Supplementary Information for:

A Kollár and Pluzsik Anisotropic Composite Beam Theory for Arbitrary Multicelled Cross Sections

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INTRODUCTION

In this supplementary information document we detail the steps required to calculate the internal loads and the shear center of our arbitrary beam.

ARBITRARY BEAM CROSS SECTION INTERNAL LOADS

Calculating internal loads in arbitrary, multicelled anisotropic beams also lacks explicit solutions such as those proposed in [1] for single cells. To deal with the conflicting counterclockwise references from each cell's internal loads we also need the interface factor $f_{k,n}$.

Internal loads from $\hat{N}_x$, $\hat{M}_y$, $\hat{M}_z$, $\hat{T}_x$

The relationship between the internal loads X_1 and X_2 of each cell n and the beam loads at the beam coordinate system can be defined as

$$\begin{Bmatrix} X_{1n} \\ X_{2n} \end{Bmatrix} = [X]_n \begin{Bmatrix} \hat{N}_{\bar{x}} \\ \hat{M}_{\bar{y}} \\ \hat{M}_{\bar{z}} \\ \hat{T}_{\bar{x}} \end{Bmatrix} \quad (1)$$

where $[X]_n$ is determined by the same solution of the $2N + 4$ system of asymmetric linear equations with unitary load cases used to determine $[\overline{W}]$.

$$[X]_n = \begin{bmatrix} \begin{Bmatrix} X_{1n} \\ X_{2n} \end{Bmatrix}_{LC1} & \begin{Bmatrix} X_{1n} \\ X_{2n} \end{Bmatrix}_{LC2} & \begin{Bmatrix} X_{1n} \\ X_{2n} \end{Bmatrix}_{LC3} & \begin{Bmatrix} X_{1n} \\ X_{2n} \end{Bmatrix}_{LC4} \end{bmatrix} \quad (2)$$

The shear flow and hoop moment in the k -th segment due to $\hat{N}_{\bar{x}}$, $\hat{M}_{\bar{y}}$, $\hat{M}_{\bar{z}}$, $\hat{T}_{\bar{x}}$ can be rewritten as a function of Equation (2) as

$$\begin{Bmatrix} N_{\xi\eta}^T \\ M_{\eta}^T \end{Bmatrix}_k = \sum_{n=1}^N f_{k,n} [X]_n \begin{Bmatrix} \hat{N}_{\bar{x}} \\ \hat{M}_{\bar{y}} \\ \hat{M}_{\bar{z}} \\ \hat{T}_{\bar{x}} \end{Bmatrix} \quad (3)$$

The expression for the k -th segment's N_{ξ} , M_{ξ} , $M_{\xi\eta}$ at a position $\eta = -0.5b_k$ to $\eta = +0.5b_k$ is adapted from [1] to our variables as

$$\begin{Bmatrix} N_{\xi} \\ M_{\xi} \\ M_{\xi\eta} \end{Bmatrix}_{k,\eta} = [\tilde{\mu}_k]^{-1} [R_{\eta}] [R_k] [\overline{W}] \begin{Bmatrix} \hat{N}_{\bar{x}} \\ \hat{M}_{\bar{y}} \\ \hat{M}_{\bar{z}} \\ \hat{T}_{\bar{x}} \end{Bmatrix} - [\tilde{\mu}_k]^{-1} [\tilde{v}_k] \begin{Bmatrix} N_{\xi\eta}^T \\ M_{\eta}^T \end{Bmatrix}_k \quad (4)$$

where

$$[R_{\eta}] = \begin{bmatrix} 1 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}; \quad [\tilde{\mu}_k] = \begin{bmatrix} \alpha_{11} & \beta_{11} & \beta_{16} \\ \beta_{11} & \delta_{11} & \delta_{16} \\ \beta_{16} & \delta_{16} & \delta_{66} \end{bmatrix}_k; \quad [\tilde{v}_k] = \begin{bmatrix} \alpha_{16} & \beta_{12} \\ \beta_{61} & \delta_{12} \\ \beta_{66} & \delta_{26} \end{bmatrix}_k \quad (5)$$

By combining Equation (3) with Equation (4) we obtain

$$\begin{Bmatrix} N_{\xi} \\ M_{\xi} \\ M_{\xi\eta} \end{Bmatrix}_{k,\eta} = [\tilde{\mu}_k]^{-1} \left([R_{\eta}] [R_k] [\overline{W}] - [\tilde{v}_k] \sum_{n=1}^N f_{k,n} [X]_n \right) \begin{Bmatrix} \hat{N}_{\bar{x}} \\ \hat{M}_{\bar{y}} \\ \hat{M}_{\bar{z}} \\ \hat{T}_{\bar{x}} \end{Bmatrix} \quad (6)$$

where η varies from $-0.5b_k$ to $+0.5b_k$.

We highlight that $[X]_n$ is null in segments that are not part of any cell and the solution simplifies to the expression presented in [2] for open cross sections.

Internal loads from $\hat{V}_y$ and $\hat{V}_z$

Internal loads on open segments from $\hat{V}_y$ and $\hat{V}_z$

To obtain the shear running load distribution in a segment k due to $\hat{V}_{\bar{y}}$ and $\hat{V}_{\bar{z}}$ we need first an expression for the variation of the axial running load in this segment due to $\hat{N}_{\bar{x}}$, $\hat{M}_{\bar{y}}$, $\hat{M}_{\bar{z}}$ and $\hat{T}_{\bar{x}}$. Analogously to the solution proposed for single cells in [2], we solve this by differentiating both sides of Equation (6) to obtain

$$\left\{ \begin{array}{c} \frac{\partial N_\xi}{\partial \xi} \\ \frac{\partial M_\xi}{\partial \xi} \\ \frac{\partial M_{\xi\eta}}{\partial \xi} \end{array} \right\}_{k,\eta} = [\tilde{u}_k]^{-1} \left([R_\eta][R_k][\overline{W}] - [\tilde{v}_k] \sum_{n=1}^N f_{k,n}[X]_n \right) \left\{ \begin{array}{c} 0 \\ \hat{V}_{\bar{z}} \\ \hat{V}_{\bar{y}} \\ 0 \end{array} \right\} \quad (7)$$

where η varies from $-0.5b_k$ to $+0.5b_k$.

If the beam contains cells, each cell needs to be cut so that the beam cross section becomes a collection of contiguous open segments. The following notation is for cuts made at segment extremities.

The open segments shear flow ($N_{\xi\eta}^{open}$) calculation starts from the free-edges and proceeds further inside the cross section every time all the shear flow arriving or leaving one segment extremity is known. This may result in a shear calculation starting at the end of a segment (position $+0.5b_k$) or at its beginning (position $-0.5b_k$).

For segments where the sum of shear flows from other segments are known at $\eta = -0.5b_k$, the expression for their $N_{\xi\eta}^{open}$ at a position η is given by

$$\left(N_{\xi\eta}^{open} \right)_{k,\eta} = - \int_{-0.5b_k}^{\eta} \left(\frac{\partial N_\xi}{\partial \xi} \right)_k d\eta + \left(N_{\xi\eta}^{open} \right)_{k,-0.5b_k} + \left(N_{\xi\eta}^{point} \right)_{k,-0.5b_k} \quad (8)$$

where $N_{\xi\eta}^{open}$ at $\eta = -0.5b_k$ denotes the sum of the shear running loads coming from other segments at this location and $N_{\xi\eta}^{point}$ is the shear flow influence from a point that contains an EA, defined in [2] by

$$N_{\xi\eta}^{point} = \frac{-(P_{3,3}\hat{V}_{\bar{z}} - P_{2,3}\hat{V}_{\bar{y}})(\bar{z}_p - z_c)EA_p - (P_{2,2}\hat{V}_{\bar{y}} - P_{2,3}\hat{V}_{\bar{z}})(\bar{y}_p - y_c)EA_p}{(P_{2,2}P_{3,3} - P_{2,3})} \quad (9)$$

Similarly, for segments where the sum of shear flows from other segments are known at $\eta = +0.5b_k$, the expression for their $N_{\xi\eta}^{open}$ at a position η is given by

$$\left(N_{\xi\eta}^{open} \right)_{k,\eta} = + \int_{\eta}^{+0.5b_k} \left(\frac{\partial N_\xi}{\partial \xi} \right)_k d\eta - \left(N_{\xi\eta}^{open} \right)_{k,+0.5b_k} - \left(N_{\xi\eta}^{point} \right)_{k,+0.5b_k} \quad (10)$$

Internal loads on segments in closed cells from $\hat{V}_y$ and $\hat{V}_z$

If the beam contains cells, the shear flow calculated above needs to be balanced by a torque that will add to each cell n a constant shear flow q_n^C and hoop moment $M_{\eta_n}^C$.

The shear flow from Equation (10) can be integrated and averaged over the segment's width b_k to determine q_k^{OP} as

$$q_k^{OP} = \frac{\int_{-0.5b_k}^{+0.5b_k} (N_{\xi\eta}^{open})_{k,\eta} d\eta}{b_k} \quad (11)$$

With $\hat{V}_y$ and $\hat{V}_z$ developing no ε_x^o , $1/\rho_{\bar{y}}$, $1/\rho_{\bar{z}}$ and $\bar{\vartheta}$, we have for each cell n

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \sum_{k=1}^K |f_{k,n}| [F_k] \left(\begin{Bmatrix} f_{k,n} q_k^{OP} + q_n^C \\ M_{\eta_n}^C \end{Bmatrix} - \sum_{m \neq n} |f_{k,m}| \begin{Bmatrix} q_m^C \\ M_{\eta_m}^C \end{Bmatrix} \right) \quad (12)$$

Equation (12) introduces 2 equations for each cell and each cell has q_n^C and $M_{\eta_n}^C$ as variables. By solving the 2N system of linear equations, the constant balancing shear and moment are given by

$$\begin{Bmatrix} N_{\xi\eta}^B \\ M_{\eta}^B \end{Bmatrix}_k = \sum_{n=1}^N f_{k,n} \begin{Bmatrix} q_n^C \\ M_{\eta_n}^C \end{Bmatrix} \quad (13)$$

The shear flow at a segment k , at a position η is finally determined by the sum of $N_{\xi\eta}^T$ from Equation (3), $N_{\xi\eta}^{open}$ from Equation (10), and $N_{\xi\eta}^B$ from Equation (13).

$$(N_{\xi\eta})_{k,\eta} = (N_{\xi\eta}^T)_k + (N_{\xi\eta}^{open})_{k,\eta} + (N_{\xi\eta}^B)_k \quad (14)$$

Similarly, $(M_{\eta})_k$ is calculated as the sum of M_{η}^T from Equation (3) and M_{η}^B from Equation (13).

$$(M_{\eta})_k = (M_{\eta}^T)_k + (M_{\eta}^B)_k \quad (15)$$

Shear center from $\hat{V}_y$ and $\hat{V}_z$ internal loads

The beam shear center coordinates relative to the beam origin is found by applying unitary shear loads to the beam and recovering the respective torsional loads at the origin from Equation (16). $\hat{T}_x^o$ becomes y_s by applying $\hat{V}_z = 1$ and z_s by applying $\hat{V}_y = 1$.

$$\hat{T}_x^o = \sum_{k=1}^K \left(\int_{-0.5b_k}^{+0.5b_k} (N_{\xi\eta})_{k,\eta} d\eta \cdot \left(\frac{(z_{+0.5b_k} - z_{-0.5b_k})}{b_k} \cdot \bar{y}_k - \frac{(y_{+0.5b_k} - y_{-0.5b_k})}{b_k} \cdot \bar{z}_k \right) \right) \quad (16)$$

Lumped EAs and GJs (booms) internal loads

The axial load of a point p is

$$\hat{N}_{x_p} = EA_p \left(\varepsilon_x^o + \frac{1}{\rho_y} (\bar{z}_p - z_c) + \frac{1}{\rho_z} (\bar{y}_p - y_c) \right) \quad (17)$$

and its torque is

$$\hat{T}_{x_p} = GJ_p \vartheta \quad (18)$$

REFERENCES

1. Kollár LP, Pluzsik A. Analysis of thin-walled composite beams with arbitrary layup. *J Reinf Plast Compos* 2002; 21(16): 1423-1465.
2. Kollár LP, Springer GS. *Mechanics of composite structures*. Cambridge: Cambridge University Press, 2003.