V_{1+l})| \leq MEV_1^2 = O\left(\frac{1}{T^2 h}\right),$$

and from assumption A.4, we have

$$J_1 = Td_T O\left(\frac{1}{T^2 h}\right) = o\left(\frac{1}{Th^2}\right).$$

Consequently,

$$P\{|R_T^*(\xi)| > \varepsilon\} \leq \frac{E[R_T^{*2}(\xi)]}{\varepsilon^2} = o(1).$$

Thus, we complete the proof of Lemma B.2.

Lemma B.3 Assume that conditions (A.1) through (A.5) are satisfied. Then we have, as $T \rightarrow \infty$

$$\frac{1}{Th} \sum_{t=1}^T K_t \psi''(0|\mathbf{X}_t, U_t) \mathbf{Z}_t^* \mathbf{Z}_t^{*T} \rightarrow f_U(u_0) \begin{pmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} \otimes \Gamma(\theta, u_0) := D(\theta, u_0), \quad (\text{A.17})$$

where $\Gamma(\theta, u_0) = 2E\left(\{\theta[1 - F_{\mathbf{X},U}(\eta(U, \mathbf{X}))] + (1 - \theta)F_{\mathbf{X},U}(\eta(U, \mathbf{X}))\} \mathbf{X}\mathbf{X}^T | U = u_0\right)$.

Proof. Note that

$$\psi'(t|\mathbf{x}, u) = E[L_\theta(Y - \eta(U, \mathbf{X}) + t) | \mathbf{X} = \mathbf{x}, U = u].$$

Hence

$$\psi''(0|\mathbf{x}, u) = 2 \left[\theta \{1 - F_{\mathbf{X},U}(\eta(U, \mathbf{X}))\} + (1 - \theta) F_{\mathbf{X},U}(\eta(U, \mathbf{X})) \right].$$

By ergodicity, we have

$$\frac{1}{Th} \sum_{t=1}^T K_t \psi''(0|\mathbf{X}_t, U_t) \mathbf{Z}_t^* \mathbf{Z}_t^{*T} - \frac{1}{h} E[K_t \psi''(0|\mathbf{X}_t, U_t) \mathbf{Z}_t^* \mathbf{Z}_t^{*T}] = o_p(1). \quad (\text{A.18})$$

It follows from simple calculation that

$$\begin{aligned}
& E[K_t \psi''(0 | \mathbf{X}_t, U_t) \mathbf{Z}_t^* \mathbf{Z}_t^{*T}] \\
= & E \left\{ K_t \begin{pmatrix} 1 & (U_t - u_0)/h \\ (U_t - u_0)/h & (U_t - u_0)^2/h^2 \end{pmatrix} 2 \left[\theta \{1 - F_{\mathbf{X}, U}(\eta(U, \mathbf{X}))\} + (1 - \theta) F_{\mathbf{X}, U}(\eta(U, \mathbf{X})) \right] \otimes \mathbf{X}_t \mathbf{X}_t^T \right\} \\
= & h f_U(u_0) \begin{pmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} \otimes \Gamma(\theta, u_0) (1 + o(1)). \tag{A.19}
\end{aligned}$$

The proof is completed by combining (A.18) and (A.19).