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Jordan-Kronecker invariants of finite-dimensional Lie algebras

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Jordan-Kronecker invariants of finite-dimensional Lie algebras

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A. Izosimov)

Integrability, Recursion, Geometry And Mechanics
Celebrating Franco Magri's 70th birthday
September 5-9, 2016 at RISM – Villa Toeplitz – Varese

- ▶ Motivation and Generalised Argument Shift Conjecture
- ▶ Jordan-Kronecker decomposition theorem
- ▶ Jordan-Kronecker invariants of a finite-dimensional Lie algebra
- ▶ General facts about JK invariants
- ▶ Polynomiality of the algebra of Ad^* -invariants
- ▶ Examples

Motivation

- ▶ It is a very natural idea.

A Lie algebra is defined by its structure tensor c_{ij}^k . Too complicated!

Classical approach: Take $\text{ad}_\xi = \sum_i c_{ij}^k \xi^i$ for generic $\xi \in \mathfrak{g}$ and then study the properties of this operator.

Another option: Take a form $\mathcal{A}_x = \sum_k c_{ij}^k x_k$ for generic $x \in \mathfrak{g}^*$.

Unfortunately, the only invariant is the rank of $\mathcal{A}_x$. But... non-trivial invariants appear if we consider a pair of forms $\mathcal{A}_x$ and $\mathcal{A}_a$, $x, a \in \mathfrak{g}^*$. Can we get anything interesting?

- ▶ Many classical facts become more transparent and some new results can be derived.
- ▶ Arbitrary Lie algebras, not necessarily semisimple.
- ▶ Generalised argument shift conjecture

Historical remarks

1. The first and main ingredient is the argument shift method suggested by [A. Mischenko](#) and [A. Fomenko](#) in 1976 as a generalisation of [S. Manakov](#) construction.
2. In 1988, [I. Gelfand](#) and [I. Zakharevich](#) observed a very important relationship between compatible Poisson brackets and Jordan–Kronecker decomposition theorem and used it to study the so-called micro-Kronecker pencils and their applications.
3. Jordan–Kronecker decomposition for a pencil of skew-symmetric forms ([C. Jordan](#), [K. Weierstrass](#), [L. Kronecker](#), [F. Gantmacher](#), [G. Gurevich](#), [R. Thompson](#), ...).
4. Transition from algebraic canonical forms of pencils in linear algebra to normal forms of compatible Poisson brackets was done by [F.-J. Turiel](#) (series of papers since 1989).
5. The idea of JK invariants was conceptualised in the framework of an informal Moscow–Loughborough research seminar for PhD students ([A. Izosimov](#), [P. Zhang](#), [A. Konjaev](#), [A. Vorontsov](#), [I. Kozlov](#), [D. Dowell](#)).
6. This talk is based on our joint paper with [Pumei Zhang](#).

Conjecture

In the two forms $\mathcal{A}_x$ $\mathcal{A}_a$, one can easily recognise two famous compatible Poisson structures on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$.

The first is the standard Lie-Poisson bracket:

$$\{f, g\}(x) = \mathcal{A}_x(df(x), dg(x)) = \sum (c_{ij}^k x_k) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

The second is:

$$\{f, g\}_a(x) = \mathcal{A}_a(df(x), dg(x)) = \sum (c_{ij}^k a_k) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad a \in \mathfrak{g}^* \text{ (fixed)}.$$

Consider the following family of functions on $\mathfrak{g}^*$ (algebra of shifts):

$$\mathcal{F}_a = \{f(x + \lambda a) \mid f \in I(\mathfrak{g}), \lambda \in \mathbb{R}\}.$$

Theorem (Mischenko, Fomenko)

- 1) $\mathcal{F}_a$ is commutative w.r.t. both $\{, \}$ and $\{, \}_a$.
- 2) If $\mathfrak{g}$ is semisimple, then $\mathcal{F}_a$ is complete.

Mischenko-Fomenko conjecture.

For any $\mathfrak{g}$, there is a complete commutative family $\mathcal{G} \subset P(\mathfrak{g})$ of polynomials in involution.

Conjecture

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- 2) If $\mathfrak{g}$ is semisimple, then $\mathcal{F}_a$ is complete.

Generalised Argument Shift Conjecture.

For any $\mathfrak{g}$, there is a complete commutative family $\mathcal{G}_a \subset P(\mathfrak{g})$ of polynomials in bi-involution.

Jordan-Kronecker decomposition

This theorem gives a classification of skew-symmetric bilinear forms $\mathcal{A}, \mathcal{B}$ on a finite-dimensional vector space by reducing them simultaneously to an elegant block-diagonal form.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two skew-symmetric bilinear forms on a complex vector space V . Then by an appropriate choice of a basis, their matrices can be simultaneously reduced to the following canonical block-diagonal form:

$$\mathcal{A} \mapsto \begin{pmatrix} \mathcal{A}_1 & & & \\ & \mathcal{A}_2 & & \\ & & \ddots & \\ & & & \mathcal{A}_k \end{pmatrix}, \quad \mathcal{B} \mapsto \begin{pmatrix} \mathcal{B}_1 & & & \\ & \mathcal{B}_2 & & \\ & & \ddots & \\ & & & \mathcal{B}_k \end{pmatrix}$$

where the pairs of the corresponding blocks $\mathcal{A}_i$ and $\mathcal{B}_i$ can be of the following three types:

Jordan-Kronecker decomposition

 $\mathcal{A}_i$
 $\mathcal{B}_i$

Jordan block
($\lambda_i \in \mathbb{C}$)

$$\begin{pmatrix} J(\lambda_i) \\ -J^\top(\lambda_i) \end{pmatrix}$$

$$\begin{pmatrix} -\text{Id} \\ \text{Id} \end{pmatrix}$$

Jordan block
($\lambda_i = \infty$)

$$\begin{pmatrix} -\text{Id} \\ \text{Id} \end{pmatrix}$$

$$\begin{pmatrix} J(0) \\ -J^\top(0) \end{pmatrix}$$

Kronecker
block

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix}} \\ \boxed{\begin{matrix} -1 & & & \\ 0 & \ddots & & \\ & \ddots & -1 & \\ & & & 0 \end{matrix}} \end{pmatrix}$$

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix}} \\ \boxed{\begin{matrix} 0 & & & \\ -1 & \ddots & & \\ & \ddots & 0 & \\ & & & -1 \end{matrix}} \end{pmatrix}$$

Some comments

- ▶ **Characteristic numbers** λ_i play the same role as “eigenvalues” in the case of linear operators (recursion operators). More precisely, λ_i are those numbers for which the rank of $\mathcal{A}_\lambda = \mathcal{A} + \lambda\mathcal{B}$ for $\lambda = \lambda_i$ is not maximal.
- ▶ If $\mu \neq \lambda_i$, then $\mathcal{A}_\mu = \mathcal{A} + \mu\mathcal{B}$ is called **regular** (in the pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$).
- ▶ Characteristic numbers are the roots of the characteristic polynomial $f_{\mathcal{P}}(\lambda) = \gcd\{\text{Pf}_{i_1 \dots i_{2k}}(\mathcal{A} + \lambda\mathcal{B})\}$, $2k = \text{rank } \mathcal{P}$.
- ▶ The sizes of Kronecker blocks are odd $2k_i - 1$, the sizes of Jordan blocks are even $2j_m$. The numbers k_i and j_m are called **Kronecker and Jordan** indices of the pencil.
- ▶ The Jordan–Kronecker decomposition theorem implies the existence of a large subspace which is isotropic simultaneously for all forms from the pencil $\mathcal{P}$ (**bi-Lagrangian subspace**).

Some more comments

Theorem

For every pencil $\mathcal{P} = \{\mathcal{A}_\lambda\}$, there is a bi-Lagrangian subspace $L \subset V$. This means that L is isotropic with respect to all $\mathcal{A}_\mu \in \mathcal{P}$ and is maximal isotropic for all regular forms $\mathcal{A}_\lambda \in \mathcal{P}$. In particular, $\dim L = \frac{1}{2}(\dim V + \text{corank } \mathcal{P})$.

There is one very special bi-isotropic subspace of V :

$$L_{\text{can}} = \sum_{\lambda} \text{Ker } \mathcal{A}_\lambda,$$

Proposition

- ▶ L_{can} is contained in any bi-Lagrangian subspace L and is the intersection of all bi-Lagrangian subspaces.
- ▶ $\dim L_{\text{can}} = \sum_{i=1}^s k_i = \frac{1}{2}(\dim V + \text{corank } \mathcal{P}) - \deg f_{\mathcal{P}}$
- ▶ $\dim L_{\text{can}}$ is bi-Lagrangian if and only if $\mathcal{P}$ is of pure Kronecker type.

Definition of Jordan-Kronecker invariants

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. We say that $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is a **generic pair** if the type of the Jordan-Kronecker decomposition of $\mathcal{A}_x = \sum_k c_{ij}^k x_k$ and $\mathcal{A}_a = \sum_k c_{ij}^k a_k$ is the same for all points in the neighborhoods of (x, a) .

Definition

The **type of the Jordan-Kronecker canonical form** for the pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ for a generic pair $(x, a) \in \mathfrak{g}^*$, is called the **Jordan-Kronecker invariant** of $\mathfrak{g}$.

In particular, a Lie algebra $\mathfrak{g}$ is

- ▶ of **Kronecker** type,
- ▶ of **Jordan (symplectic)** type,
- ▶ of **mixed** type,

if the Jordan-Kronecker decomposition of a (generic) pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ consists of

- ▶ only Kronecker blocks,
- ▶ only Jordan blocks
- ▶ both Jordan and Kronecker blocks.

The Kronecker and Jordan indices of a generic pencil $\{\mathcal{A}_x + \lambda \mathcal{A}_a\}$ are said to be the **Kronecker and Jordan indices** of $\mathfrak{g}$.

Prerequisites: basic notions

- ▶ Finite-dimensional Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$
- ▶ Lie-Poisson bracket on $\mathfrak{g}^*$:

$$\{f, g\}(x) = \langle x, [df(x), dg(x)] \rangle$$

- ▶ Annihilator of $a \in \mathfrak{g}^*$:

$$\text{Ann } a = \{\xi \in \mathfrak{g} \mid \text{ad}_\xi^* a = 0\}$$

- ▶ $f_{\mathfrak{g}}(x) = \gcd \{ \text{Pf}_{i_1 \dots i_{2k}}(\mathcal{A}_x) \}$, the fundamental semi-invariant of $\mathfrak{g}$.
- ▶ Index of $\mathfrak{g}$ is the codimension of generic Ad^* -orbits:

$$\text{ind } \mathfrak{g} = \min_{x \in \mathfrak{g}^*} \dim \text{Ann } x$$

If $\text{ind } \mathfrak{g} = 0$, then the Lie algebra $\mathfrak{g}$ is called Frobenius.

- ▶ Singular set

$$\text{Sing} = \{y \in \mathfrak{g}^* \mid \dim \text{Ann}(y) > \text{ind } \mathfrak{g}\}$$

Kronecker case

Theorem

The following properties of a Lie algebra $\mathfrak{g}$ are equivalent

1. $\mathfrak{g}$ is of Kronecker type, i. e. the Jordan–Kronecker decomposition of the (generic) pencil $A_x + \lambda A_a$ consists of Kronecker blocks only,
2. $\text{codim Sing} \geq 2$, where

$$\text{Sing} = \{y \in \mathfrak{g}^* \mid \dim \text{Ann } y > \text{ind } \mathfrak{g}\} \subset \mathfrak{g}^*$$

is the singular subset of $\mathfrak{g}^$,*

3. *the algebra of shifts $\mathcal{F}_a$ is complete.*

Example

Let $\mathfrak{g}$ be semisimple. Then $\mathfrak{g}$ is of Kronecker type and the Kronecker indices $k_1, \dots, k_s$, $s = \text{ind } \mathfrak{g}$, are exactly the degrees of the basic Casimirs $f_1, \dots, f_s$ (invariants of the adjoint representation).

This property holds for many other classes of Lie algebras, e. g., $e(n) = so(n) + \mathbb{R}^n$.

Theorem

The following properties of a Lie algebra $\mathfrak{g}$ are equivalent

- 1. $\mathfrak{g}$ is of Jordan type, i.e. the Jordan–Kronecker decomposition of the generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ consists of Jordan blocks only,*
- 2. a generic form $\mathcal{A}_x$ is non-degenerate, i.e., $\text{ind } \mathfrak{g} = 0$ and $\mathfrak{g}$ is a Frobenius Lie algebra,*
- 3. $\mathcal{F}_a$ is trivial, i.e., $\mathcal{F}_a = \mathbb{C}$.*

Proposition

- 1) The number of Kronecker blocks in the JK decomposition is equal to the index of $\mathfrak{g}$.
- 2) The number of trivial Kronecker blocks is greater or equal to the dimension of the center of $\mathfrak{g}$.
- 3) The number of independent functions in the family of shifts $\mathcal{F}_a$ is equal to $\sum k_i$ where k_i are the Kronecker indices of $\mathfrak{g}$ or, equivalently,

$$\text{tr.deg.}\mathcal{F}_a = \frac{1}{2}(\dim \mathfrak{g} + \text{ind } \mathfrak{g}) - \deg \mathfrak{p}_{\mathfrak{g}}.$$

Theorem (A. Vorontsov, 2011)

Let $f_1(x), f_2(x), \dots, f_s(x) \in P(\mathfrak{g})$ be algebraically independent polynomial Ad^* -invariants of $\mathfrak{g}$, $s = \text{ind } \mathfrak{g}$, and $m_1 \leq m_2 \leq \dots \leq m_s$ be their degrees, $m_i = \deg f_i$. Then

$$m_i \geq k_i,$$

where $k_1 \leq k_2 \leq \dots \leq k_s$ are Kronecker indices of the Lie algebra $\mathfrak{g}$.

In the semisimple case (but not only!): $m_i = k_i$.

Polynomiality of the algebra $I(\mathfrak{g})$ of Ad^* -invariants

Let $f_1, \dots, f_s$, $s = \text{ind } \mathfrak{g}$, be algebraically independent Ad^* -invariant polynomials. Then

$$\sum_{i=1}^s \deg f_i \geq \sum_{i=1}^s k_i = \frac{1}{2}(\dim \mathfrak{g} + \text{ind } \mathfrak{g}) - \deg f_{\mathfrak{g}}.$$

For many classes of Lie algebras, this estimate becomes an equality (known as a *sum rule*) which (surprisingly) implies that the differentials $df_1, \dots, df_s$ are linearly independent at $x \in \mathfrak{g}^*$ if and only if $x \notin \text{Sing}_1$.

Theorem

Let $k_1 \leq \dots \leq k_s$ be the Kronecker indices of $\mathfrak{g}$ and $f_1, \dots, f_s \in I(\mathfrak{g})$ be algebraically independent Ad^* -invariant polynomials with $\deg f_1 \leq \deg f_2 \leq \dots \leq \deg f_s$, $s = \text{ind } \mathfrak{g}$. Assume that $\mathfrak{g}$ is unimodular and $f_{\mathfrak{g}} \in I(\mathfrak{g})$. Then the following conditions are equivalent:

1. $k_i = \deg f_i$, $i = 1, \dots, s$;
2. $\sum_{i=1}^s \deg f_i = \frac{1}{2}(\dim \mathfrak{g} + \text{ind } \mathfrak{g}) - \deg f_{\mathfrak{g}}$;
3. $I(\mathfrak{g})$ is polynomial on $f_1, \dots, f_s$.

Characteristic numbers and singular set

- ▶ The **characteristic numbers** λ_i of the Lie algebra $\mathfrak{g}^*$ are, by definition, the characteristic numbers for a generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$, $(x, a) \in \mathfrak{g}^* \times \mathfrak{g}^*$.
- ▶ The characteristic numbers $\lambda_i = \lambda_i(x, a)$ are defined by the following algebraic condition $x + \lambda_i a \in \text{Sing}$.
- ▶ The characteristic numbers exist if and only if $\text{codim Sing} = 1$.
- ▶ The codimension one component of the singular set Sing is defined by one polynomial equation $f_g(x) = 0$, where $f_g(x)$ is the fundamental invariant of $\mathfrak{g}$. The rest of Sing will be denoted by Sing_1 .
- ▶ The characteristic numbers of $\mathfrak{g}$ are the roots of $f_g(x + \lambda a) = 0$.
- ▶ The characteristic numbers are in bi-involution.

Frobenius Lie algebras

Let $\mathfrak{g}$ be Frobenius, i.e., $\text{ind } \mathfrak{g} = 0$. Then Sing is defined by one polynomial, namely, $f_{\mathfrak{g}}(x) = \text{Pf}(\mathcal{A}_x) = \sqrt{\det \left(c_{ij}^k x_k \right)}$. The degree of this polynomial is $\frac{1}{2} \dim \mathfrak{g}$.

Theorem

Let $\mathfrak{g}$ be a Frobenius Lie algebra, and the (geometric) degree of $\text{Sing} \subset \mathfrak{g}^$ be equal to $k = \frac{1}{2} \dim \mathfrak{g}$.*

Then a generic pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ is diagonalisable (i.e. has no Jordan blocks of size greater than 2×2), all characteristic numbers are distinct, and the coefficients of the “characteristic polynomial” $p(\lambda) = \text{Pf } \mathcal{A}_{x+\lambda a}$ form a complete family of polynomials in bi-involution.

Mixed case

Consider the polynomial $f_g(x)$, substitute $x + \lambda a$ and consider it as a polynomial in λ :

$$f(\lambda) = f_g(x + \lambda a) = g_0(x) + \lambda g_1(x) + \lambda^2 g_2(x) + \cdots + \lambda^m g_m(x).$$

The homogeneous polynomials $g_0(x), \dots, g_m(x)$ are obviously the symmetric polynomials of characteristic numbers. So they are in bi-involution and, moreover, they are in bi-involution with the algebra of shifts $\mathcal{F}_a$.

Combining the collection of g_k 's with the algebra of shifts $\mathcal{F}_a$, we obtain an extended algebra of functions in bi-involution $\mathcal{G}_a$.

Question. Is $\mathcal{G}_a$ complete?

Theorem (A.Isosimov)

The algebra $\mathcal{G}_a$ is complete if and only if, the annihilator of a generic singular element is isomorphic to $\mathfrak{a}_2 \oplus \text{centre}$, where $\mathfrak{a}_2$ is the two-dimensional non-Abelian Lie algebra.

Elashvili conjecture

Let $\text{Ann } a = \{\xi \in \mathfrak{g} \mid \text{ad}_\xi^* a = 0\}$ be the stationary subalgebra of $a \in \mathfrak{g}^*$ with respect to the coadjoint representation. The following estimate is well-known:

$$\text{ind Ann } a \geq \text{ind } \mathfrak{g}$$

Elashvili conjecture: if $\mathfrak{g}$ is semisimple, then $\text{ind Ann } a = \text{ind } \mathfrak{g}$ for all $a \in \mathfrak{g} = \mathfrak{g}^*$.

Interpretation in terms of Jordan-Kronecker decomposition:

Proposition

Let $a \in \mathfrak{g}^*$ be fixed and $x \in \mathfrak{g}^*$ is generic in the sense that the type of the Jordan-Kronecker decomposition of $\mathcal{A}_x$ and $\mathcal{A}_a$ does not change in a certain neighborhood of x . Then

$$\text{ind Ann } a = \text{ind } \mathfrak{g}$$

if and only if the Jordan-Kronecker decomposition does not contain any non-trivial Jordan blocks, i.e., the Jordan part is diagonalisable.

Otherwise, i.e. if there are non-trivial Jordan blocks, we have strong inequality:

$$\text{ind Ann } a > \text{ind } \mathfrak{g}$$

Examples

- ▶ Semisimple Lie algebras:

Pure Kronecker case, the Kronecker indices k_i coincide with the degrees of m_i of basis Casimirs:

$$k_i = m_i.$$

For classical series these numbers m_i are:

- ▶ A_n : $2, 3, 4, \dots, n+1$;
- ▶ B_n : $2, 4, 6, \dots, 2n$;
- ▶ C_n : $2, 4, 6, \dots, 2n$;
- ▶ D_n : $2, 4, 6, \dots, 2n-2$ and n .
- ▶ Some semidirect sums.
 - ▶ $\mathfrak{e}(n) = \mathfrak{so}(n) + \mathbb{R}^n$:
JK invariants are the same as those for $\mathfrak{so}(n+1)$.
 - ▶ $\mathfrak{g} = \mathfrak{sl}(n) + \mathbb{R}^n$:
Pure Kronecker type with one single Kronecker block, i.e.,
 $k_1 = \frac{1}{2}(\dim \mathfrak{g} + 1)$.
 - ▶ $\mathfrak{k} +_\rho V$ with $\mathfrak{k}$ simple and ρ irreducible:
Pure Kronecker type
(F. Knop, P. Littelmann, B. Priwitzer, AB)
- ▶ Lie algebras of low dimension ≤ 5 . The complete list of JK invariants is obtained by P. Zhang.

Examples

- Affine Lie algebra $\mathfrak{aff}(n) = \mathfrak{gl}(n) + \mathbb{R}^n$. The Pfaffian $\text{Pf}(\mathcal{A}_x) = \sqrt{\det(c_{ij}^k x_k)}$ is irreducible and we can apply one of above theorems. The pencil $\mathcal{A}_x + \lambda \mathcal{A}_a$ is of Jordan type, diagonalisable, with $\frac{1}{2} \dim \mathfrak{g}$ distinct characteristic number. In other words, the Jordan indices of $\mathfrak{g}$ are

$$\underbrace{1, 1, \dots, 1}_k, \quad k = \frac{1}{2}(n^2 + n) = \frac{1}{2} \dim \mathfrak{aff}(n).$$

- Another interesting example $\mathfrak{g} = \mathfrak{gl}(n) + \mathbb{R}^{n^2}$, where $\mathbb{R}^{n^2}$ is realised as the space of $n \times n$ -matrices, and the action of $\mathfrak{gl}(n)$ on it is left multiplication. The matrix realisation of $\mathfrak{g}$ is: $\begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix}$.

This Lie algebra is Frobenius, the singular set is defined by $\det C = 0$. Hence, we have n distinct characteristic numbers, the multiplicity of each of them is $2n$, and the Jordan indices are

$$\underbrace{1, 1, \dots, 1}_{n-2}, 2$$

Triangular Lie algebra

Let $\mathfrak{t}_n$ be the Lie algebra of upper triangular $n \times n$ matrices.

The description of Jordan-Kronecker invariants for $\mathfrak{t}_n$ easily follows from the results by [A.Arkhangel'skii](#).

If n is even, then $\mathfrak{t}_n$ is of mixed type. The coadjoint invariants are rational functions $f_k = \frac{P_k}{Q_k}$, $k = 1, \dots, \frac{n}{2}$ with $\deg P_k = k$ and $\deg Q_k = k - 1$. The Kronecker indices are exactly $\deg P_k + \deg Q_k$, namely

$$1, 3, 5, \dots, n - 1.$$

The singular set $\text{Sing} \subset \mathfrak{t}_n^*$ is defined by an irreducible polynomial f of degree $\frac{n}{2}$. Therefore, $\mathfrak{t}_n$ possesses $\frac{n}{2}$ distinct characteristic numbers, each of multiplicity one. In particular, the Jordan part of a generic pencil $\mathcal{A}_{x+\lambda a}$ is diagonalisable and Jordan indices are $1, \dots, 1$ ($\frac{n}{2}$ times).

If n is odd, then $\mathfrak{t}_n$ is of Kronecker type and the Kronecker indices are $1, 3, 5, \dots, n$.

Important: For all these Lie algebras, the Generalised Argument Shift Conjecture holds.

Many thanks
for your attention