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On Billaud Words and Their Companions

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Abstract. The Billaud Conjecture, which has been open since 1993, is a fundamental problem on finite words w and their heirs, i.e., the words obtained by deleting every occurrence of a given letter from w . It posits that every morphically primitive word, i.e. a word which is a fixed point of the identity morphism only, has at least one morphically primitive heir. In this paper, we introduce and investigate the related class of so-called Billaud words, i.e. words whose all heirs are morphically imprimitive. We provide a characterisation of morphically imprimitive Billaud words, using a new concept. We show that there are two phenomena through which words can have morphically imprimitive heirs, and we highlight that only one of those occurs in morphically primitive words. Finally, we examine our concept further, use it to rephrase the Billaud Conjecture and study its difficulty.

Keywords: Billaud Conjecture · Morphic primitivity · Fixed point

1 Introduction

In this paper we study the notion of morphic primitivity of words: a word is morphically primitive if the only morphism for which it is a fixed point is the identity morphism. The context of our research is the conjecture posed by Billaud [1] in 1993, which is still open today:

Conjecture 1 (The Billaud Conjecture). There exists at least one letter x in every morphically primitive word w such that the word obtained by deleting all occurrences of x in w is also morphically primitive.

We shall call the word w the *parent*, and the words obtained through a deletion of a letter from w *heirs*. As a simple example, consider the morphically primitive word **abcbac**: the word resulting from the deletion of the letter **c** is **abba**, and it is morphically primitive. The Billaud Conjecture, however, is an implication and not a characterisation. Hence, while there exist words, such as $(abc)^2$, where the deletion of each of the letters leads to a morphically imprimitive heir, there are also words such as **abcbca**, which are morphically imprimitive, yet have morphically primitive heirs (here **abba**).

Despite being an open problem, a number of results in relation to the Billaud Conjecture have been established. In the direct response to the question posed by Billaud, the initial insight that we need only to consider idempotent morphisms was due to Geser [3]; an extended version of this statement was proved by Levé and Richomme [9]. Several special cases where the conjecture holds were identified: Zimmermann [15] showed that the conjecture holds for the alphabet size of 3. (We note the case of alphabet size of 2 is trivial.) Levé and Richomme [9], working with the contrapositive of the conjecture, proved that if all morphically imprimitive heirs of a word are fixed points of non-trivial morphisms with exactly one expanding letter each, then the same holds for the parent word; the letter x is expanding if the morphism ϕ for which the word is a fixed point is such that $|\phi(x)| \geq 2$. Walter [14] identified and proved some of the cases necessary for showing that the conjecture holds for alphabet size of 4. Nevisi and Reidenbach [11] proved that the conjecture is correct for all words (with three or more different letters) if they contain each letter exactly twice.

More generally, the study of morphic primitivity of words has been of interest in a variety of contexts. Finite fixed points of morphisms were first characterised by Head [5] in the context of L-systems. An alternative characterisation of the finite fixed points was given by Hamm and Shallit [4], and in the context of erasing pattern languages by Reidenbach [12]. Reidenbach and Schneider [13] introduced the concept of ‘morphic imprimitivity’ itself, and described a factorisation characteristic to finite fixed points of morphisms. In relation to terminal-free erasing pattern languages, a morphically primitive word is the shortest generator of a pattern language, as shown by Filè [2], and Jiang et al. [7]. Holub [6] gave a polynomial-time algorithm (further refined and analysed by Matocha and Holub [10]) to decide whether or not a word w is morphically imprimitive; the algorithm can yield the imprimitivity factorisation of a morphically imprimitive w . Kociumaka et al. [8] further improved the algorithm to work in linear time.

In this paper we introduce and explore a class of words related to the Billaud Conjecture, the words whose all heirs are morphically imprimitive. We shall call these *Billaud words*, and as an example consider the word **abcabc** again, whose three heirs **abab**, **acac**, and **cbcb** are all morphically imprimitive. In particular, if the Billaud Conjecture holds, then all Billaud words are morphically imprimitive. After providing some basic definitions, in Section 3 we focus on morphically imprimitive Billaud words. We shall give more complex examples of these words and show that there are two fundamental phenomena leading to morphically imprimitive heirs that occur on deletion of a letter. Finally, we provide a characterisation of morphically imprimitive Billaud words. In the subsequent Section 4, we shift our focus to examining different aspects surrounding one of the given phenomena. To learn more about the reasons for the difficulty of the Billaud Conjecture, we provide an alternative statement of it and discuss why certain approaches to solving it are unsuitable.

We note that due to space constraints most of the proofs have been omitted.

2 Preliminaries

We shall denote the set of all positive integers with $\mathbb{N}_+$, and non-negative integers with $\mathbb{N}_0$. Moreover, for any two integers m, n , $m \geq n$, we shall denote with $\llbracket n, m \rrbracket$ the set of all integers i such that $n \leq i \leq m$. An *alphabet* Σ is an enumerable set of *symbols* or *letters*. Letters typeset in a sans-serif font, i.e. $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$, should always be understood as distinct; in other cases, we shall explicitly state such assumptions. As a *word* (over an alphabet Σ) we refer to a finite sequence of elements, symbols, of Σ . The cardinality of a set A shall be denoted by $|A|$, similarly the length of a word w is $|w|$. The *empty word*, denoted by λ , is the special word for which $|\lambda| = 0$. The *concatenation* of two words w and v shall be written as $w \cdot v$ or wv . Given a word w and $n \in \mathbb{N}_0$, w^n denotes the concatenation of n copies of the word w . We say that a word v is a *factor* of w , denoted $v \sqsubseteq w$, if there exist words w_1 and w_2 such that $w = w_1 \cdot v \cdot w_2$. Given words w, v and $i \in \mathbb{N}_0$, we shall call the tuple $\langle i, v \rangle$ an *occurrence* (of v) in w (at position i) if there are words w_1, w_2 such that $w = w_1 \cdot v \cdot w_2$ and $|w_1| = i$. The number of occurrences of v in w is denoted by $|w|_v$. Let $\text{symb}(w)$ denote the set of all letters $x \sqsubseteq w$.

For any alphabets A and B and all words $w, v \in A^*$, a (*homo*-)*morphism* $\phi : A^* \rightarrow B^*$ is a function that satisfies $\phi(w)\phi(v) = \phi(wv)$. A morphism $\phi : A^* \rightarrow A^*$ is *idempotent* if $\phi = \phi \circ \phi$. A word w is a (*finite*) *fixed point* of a morphism ϕ if $\phi(w) = w$. If a morphism ϕ is not idempotent, and there exists a finite fixed point word w of ϕ , then, as shown by Geser [3], there exists an idempotent morphism ϕ' such that $\phi'(w) = w$. There also exists an integer $i \in \mathbb{N}_+$ such that $\phi^i = \phi'$, and we shall call i the *mortality exponent* of ϕ . If x is a letter, then we denote with π_x the morphism *deleting* x , i.e. $\pi_x(y) = y$ for all $y \neq x$, and $\pi_x(x) = \lambda$. The morphism $\iota : A^* \rightarrow A^*$ is the *identity* (or *trivial*) if for every $x \in A$, $\iota(x) = x$.

A word w is *morphically primitive* [13] if there is no word w' with $|w'| < |w|$ such that w and w' can be mapped onto each other by morphisms; otherwise w is *morphically imprimitive*. At the end of the present section, we shall explain that this definition is equivalent to the notion of morphic primitivity used at the beginning of Section 1.

Let $\phi : \Sigma^* \rightarrow \Sigma^*$ be an idempotent morphism; when discussing morphic imprimitivity we shall restrict ourselves only to idempotent morphisms, unless explicitly stated otherwise. We define the following three sets, similarly to Levé and Richomme [9], which form a partition on $\text{symb}(w)$, in the context of ϕ :

- the set of *expanding letters*, $E_\phi = \{x \in \Sigma \mid |\phi(x)| \geq 2\}$,
- the set of *mortal letters*, $M_\phi = \{x \in \Sigma \mid \phi(x) = \lambda\}$,
- the set of *constant letters*, $C_\phi = \{x \in \Sigma \mid \phi(x) = x\}$.

A letter $x \in \Sigma$ is *necessarily expanding* (or *mortal*, or *constant*) if $x \in E_\phi$ (or M_ϕ , or C_ϕ respectively) for all (idempotent) non-identity $\phi : \text{symb}(w)^* \rightarrow \text{symb}(w)^*$ with a fixed point w . Let us denote the set of (*all*) *expanding letters in a word* w with E_w : the letter x is in E_w if there is a morphism ϕ such that $x \in E_\phi$, and

$\phi(w) = w$. Then we say that x is *expanding in (the word) w* . We define M_w and C_w analogously.

An *imprimitivity factorisation* f of w is a tuple $\langle x_1, x_2, \dots, x_n; v_1, v_2, \dots, v_n \rangle$, where $v_1, v_2, \dots, v_n \in \Sigma^+$, $x_1, x_2, \dots, x_n \in \Sigma$, $n \in \mathbb{N}_+$, such that w can be written as $u_0 v_1 u_1 v_2 u_2 \dots v_n u_n$, for some $u_0, u_1, \dots, u_n \in \Sigma^*$, and there is a non-trivial morphism ϕ with $\phi(w) = w$ such that for all $i \in \mathbb{N}_+$, $|v_i|_{x_i} = 1$, $v_i = \phi(x_i)$ and $u_0, u_i \in C_\phi^*$. We call the word v_i an *imprimitivity factor of f (corresponding to x_i)*. We shall say that ϕ *determines f* , and define sets E_f, M_f , and C_f to be equal to E_ϕ, M_ϕ , and C_ϕ .

Finally, as we shall operate with the following concepts interchangeably, we restate the following result by Reidenbach and Schneider [13]. For every word w , the following statements are equivalent: w is morphically imprimitive, w is a fixed point of a non-trivial morphism, and w has an imprimitivity factorisation.

3 Morphically Imprimitive Billaud Words

Let us recall that the Billaud Conjecture is stated merely as an implication, and not as a characterisation. In other words, there exist morphically imprimitive words with morphically primitive heirs. In this section, we shall provide a characterisation of morphically imprimitive Billaud words, i.e. words whose all heirs are morphically imprimitive. In the context of the conjecture, this means partitioning the set of all morphically imprimitive words into those relevant to the Billaud Conjecture, hence our name: Billaud words, and the morphically imprimitive words with morphically primitive heirs. Moreover, it is also worth noting that the Billaud Conjecture in this context posits that all Billaud words are morphically imprimitive, i.e. there are no morphically primitive Billaud words in general. We look into morphically primitive words in more detail in Section 4. We shall now proceed to formally define the notion of Billaud words:

Definition 1. *The word w is a Billaud word if $\pi_x(w)$ is morphically imprimitive for all $x \in \text{symb}(w)$.*

In order to better understand what Billaud words look like, and to motivate our reasoning thereafter, we consider the following example:

Example 1. Let us define the following words w_1 and w_2 :

$$w_1 := \text{ab c ab dea c ab dea c ab}, \quad w_2 := \text{ab c ab de}^2\text{a c ab de}^2\text{a c ab}$$

The words w_1, w_2 are fixed points of the following morphisms ϕ_1, ϕ_2 respectively:

$$\phi_1 : \text{b} \mapsto \text{ab}, \text{d} \mapsto \text{dea}, \text{c} \mapsto \text{c}, \text{ae} \mapsto \lambda, \quad \phi_2 : \text{b} \mapsto \text{ab}, \text{d} \mapsto \text{de}^2\text{a}, \text{c} \mapsto \text{c}, \text{ae} \mapsto \lambda$$

Neither ϕ_1 nor ϕ_2 is trivial, and the words w_1 and w_2 are morphically imprimitive. The word w_1 is a Billaud word, as all of its heirs are morphically imprimitive. This can be verified by finding a suitable imprimitivity factorisation of each of these heirs or by finding morphisms for which the words are fixed points. On the other hand, the word w_2 is not a Billaud word, as it has an heir $\pi_d(w_2)$ which is morphically primitive. $\diamond$

The words in Example 1 are very similar in many respects, and differ superficially by one square. In particular, we can see that the sets of expanding, constant, and mortal letters are equal for ϕ_1 and ϕ_2 . Nevertheless, a more detailed consideration of letter roles in a word is centre to our characterisation of Billaud words. In the remainder of this section we shall prove our characterisation of morphically imprimitive Billaud words, which shall culminate with Theorem 1, by systematically considering deletion of letters of different roles.

We shall start by giving a lemma with a basic necessary condition for a word to be a morphically imprimitive Billaud word, namely that a word has to have at least one imprimitivity factorisation with more than one mortal letter in it.

Lemma 1. *Let w be a morphically imprimitive word. If every imprimitivity factorisation of w has exactly one mortal letter, then $\pi_m(w)$ is morphically primitive for every $m \in M_w$.*

We shall note that the class of words which have one mortal letter in each of their imprimitivity factorisations is severely restricted. Generally speaking, if a word has one mortal letter per imprimitivity factorisation, then it is the same mortal letter across all imprimitivity factorisations. There is an exception to this rule, illustrated by the word $(ab)^2$, where the roles of the letters a, b can ‘flip’ between the two factorisations of the word. The following proposition illustrates this:

Proposition 1. *Let w be a morphically imprimitive word where every imprimitivity factorisation has exactly one mortal letter. Let w have no pair of imprimitivity factorisations f, g such that $E_f \cup M_f = E_g \cup M_g$. Then the word has exactly one mortal letter.*

We shall now commence with the key part of the proof of our characterisation of morphically imprimitive Billaud words. Let w be an arbitrary morphically imprimitive word. We shall investigate what happens when we delete letters from w , and in particular if the resulting heirs are morphically imprimitive, in order to determine whether or not w is a Billaud word. In particular we note that if a letter is neither mortal nor constant in w , it is necessarily expanding.

We introduce Lemmata 2, 3, and 6, each giving a necessary and sufficient condition for the morphic imprimitivity of the heirs obtained by deletions of different classes of letters. The subsequent two lemmata deal with the simpler cases of deletion of constant or mortal letters from w .

Lemma 2 (Any Constant Letter). *Let w be a morphically imprimitive word. For all $x \in C_w$, the word $\pi_x(w)$ is morphically imprimitive.*

Lemma 3 (Any Mortal Letter). *Let w be a morphically imprimitive word. For all $x \in M_w$, the word $\pi_x(w)$ is morphically imprimitive if and only if there is at least one imprimitivity factorisation f of w where $|M_f| \geq 2$.*

So far, we have dealt with situations where the imprimitivity factorisations of the parent and the heir are in a way ‘related’ to each other. These are the cases

when the morphisms determining the imprimitivity of the parent and the heir, let us call them ϕ and χ respectively, are such that either $\chi = \phi$ or $\chi = \pi_x \circ \phi$.

In order to finalise our characterisation of morphically imprimitive Billaud words, the remainder of our reasoning deals with those situations in the word when, as a result of deletion of a letter, we obtain a word which has an imprimitivity factorisation ‘unrelated’ to the parent word. As we have considered the deletion of constant, and mortal letters, we now address the final class of necessarily expanding letters. We observe that when we delete a necessarily expanding letter in the word, the resulting heir has an alternative parent, in which the deleted letter is not necessarily expanding. The following is an example of such a word, where there is a letter that is necessarily expanding.

Example 2. Let $w := (\text{abacb})^2$; then it can be verified that $\phi : c \mapsto \text{abacb}, \text{ab} \mapsto \lambda$ is the only non-trivial morphism for which w is a fixed-point. Note that the letter c is thus necessarily expanding in w . The heir obtained by deleting c from w , $\pi_c = (\text{ab})^2$, is trivially morphically imprimitive, as it is a fixed point of, e.g., the morphism $\chi : a \mapsto \text{ab}, b \mapsto \lambda$. Interestingly, χ is not related to ϕ – there is no simple way in which we could uniformly ‘add’ letters to χ , such that the new morphism determines the morphic imprimitivity of w .

However, let us observe what happens when we apply χ (redefined to map c to itself) to w : $w' := \chi(w) = ((\text{ab})^2 c)^2$. The ‘new’ word w' , which shares an heir with w , still has c as an expanding letter, but it is no longer necessarily expanding, as a result of having another imprimitivity factorisation determined by χ . In fact, with Lemma 6, we show that the existence of a morphism where c is a constant letter, and mapping w to w' is characteristic for the word $\pi_c(w)$ to be morphically imprimitive. $\diamond$

We shall define this specific relationship between the ‘original’, and the ‘new’ parent as follows:

Definition 2. Let w and w' be words. We call w' a companion of w (with respect to x) if $\pi_x(w) = \pi_x(w')$ and there exists a non-identity idempotent morphism ϕ such that $\phi(w) = w'$ and $\phi(x) = x$. We shall refer to such morphism ϕ as a companion morphism.

Firstly, we present a fundamental lemma that motivates our use of companions, which demonstrates that these words are always morphically imprimitive:

Lemma 4. Let w' be a companion of some word w , and let ϕ be the corresponding companion morphism. Then the word w' is morphically imprimitive, and ϕ determines an imprimitivity factorisation of w' .

Having defined this notion of a companion, we now postulate that a word having a companion for a letter is characteristic for it having a morphically imprimitive heir under the deletion of said letter:

Lemma 5. Let w be a word, and let x be a letter in w . The word $\pi_x(w)$ is morphically imprimitive if and only if the word w has a companion with respect to x .

However, it can be seen from Lemma 5 that the present definition of companions is quite general and includes the simpler cases otherwise covered in Lemmata 2 and 3. For example, it is worth noting that any morphically imprimitive word which has a constant letter is its own companion with respect to that letter, as trivially the morphism for which the word is a fixed point maps the letter to itself. Moreover, the phenomenon described before, where a new imprimitivity factorisation appears in the word upon deletion of a letter can also happen, even when the letter is constant or mortal. Since we have Lemmata 2 and 3, such an imprimitivity factorisation is in a way redundant to determining the morphic imprimitivity of the heir. Thus, it is advisable to strengthen the definition of a companion to include only those cases when we do not have a trivial imprimitivity factorisation. Definition 3 accomplishes that, but before we shall introduce it, we present an example to better illustrate the discussed phenomena.

Example 3. Let $w := \text{abc abxdebxc ebd}$. The word has only one imprimitivity factorisation determined by the following morphism:

$$\phi : a \mapsto ab, e \mapsto eb, b \mapsto \lambda, c \mapsto c, d \mapsto d, x \mapsto x$$

In particular the letter x is a constant letter in w , and thus trivially by Lemma 2 the heir $\pi_x(w)$ is morphically imprimitive. It can then be seen that the morphism ϕ satisfies the definition of a companion morphism, and as such the word w is its own companion. However, w has another companion w' with respect to the following morphism:

$$\chi : c \mapsto bc, d \mapsto bd, b \mapsto \lambda, a \mapsto a, e \mapsto e, x \mapsto x$$

Thus $w' := \chi(w) = \text{abc axbd exbc ebd}$, or in other words we have swapped neighbouring b and x . The word w' has an imprimitivity factorisation determined by χ which is ‘new’: neither the letter c nor d is expanding in w . However, even though this imprimitivity factorisation of w' cannot be derived from any imprimitivity factorisations of w and is as such an interesting case, it is in a way redundant, as already by Lemma 2 we have determined that $\pi_x(w)$ is morphically imprimitive. $\diamond$

In other terms, the word in Example 3 possesses a dual nature: it is a witness to both of the previously discussed phenomena simultaneously. The heir of $\pi_x(w)$ of w has an imprimitivity factorisation derived from the imprimitivity factorisation of w , but also has a new unrelated one. In light of this, in order to strengthen the notion of a companion, we present the following definition, which aims to encompass only those companions that are essential for determining the morphic imprimitivity of the heir:

Definition 3. Let w be a word and $x \in \text{ymb}(w)$. A companion of w with respect to x is called essential if there do not exist a morphism ϕ and a companion morphism χ of w with respect to x such that $\phi(w) = w$, and $E_\phi = E_\chi$.

In the context of the previous example it can be seen that the word w is not an essential companion, as it is a companion of itself, and has a companion morphism ϕ equal to the morphism for which it is a fixed point.

We are now ready to state the third main lemma, in which we characterise the morphic imprimitivity of an heir under deletion of a necessarily expanding letter:

Lemma 6 (Necessarily Expanding Letter). *Let w be a morphically imprimitive word, and let x be a necessarily expanding letter in w . The word $\pi_x(w)$ is morphically imprimitive if and only if the word w has an essential companion with respect to x .*

From the above lemmata we can now see that there are two conditions that together are necessary and sufficient for an arbitrary morphically imprimitive word to be a Billaud word, which we formally express in Theorem 1. This is the characterisation for morphically *imprimitive* Billaud words only, particularly due to Lemma 2, and the somewhat unclear nature of the notion of letter roles in morphically primitive words.

Theorem 1. *A morphically imprimitive word w is a Billaud word if and only if it has at least one imprimitivity factorisation with two or more mortal letters, and for every necessarily expanding letter it has an essential companion.*

Proof. If direction. Let w have at least one imprimitivity factorisation with two or more mortal letters, and let it have an essential companion with respect to each of its necessarily expanding letters.

Since the word w is morphically imprimitive, due to Lemma 2, $\pi_x(w)$ is morphically imprimitive for all letters $x \in C_w$. Given that the word w has at least one imprimitivity factorisation with more than one mortal letter, by Lemma 3, $\pi_x(w)$ is morphically imprimitive for all letters $x \in M_f$. Finally, as w has an essential companion with respect to every necessarily expanding letter, by Lemma 6, $\pi_x(w)$ is morphically imprimitive for all letters $x \in E_w \setminus (C_w \cup M_w)$.

Therefore, as every letter from $\text{symb}(w)$ is constant, mortal, or necessarily expanding, all heirs of w are morphically imprimitive.

Only-if direction. Let w be a morphically imprimitive Billaud word, and thus let all of its heirs be morphically imprimitive.

In particular, all heirs $\pi_x(w)$, where x is a mortal letter in w , are morphically imprimitive, and thus by Lemma 1 there exists an imprimitivity factorisation f of w , such that f has two or more mortal letters. Moreover, if y is a necessarily expanding letter in w , then by Lemma 6, w has an essential companion with respect to x .

Therefore, w has at least one imprimitivity factorisation with more than one mortal letter, and for every necessarily expanding letter in w (the count of which could be zero), the word w has an essential companion. $\square$

Theorem 1 concludes our characterisation of morphically imprimitive Billaud words. We have shown precisely which morphically imprimitive words have morphically imprimitive heirs only.

4 Some Technical Considerations on Companions

Various questions can be posed about the nature of companion words, and in particular about the essential companions. For instance, after examining our examples, one might be tempted to conclude that necessarily expanding letters are relatively infrequent in words, with many words not having any. The below proposition shows that such a conclusion is not necessarily true:

Proposition 2. *For every $n \in \mathbb{N}_+$ there exists a morphically imprimitive Billaud word with n necessarily expanding letters and $2n + 1$ letters that are not necessarily expanding.*

Proof idea. We examine the set of words $w_i := (a_1 b_1 c a_1 d_1 b_1)^2 \cdots (a_i b_i c a_i d_i b_i)^2$, and show that for every $i \in \mathbb{N}_+$, the word w_i is a Billaud word and has i necessarily expanding letters, and $2i + 1$ letters that are not necessarily expanding. $\square$

We now present another observation pertaining to the companions and why another approach, or a tool, previously used to prove a special case of the conjecture might not be suitable in the general case. Before we state our proposition, it is necessary for us to define the notion of a letter interrupting an occurrence of an imprimitivity factor:

Definition 4. *Let w' be a companion of some word w with respect to a letter x , and let ϕ be the corresponding companion morphism. We say that x interrupts an occurrence of an imprimitivity factor $\phi(y)$ (at position(s) $i_1, i_2, \dots$), for some $y \in E_\phi$, if there exists a factor $v \sqsubseteq w$ such that $\pi_x(v) = \phi(y)$, and such that for all $i_j, j \in \mathbb{N}_+, j \leq |w|_x$, v can be factorised as $v_j x v'_j$, for some words v_j, v'_j , such that $i_j = |\pi_x(v_j)|$.*

We point out that every collection of positions mentioned in Definition 4 refers to an *occurrence* of an imprimitivity factor:

Example 4. Let $w := \text{abxc axbxc}$ and let $w' := \text{abcx abc}^2$ be a companion of w . The companion morphism ϕ can be defined as follows: $\phi : a \mapsto \text{abc}, b, c \mapsto \lambda, x \mapsto x$. We say that x interrupts an occurrence of an imprimitivity factor $\phi(a) = \text{abc}$ at position 2, as $\text{abxc} = v' x v'' \sqsubseteq w$, for $v' = \text{ab}$, and $v'' = \text{c}$, and $|v'| = 2$. The letter x also interrupts an occurrence of an imprimitivity factor $\phi(a) = \text{abc}$ at positions 1 and 2, as $\text{axbxc} = v' x v'' \sqsubseteq w$, for $v' = \text{a}$, and $v'' = \text{bxc}$, and for $v' = \text{axb}$, and $v'' = \text{c}$. $\diamond$

Levé and Richomme [9], in their proof of a special case of the Billaud Conjecture, work with its contrapositive statement. One of the observations, in their special case, is that if a letter interrupts an imprimitivity factor of a word, it does so once and at the same position for every occurrence of the imprimitivity factor of the heir (and thus the companion). We can show that this is not generally true when considering companions:

Proposition 3. *There exists a word w that has an essential companion $w' = \chi(w)$ with respect to some necessarily expanding letter x , where χ is the companion morphism, such that x interrupts two occurrences of an imprimitivity factor of χ at different positions $i, j \in \mathbb{N}_+$.*

Proof idea. The word $w := \text{abcdbec becd bcf abcd bcf}$ is a fixed point of $\phi : a \mapsto \text{abc}, e \mapsto \text{bec}, f \mapsto \text{bcf}, bc \mapsto \lambda, d \mapsto d$. There is also a companion $\chi(w)$ of w with respect to e , where $\chi : d \mapsto \text{bcd bc}, bc \mapsto \lambda, a \mapsto a, e \mapsto e, f \mapsto f$. The letter e interrupts two occurrences of the imprimitivity factor $\chi(d)$, once at position 4 and once at position 1. $\square$

Finally, when examining some of the simpler presented examples of morphically imprimitive words, such as the word abaxb , which has a companion ababx with respect to x , one might spot that the necessarily expanding letter in the parent is also a candidate expanding letter in the companion. We can show that such a conclusion does not hold universally:

Proposition 4. *There exists a morphically imprimitive word w that has an essential companion w' with respect to a necessarily expanding letter x , and which has a necessarily expanding letter y (not necessarily different to x) that is not expanding in w' .*

Proof idea. Consider the example word in the proof of Proposition 3. The letters a, e, f are necessarily expanding in w due to ϕ being the only non-trivial morphism for which w is a fixed point, but they are not in E_χ . $\square$

We do note, however, that the word used in the above propositions is not a Billaud word. It could be the case that the above propositions are not satisfied by any such word: if that is the case, such observation would serve as a witness to the complexity of Billaud words.

So far we have discussed the morphically imprimitive Billaud words, that is the morphically imprimitive words without morphically primitive heirs. In the introduction to the previous section we mentioned that the Billaud Conjecture is equivalent to stating that there are no morphically primitive Billaud words. We present this alternative statement of the Billaud Conjecture as follows, using our previously introduced concept of essential companions:

Conjecture 2. Let w be a morphically primitive word. Then there is at least one letter $x \in \text{symb}(w)$ such that w does not have an essential companion with respect to x .

Since morphically primitive words do not have any imprimitivity factorisations, the only way an heir of such a word can be morphically imprimitive is if the latter of the phenomena central to Theorem 1 occurs: by default every imprimitivity factorisation of an heir of a morphically primitive word is unrelated to the parent word. We can use this link between the phenomena occurring in morphically imprimitive and in morphically primitive words to demonstrate the inherent complexity of the Billaud Conjecture. We begin by giving the following proposition, which shows that a morphically primitive word can only have essential companions:

Proposition 5. *Let w be a morphically primitive word. If w has a companion w' with respect to some letter x , then w' is essential.*

Knowing the above, and for the sake of completeness, we can show the equivalence of Conjecture 2 and the Billaud Conjecture:

Proposition 6. *Conjecture 2 is true if and only if Conjecture 1 is true.*

An immediate idea for solving Conjecture 2 could be a numerical argument: one could conjecture that a morphically primitive word w has at most $|\text{symb}(w)| - 1$ companions. This could be a viable approach if there was only one and unique companion with respect to every letter. In such a case, showing that a morphically primitive word w has less companions than letters would be sufficient to prove the Billaud Conjecture. However, it is not universally true that companions have to be unique, or that there is only one with respect to a given letter. The following propositions demonstrate these claims, and they therefore show that such an approach would be futile.

In order to show the first claim, we refer to the example word **abcacb**:

Proposition 7. *There exists a morphically primitive word w which has more than $|\text{symb}(w)|$ essential companions.*

Proof idea. Let $w := \text{abcacb}$. There are 4 essential companions of w : $w_1 = \text{abcabc}$ and $w_2 = \text{abccab}$, which are companions with respect to **c**, and $w_3 = \text{bacacb}$ and $w_4 = \text{acbacb}$, which are companions with respect to **b**. $\square$

Moreover, we present a related proposition, where we claim that there can be arbitrarily many essential companions with respect to the same letter:

Proposition 8. *For every alphabet Σ with $|\Sigma| \geq 3$ there exists a morphically primitive word $w \in \Sigma^*$ and a letter $x \in \Sigma$ such that w has $|\Sigma| - 1$ essential companions with respect to x .*

Proof idea. We consider the recursively defined words $w_1, w_2, \dots$ such that, for any $i \in \mathbb{N}_+$ we have $w_i := ux^2u'$ where $uu' = v_i$, $|u| = 1$, and where $v_0 := \mathbf{a}_0$, $v_j := (v_{j-1}\mathbf{a}_i)^2$ for $j \in \mathbb{N}_+$. For instance, $w_3 = \mathbf{a}_0\mathbf{x}\mathbf{a}_1\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2 (\mathbf{a}_0\mathbf{a}_1)^2\mathbf{a}_2$. Every word w_i can be shown to have $i + 1 = |\text{symb}(w_i)| + 2$ essential companions with respect to **x**, corresponding to every $\mathbf{a}_k$, $k \in \llbracket 0, j \rrbracket$, being the expanding letter in the given companion. $\square$

The following proposition demonstrates our other claim, namely that companions need not be unique:

Proposition 9. *There exists a morphically primitive word w which has the same word as an essential companion with respect to two different letters.*

Proof. Let $w := \text{abcdacbd}$, and let $w' := \text{abcdabcd}$. The word w' is an essential companion with respect to the letter **c**, as there is a morphism $\chi_c : \mathbf{a} \mapsto \mathbf{ab}, \mathbf{b} \mapsto \lambda, \mathbf{c} \mapsto \mathbf{c}, \mathbf{d} \mapsto \mathbf{d}$, such that $\chi_c(w) = w$, χ_c is non-trivial and idempotent, and $\pi_c(w) = \pi_c(w')$. The word w' is also an essential companion with respect to the letter **b**, as there is a morphism $\chi_b : \mathbf{a} \mapsto \mathbf{a}, \mathbf{b} \mapsto \mathbf{b}, \mathbf{c} \mapsto \lambda, \mathbf{d} \mapsto \mathbf{cd}$, such that $\chi_b(w) = w$, χ_b is non-trivial and idempotent, and $\pi_b(w) = \pi_b(w')$. $\square$

Finally, to conclude this section, we present a collection of examples demonstrating the varied ways in which roles of letters can differ between the imprimitivity factorisations of the parents and their essential companion morphisms. We consider companions with respect to necessarily expanding letters, as these are of particular interest due to Theorem 1. These examples illustrate vital phenomena that need to be considered when exploring the existence of companions. While our overview naturally refers to morphically imprimitive parents, we anticipate that further progress on Conjecture 2 will be related to the question of whether there exist any words that would fill the gaps in the below table.

We consider words and their companion morphisms with respect to a necessarily expanding letter x . We note the claims of the table can be verified using the definitions of morphic imprimitivity and companion morphisms. For cells marked with a question mark we conjecture the words do not exist, otherwise we provide an example word witnessing a given letter role change; we indicate such letters in square brackets. We shall denote by NE the necessarily expanding letters. In companions this shall be understood as expanding in all companion morphisms; we define NM and NC similarly. The backslash should be read out as ‘but not’, i.e. $E \backslash NE$ stands for ‘expanding, but not necessarily expanding letter’. Some words exhibit multiple changes: $w_1 := (abm^2axb)^2(ymzb)^2$, $w_2 := (abaxb)^2(ybz)^2$, $w_3 := abc^2bc^2m^2abc^2bxc^2$.

Parent Role $\rightarrow$ $\downarrow$ Comp. Mor.	NE [y]	$E \backslash NE$	NM	$M \backslash NM$	NC [c]	$C \backslash NC$
NE	$(abaxb)^2(yb)^2$	w_1 [z]	ab^2axb^2 [a]	w_1 [z]	$axcax(yacax)^2$	?
$E \backslash NE$	?	w_2 [yz]	w_3 [ab]	w_2 [yz]	?	w_2 [yz]
NM	?	?	w_3 [c]	?	?	?
$M \backslash NM$	?	w_2 [yz]	w_3 [b]	w_2 [yz]	?	w_2 [yz]
NC	$(abm^2axb)^2(ym)^2$	w_1 [y]	w_3 [m]	w_1 [y]	$(abaxb)^2c^2$	w_1 [y]
$C \backslash NC$	?	w_2 [yz]	w_3 [a]	w_2 [yz]	?	w_2 [yz]

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