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Korteweg – de Vries equation: solitons and undular bores

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Abstract

The Korteweg – de Vries (KdV) equation is a fundamental mathematical model for the description of weakly nonlinear long wave propagation in dispersive media. It is known to possess a number of families of exact analytic solutions. Two of them: solitons and nonlinear periodic travelling waves – are of particular interest from the viewpoint of fluid dynamics applications as they occur as typical asymptotic outcomes in a broad class of initial/boundary-value problems. Two different major approaches have been developed in the last four decades to deal with the problems involving solitons and nonlinear periodic waves: inverse scattering transform and the Whitham method of slow modulations. We review these methods and show relations between them. Emphasis is made on solving the KdV equation with large-scale initial data. In this case, the long-time evolution leads to formation of an expanding undular bore, a modulated travelling wave connecting two different non-oscillating flows. Another problem considered is propagation of a solitary wave through a variable environment in the framework of the variable-coefficient KdV equation. If the background environment varies slowly, the solitary wave deforms adiabatically and an extended small-amplitude trailing shelf is generated ensuring conservation of mass. On a long-time scale, the trailing shelf evolves, via an intermediate stage of an undular bore, into a secondary soliton train.

1 Introduction

The Korteweg - de Vries (KdV) equation

$$u_t + \alpha uu_x + \beta u_{xxx} = 0, \quad (1)$$

is a universal mathematical model for the description of weakly nonlinear long wave propagation in dispersive media. Here $u(x, t)$ is an appropriate field variable and x, t are space coordinate and time respectively. The coefficients α and β are determined by the medium properties and can be either constants or functions of x, t . An incomplete list of physical applications of the KdV equation includes shallow-water gravity waves, ion-acoustic waves in collisionless plasma, internal waves in the atmosphere and ocean, and waves in bubbly fluids. This broad range of applicability is explained by the fact that the KdV equation (1) describes a combined effect of the lowest-order, quadratic, nonlinearity (term uu_x) and the simplest long-wave dispersion (term u_{xxx}). One can find derivation of the KdV equation for different physical contexts in the books of Dodd et al (1982), Drazin and Johnson (1989), Newell (1985), Karpman (1975) and many others.

Although equation (1) with constant coefficients was originally derived in the second half of nineteenth century, its real significance as a fundamental mathematical model for the generation and propagation of long nonlinear waves of small amplitude has been understood only after the seminal works of Zabusky and Kruskal (1965), Gardner, Greene, Kruskal and Miura (1967) and Lax (1968). These authors showed that the KdV equation (unlike a ‘general’ nonlinear dispersive equation) can be solved *exactly* for a broad class of initial/boundary conditions and, importantly, the solutions often contain a combination of localized wave states which preserve their ‘identity’ in the interactions with each other pretty much as classical particles do. In the long-time asymptotic solutions, such localized states represent solitary waves well known due to original works of Russel, Boussinesq, Rayleigh, and Korteweg and de Vries. Such solitary wave solutions of the KdV equation have been called *solitons* by Zabusky and Kruskal (1965) owing to their unusual particle-like behaviour in the interactions with other solitary waves and nonlinear radiation. It turned out that the KdV equation possesses the family of N -soliton solutions where $N \in \mathbb{N}$ can be arbitrary. Later, similar multisoliton solutions have been found for other equations which share with the KdV equation the remarkable property of *complete integrability*.

2 Periodic travelling wave solutions and solitary waves

We start with an account of elementary properties of the travelling wave solutions of the KdV equation with constant coefficients α , β . Making in (1) a simple change of variables

$$u' = \frac{1}{6}\alpha u, \quad x' = \frac{x}{\sqrt{\beta}} \quad t' = \frac{t}{\sqrt{\beta}} \quad (2)$$

we, on omitting primes, cast the KdV equation into its canonical dimensionless form

$$u_t + 6uu_x + u_{xxx} = 0. \quad (3)$$

We shall look for a solution of (3) in the form of a single-phase periodic wave of a permanent shape, i.e. in the form $u(x, t) = u(\theta)$, where $\theta = x - ct$ is the travelling phase and $c = \text{constant}$ is the phase velocity. For such solutions the KdV equations reduces to the ordinary differential equation

$$-cu_\theta + 6uu_\theta + u_{\theta\theta\theta} = 0, \quad (4)$$

which is integrated twice (the second integration requires an integrating factor u_θ) to give

$$(u_\theta)^2 = -G(u), \quad G(u) = 6(u - b_1)(u - b_2)(u - b_3), \quad (5)$$

where $b_3 \geq b_2 \geq b_1$ are constants and

$$c = 2(b_1 + b_2 + b_3). \quad (6)$$

Equation (5) describes periodic motion of a ‘particle’ with co-ordinate u and time θ in the potential $-G(u)$. Since $-G(u) > 0$ for $u \in [b_2, b_3]$, the ‘particle’ oscillates between the endpoints b_2 and b_3 and the period of the oscillations (the wavelength of the travelling wave $u(x - ct)$) is

$$L = \int_0^L d\theta = 2 \int_{b_2}^{b_3} \frac{du}{\sqrt{-G(u)}} = \frac{2\sqrt{2}K(m)}{(b_3 - b_1)^{1/2}}, \quad (7)$$

where $K(m)$ is the complete elliptic integral of the first kind and m is the modulus,

$$m = \frac{b_3 - b_2}{b_3 - b_1}, \quad 0 \leq m \leq 1. \quad (8)$$

The wavenumber and the frequency of the travelling wave $u(x, t)$ are

$$k = 2\pi/L, \quad \omega = kc. \quad (9)$$

Equation (5) is integrated in terms of the Jacobian elliptic cosine function $\text{cn}(\xi; m)$ (see Abramowitz & Stegun 1965, for instance) to give

$$u(x, t) = b_2 + (b_3 - b_2) \text{cn}^2 \left(\sqrt{2(b_3 - b_1)}(x - ct - x_0); m \right), \quad (10)$$

where x_0 is an initial phase. Solution (10) is often referred to as *cnoidal wave*. According to the properties of elliptic functions, when $m \rightarrow 0$ ($b_2 \rightarrow b_3$), the cnoidal wave converts into the vanishing amplitude harmonic wave

$$u(x, t) \approx b_3 - a \sin^2[k_0(x - c_0t - x_0)], \quad a = b_3 - b_2 \ll 1, \quad (11)$$

where $k_0 = k(b_1, b_3, b_3)$, $c_0 = c(b_1, b_3, b_3)$. The relationship between c_0 and k_0 is obtained from Eqs. (6), (9) considered in the limit $b_2 \rightarrow b_3$:

$$c_0 = 6b_3 - k_0^2, \quad (12)$$

– and agrees with the KdV linear dispersion relation $\omega(k) = 6ku_0 - k^3$ for linear waves propagating against the background $u_0 = b_3$.

When $m \rightarrow 1$ (i.e. $b_2 \rightarrow b_1$), the cnoidal wave (10) turns into a solitary wave

$$u_s(x, t) = b_1 + a_s \text{sech}^2[\sqrt{a_s/2}(x - c_s t - x_0)], \quad (13)$$

whose speed of propagation $c_s = c(b_1, b_1, b_3)$ is connected with the amplitude $a_s = b_3 - b_1$ by the relation

$$c_s = b_1 + 2a_s. \quad (14)$$

Here $u = b_1$ is a background flow: $u_s(x, t) \rightarrow b_1$ as $|x| \rightarrow \infty$. We note that, although the background flow in (11), (13) can be eliminated by a passage to a moving reference frame and using the invariance of the KdV equation with respect to Galilean transformation,

$$u \rightarrow u + d, \quad x \rightarrow x - 6dt, \quad d = \text{constant}, \quad (15)$$

– it is instructive to retain the full set of free parameters in the solution as they will be important in the study of the slowly varying solutions of the KdV equation, where b_1, b_3 are no longer constants and, therefore, cannot be eliminated by the transformation (15).

To sum up, the cnoidal waves form a *three-parameter* family of the KdV solutions while the linear waves and solitary waves are characterised by only *two* independent parameters (with an account of background flow). We remark that the asymptotic limits (11) and (13) could be obtained directly from the basic equation (5) which is easily integrated in terms of elementary functions when $b_2 = b_3$ or $b_2 = b_1$.

3 Inverse scattering transform method and solitons

Although the existence of the *particular* permanent shape travelling wave solutions such as (10), (13) for a nonlinear partial differential equation is a nontrivial fact on its own, these solutions would have had very limited applicability if they would not appear in some reasonable class of initial-value or boundary-value problems. The real significance of these particular solutions becomes clear when one realises that solitary waves and periodic travelling waves naturally occur in the asymptotic solutions of a broad class of the initial-value problems for the KdV equation and, moreover, the methods exist enabling one to construct these solutions analytically. In this section, we will discuss some remarkable properties of the KdV solitary waves and the method for solving the problems involving their formation and evolution.

We will be interested in solving the KdV equation (3) in the class of functions decaying sufficiently fast together with their first derivatives far from the origin. With this aim in view we consider initial data

$$u(x, 0) = u_0(x), \quad u_0(x) \rightarrow 0, \quad u'_0(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (16)$$

The properties we are going to consider and the existence of the method of exact integration of the Cauchy problem (3), (16) reflect the fundamental fact of the complete integrability of the KdV equation. Although the notion of complete integrability for a nonlinear partial differential equation has the exact mathematical meaning, which is formulated in terms of dynamics of infinite-dimensional Hamiltonian systems (see for instance Novikov et al 1984, or Newell 1985) we, for practical purposes of this text, will broadly mean by this the ‘solvability’ of a certain class of initial/boundary-value problems. Still, it is instructive to mention that integrability or solvability of a finite-dimensional Hamiltonian system is intimately connected with the existence of ‘higher than usual’ number of conserved quantities.

3.1 Conservation laws

First we note that the KdV equation (3) can be represented in the form of a *conservation law*

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (3u^2 + u_{xx}) = 0. \quad (17)$$

Indeed, equation (17) implies conservation of the integral ‘mass’,

$$\frac{d}{dt} \int_{-\infty}^{+\infty} u dx = 0 \quad (18)$$

provided the function $u(x, t)$ vanishes together with its spatial derivatives as $x \rightarrow \pm\infty$ (another class of admissible functions is provided by periodic functions, in this case the integral in (18) should be taken over the period). Using simple algebra one can also obtain conservation equations for the ‘momentum’

$$\frac{\partial}{\partial t} \frac{u^2}{2} + \frac{\partial}{\partial x} \left(2u^3 + uu_{xx} - \frac{1}{2}u_x^2 \right) = 0, \quad (19)$$

and for the ‘energy’

$$\frac{\partial}{\partial t} \left(u^3 - \frac{1}{2}u_x^2 \right) + \frac{\partial}{\partial x} \left(\frac{9}{2}u^4 + 3u^2u_{xx} + u_tu_x + \frac{1}{2}u_{xx}^2 \right) = 0. \quad (20)$$

Now we will show that the KdV equation actually possesses an *infinite number* of conservation laws. For that, we, following Gardner, introduce the differential substitution

$$u = w + i\epsilon w_x + \epsilon^2 w^2, \quad (21)$$

where ϵ is an arbitrary parameter. Setting (21) into the KdV equation (3) we obtain

$$u_t + 6uu_x + u_{xxx} = (1 + 2\epsilon^2 w + i\epsilon \frac{\partial}{\partial x})(w_t + (3w^2 + 2\epsilon^2 w^3 + w_{xx})_x) = 0. \quad (22)$$

Now it follows from (22), that if $w(x, t)$ satisfies the conservation equation

$$w_t + (3w^2 + 2\epsilon^2 w^3 + w_{xx})_x = 0, \quad (23)$$

then $u(x, t)$ is the solution of the KdV equation (3). We represent w in the form of an infinite asymptotic series in powers of ϵ (we do not require formal

convergence though)

$$w \sim \sum_{n=0}^{\infty} \epsilon^n w_n. \quad (24)$$

Then, solving (21) by iterations for $\epsilon \ll 1$ we get subsequently:

$$w_0 = u, \quad w_1 = u_x, \quad w_2 = u^2 - u_{xx}, \quad \dots \quad (25)$$

Now, setting (24), (25) into the conservative equation (23) we obtain an infinite number of the KdV conservation laws as coefficients at even powers of ϵ . It can be shown that the conservation laws corresponding to odd powers of ϵ represent the x -derivatives of the conservation laws corresponding to preceding even powers and thus, do not carry any additional information. The values w_{2n} , $n = 0, 1, 2, \dots$ are often called the *Kruskal integrals*.

Using the infinite set of the Kruskal integrals and the Hamiltonian structure it was established that the KdV equation represents an infinite-dimensional integrable system. Practical realisation of this ‘abstract’ integrability is achieved through the inverse scattering transform method.

3.2 Lax pair

The method of integrating the KdV equation discovered by Gardner, Greene, Kruskal and Miura (1967) and put into general mathematical context by Lax (1968) is based on the possibility to represent Eq. (3) as a compatibility (integrability) condition for two *linear* differential equations for the *same* auxiliary function $\phi(x, t; \lambda)$:

$$L\phi \equiv (-\partial_{xx}^2 - u)\phi = \lambda\phi, \quad (26)$$

$$\phi_t = A\phi \equiv (-4\partial_{xxx}^3 - 6u\partial_x - 3u_x + C)\phi \quad (27)$$

$$= (u_x + C)\phi + (4\lambda - 2u)\phi_x. \quad (28)$$

Here λ is a complex parameter and $C(\lambda, t)$ is determined by the normalization of ϕ . Equation (26) constitutes the spectral problem and Eq. (27) the evolution problem. Direct calculation shows that the compatibility condition $(\phi_{xx})_t = (\phi_t)_{xx}$ yields the KdV equation (3) for $u(x, t)$ provided

$$\lambda_t = 0, \quad (29)$$

i.e. the evolution according to the KdV equation preserves the spectrum λ of the operator L in (26). This *isospectrality* property is very important for the further analysis.

The operators L and A in (26), (27) are often referred to as the *Lax pair*. It can be seen that the KdV equation (3) can be represented in an operator form $L_t = LA - AL \equiv [LA]$. This Lax representation provides a route for constructing further generalisations by appropriate choice of the operators L and A . For instance, it is clear that given the L -operator (26) the A -operator in the Lax pair is determined up to an operator commuting with L , which makes it possible to construct a *hierarchy* of equations associated with the same spectral problem but having different evolution properties. Such ‘higher’ KdV equations play important role in constructing the nonlinear multiperiodic (multiphase) solutions of the original KdV equation (3) (see Novikov et al 1984).

3.3 Direct scattering transform and evolution of spectral data

We consider the KdV equation in the class of functions sufficiently rapidly decaying as $|x| \rightarrow \infty$. To be more precise, we require boundedness of the integral (Faddeev’s condition)

$$\int_{-\infty}^{+\infty} (1 + |x|)|u(x)|dx < \infty, \quad (30)$$

which ensures applicability of the scattering analysis in the sequel. Now we turn to the first Lax equation (26), which represents the *time-independent Schrödinger equation*. This equation plays central role in quantum mechanics (see for instance Landau & Lifshitz 1977) and describes behaviour of the wave function $\phi(x; \lambda)$ of a particle moving through the potential $V(x) = -u(x)$. In this interpretation, $E = -\lambda$ is the energy of the particle. For a given potential $-u(x)$ the problem is to find the *spectrum* of the linear operator L , i.e a set $\{\lambda\}$ of admissible values for λ , and to construct the corresponding functions $\phi(x; \lambda)$. Depending on the concrete form of the potential $-u(x)$, there are two different types of such solutions characterised by different types of the spectral set $\{\lambda\}$.

i) Continuous spectrum $\lambda > 0$: *scattering solutions*.

We introduce $k^2 \equiv \lambda$, $k \in \mathbb{R}$ and, assuming that $u \rightarrow 0$ as $x \rightarrow \pm\infty$ such that condition (30) is satisfied, fix an asymptotic behaviour of the function $\phi(x; k^2)$ far from origin :

$$\phi \sim \exp(-ikx) + R(k) \exp(ikx) \quad \text{as } x \rightarrow +\infty, \quad (31)$$

$$\phi \sim T(k) \exp(-ikx) \quad \text{as } x \rightarrow -\infty. \quad (32)$$

This asymptotic solution of Eq. (26) describes scattering *from the right* of the incident wave $\exp(-ikx)$ on the potential $-u(x)$. Then $R(k)$ represents

a *reflection coefficient* and $T(k)$ a *transmission coefficient*. These are called *scattering data* and the problem of their determination for a given potential constitutes a direct scattering problem: $u \mapsto \{R(k), T(k)\}$. We note that owing to analytic properties of the solutions of the Schrödinger equation (see for instance Dodd et al 1984), the functions $R(k)$ and $T(k)$ are not independent and the scattering data can be characterized by a single function $R(k)$. In particular, we mention the physically transparent [total probability] relationship $|T|^2 + |R|^2 = 1$ following from the constancy of the Wronskian for two independent scattering solutions with the asymptotic behaviour (31), (32).

Let the potential $-u(x, t)$ evolve according to the KdV equation (3). Then the corresponding evolution of the scattering data is found by substituting Eqs. (31), (32) into the second Lax equation (27) (note that for the continuous spectrum the parameter λ can always be viewed as a constant). As a result, assuming $u(x), u'(x) \rightarrow 0$ as $x \rightarrow \pm\infty$ we obtain

$$C(\lambda, t) = 4ik^3, \quad \frac{dR}{dt} = 8ik^3 R, \quad \frac{dT}{dt} = 0. \quad (33)$$

Hence

$$R(k, t) = R(k, 0) \exp(8ik^3 t), \quad T(k, t) = T(k; 0). \quad (34)$$

ii) Discrete spectrum $\lambda = \lambda_n < 0$: *bound states*

If the potential $-u(x)$ is sufficiently negative near the origin of the x -axis, the scattering problem (26) implies existence of finite number of bound states $\phi = \phi_n(x; \lambda)$, $n = 1, \dots, N$ corresponding to the discrete admissible values of the spectral parameter $\lambda = \lambda_n = -\eta_n^2$, $\eta_n \in \mathbb{R}$, $\eta_1 < \eta_2 < \dots < \eta_N$. We require the following asymptotic behaviour as $x \rightarrow \pm\infty$, consistent with Eq. (26) for $u \rightarrow 0$:

$$\phi_n \sim \beta_n \exp(-\eta_n x) \quad \text{as } x \rightarrow +\infty, \quad (35)$$

$$\phi_n \sim \exp(\eta_n x) \quad \text{as } x \rightarrow -\infty. \quad (36)$$

Thus, for the case of discrete spectrum we have an analog of the scattering transform: $u \mapsto \{\eta_n, \beta_n\}$. Again, we shall be interested in the evolution of the spectral parameters when the potential $-u(x, t)$ evolves according to the KdV equation (3).

We first substitute Eq. (36) into (28) to obtain $C = C_n = 4\eta_n^3$ (cf. (33)). Then, setting (35) into (28) and using the isospectrality condition (29) we obtain

$$\frac{d\beta_n}{dt} = 4\eta_n^3 \beta_n \text{ so that } \beta_n(t) = \beta_n(0) \exp(4\eta_n^3 t). \quad (37)$$

We note that the bound state problem can be viewed as the analytic continuation of the scattering problem, defined on the real k -axis, to the upper half of the complex k -plane. Then the discrete points of the spectrum are found as *simple* poles $k = i\eta_n$ of the reflection coefficient $R(k)$ and $R \rightarrow 1$ as $|k| \rightarrow \infty$ (see details in Dodd et al 1982 for instance).

Thus, for a general potential $-u(x)$, decaying as in (30), we can introduce the direct scattering transform by the mapping

$$u \mapsto S = \{(\eta_n, \beta_n); R(k), k \in \mathbb{R}, n = 1, \dots, N\}. \quad (38)$$

Now, if the potential $u(x, t)$ evolves according to the KdV equation (3) then the scattering data S evolve according to simple equations

$$\eta_n = \text{constant}, \quad \beta_n(t) = \beta_n(0) \exp(4\eta_n^3 t), \quad R(k, t) = R(k, 0) \exp(8ik^3 t). \quad (39)$$

Equations (39) are often referred to as Gardner-Greene-Kruskal-Miura (GGKM) equations.

3.4 Inverse scattering transform: Gelfand - Levitan - Marchenko equation

It was established in 1950s that the potential $-u(x)$ of the Schrödinger equation can be completely reconstructed from the scattering data S . The corresponding mapping $S \mapsto u$ is called *inverse scattering transform* (IST) and is accomplished through the Gelfand - Levitan - Marchenko (GLM) linear integral equation. The derivation of this equation is beyond the scope of this text and can be found elsewhere (see for instance Whitham 1974, Drazin & Johnson 1989, Scott 2003). Here we present only the resulting formulae and show some of their important consequences.

We define the function $F(x)$ as

$$F(x, t) = \sum_{n=1}^N \beta_n^2(t) \exp(-\eta_n x) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(k, t) \exp(ikx) dk. \quad (40)$$

Then the potential $-u(x, t)$ is restored from the formula

$$u(x, t) = 2 \frac{\partial}{\partial x} K(x, x, t), \quad (41)$$

where the function $K(x, y, t)$ is found from the linear integral (Gelfand-Levitan-Marchenko) equation

$$K(x, y) + F(x + y) + \int_x^{+\infty} K(x, z)F(y + z)dz = 0 \quad (42)$$

defined for any moment t .

Thus, we have the scheme of integration of the KdV equation by the IST method:

$$u(x, 0) \mapsto S(0) \rightarrow S(t) \mapsto u(x, t). \quad (43)$$

It is essential that at each step of this algorithm one has to solve a linear problem. One can notice that the described method of integration of the KdV equation is in many respects analogous to Fourier method for integrating linear partial differential equations with the role of direct and inverse Fourier transform played by the direct and inverse scattering transform. Moreover, it can be shown (see for instance Ablowitz & Segur 1981) that for linear problems the scattering transform indeed converts into usual Fourier transform.

3.5 Reflectionless potentials and N -soliton solutions

There exists a remarkable class of potentials characterised by zero reflection coefficient, $R(k) = 0$. Such potentials are called *reflectionless* and can be expressed in terms of elementary functions. We start with the simplest case $N = 1$. In this case, since $R(k) = 0$, we have from (40), (39): $F(x, t) = \beta(0)^2 \exp(-\eta x + 8\eta^3 t)$, where $\beta \equiv \beta_1$, $\eta \equiv \eta_1$. Then solution of Eq. (42) can be sought in the form $K(x, y, t) = M(x, t) \exp(-\eta y)$. After simple algebra we get

$$M(x, t) = \frac{-2\eta\beta(0)^2 \exp(-\eta x + 8\eta^2 t)}{2\eta + \beta(0)^2 \exp(-2\eta x + 8\eta^2 t)}. \quad (44)$$

As a result, we obtain from (41)

$$u = 2\eta^2 \operatorname{sech}^2(\eta(x - 4\eta^2 t - x_0)), \quad (45)$$

which is just the solitary wave (13) of the amplitude $a_s = 2\eta^2$ propagating on a zero background ($b_1 = 0$) to the right with the velocity $c_s = 4\eta^2$ and

having the initial phase

$$x_0 = \frac{1}{2\eta} \ln \frac{\beta(0)^2}{2\eta}. \quad (46)$$

For arbitrary $N \in \mathbb{N}$ and $R(k) \equiv 0$ we have from (40)

$$F(x) = \sum_{n=1}^N \beta_n(t)^2 \exp(-\eta_n x),$$

and therefore, seek the solution of the GLM equation (42) in the form

$$K(x, y, t) = \sum_{n=1}^N M_n(x, t) \exp(-\eta_n y). \quad (47)$$

Now, on using (41), one arrives, after some algebra (see for instance Novikov et al 1984), at the general representation for the reflectionless potential $V(x, t) = -u_N(x, t)$,

$$u_N(x, t) = 2 \frac{\partial^2}{\partial x^2} \ln \det A(x, t). \quad (48)$$

Here A is the $N \times N$ matrix given by

$$A_{mn} = \delta_{mn} + \frac{\beta_n(t)^2}{\eta_n + \eta_m} e^{-(\eta_n + \eta_m)x}, \quad (49)$$

δ_{km} being the Kronecker delta.

Analysis of formulae (48), (49) (see Karpman 1975, Novikov et al 1984, Drazin & Johnson 1989) shows that for $t \rightarrow \pm\infty$ the solution of the KdV equation corresponding to the reflectionless potential can be asymptotically represented as a superposition of N single-soliton solutions propagating to the right and ordered in space by their speeds (amplitudes):

$$u_N(x, t) \sim \sum_{n=1}^N 2\eta_n^2 \operatorname{sech}^2[\eta_n(x - 4\eta_n^2 t \mp x_n)], \quad \text{as } t \rightarrow \pm\infty, \quad (50)$$

where the amplitudes of individual solitons are given by $a_n = 2\eta_n^2$ and the positions $\mp x_n$ of the n -th soliton as $t \rightarrow \mp\infty$ are given by the relationship (cf. (46) for a single soliton)

$$x_n = \frac{1}{2\eta_n} \ln \frac{\beta_n^2(0)}{2\eta_n} + \frac{1}{2\eta_n} \left\{ \sum_{m=1}^{n-1} \ln \left| \frac{\eta_n - \eta_m}{\eta_n + \eta_m} \right| - \sum_{m=n+1}^N \ln \left| \frac{\eta_n - \eta_m}{\eta_n + \eta_m} \right| \right\} \quad (51)$$

One can infer from Eq. (50) that at $t \gg 1$, the tallest soliton with $n = N$ is at the front followed by the progressively shorter solitons behind, forming

thus the triangle amplitude (velocity) distribution characteristic for noninteracting particles (see Whitham 1974). At $t \rightarrow -\infty$ we get the reversed picture. The full solution (48), (49) thus describes the interaction (collision) of N solitons at finite times. For this reason it is called N -soliton solution. The N -soliton solution is characterised by $2N$ parameters $\eta_1, \dots, \eta_N, \beta_1(0), \dots, \beta_N(0)$. Owing to isospectrality ($\eta_n = \text{constant}$), the solitons preserve their amplitudes (and velocities) in the interactions, the only change they undergo is an additional phase shift $\delta_n = 2x_n$ due to collisions.

Say, for a two-soliton collision with $\eta_1 > \eta_2$ the phase shifts as $t \rightarrow +\infty$ are

$$\delta_1 = 2x_1 = \frac{1}{\eta_1} \ln \left| \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right|, \quad \delta_2 = 2x_2 = -\frac{1}{\eta_2} \ln \left| \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right|. \quad (52)$$

It follows from the formula (52) that, as a result of the collision, the taller soliton gets an additional shift forward by the distance x_1 while the shorter soliton is shifted backwards by the distance x_2 . One should also keep in mind that the general formula (51) and its 2-soliton reduction (52) are relevant only for sufficiently large times when individual solitons are separated enough for the asymptotic representation (50) to be applicable.

In conclusion of this section, we note that one of the remarkable consequences of the formula (51) for the phase shifts is that the solitons in the N -soliton solution of the KdV equation interact only pairwise, i.e. the ‘multi-particle’ effects in the soliton interactions are absent.

3.6 Purely reflective potentials: nonlinear radiation

Opposite to the reflectionless potentials, the *purely reflective* potentials are characterised by zero transmission coefficient $T(k) \equiv 0$. This is evidently the case for all positive potentials $V(x) = -u_0(x) \geq 0$ characterised by pure continuous spectrum. Now one has to deal with the second term alone in formula (40). In this case, the general expression for the solution similar to the N -soliton solution is not available. An asymptotic analysis (see Dodd et al (1984), Ablowitz & Segur (1981) and references therein) shows that, under the long-time evolution the purely reflective potential transforms into the linear dispersive wave (the radiation) described locally by the linearised KdV equation but, unlike that in the solution of the initial-value problem for the linearised KdV equation, its amplitude decays at each fixed point at $x < 0$ with the rate greater or equal $t^{-1/2}$ rather than $t^{-1/3}$. The detailed structure of this wave and its dependence on the initial data are very complicated. However, the local qualitative behaviour is physically transparent:

the linear radiation propagates to the left with the velocity close to the group velocity $c_g = -3k^2$ of a linear wavepacket and the lowest rate of the amplitude decay is consistent with the momentum conservation law in linear modulation theory (see Whitham 1974).

3.7 Evolution of an arbitrary decaying potential

Now we are able to qualitatively describe an asymptotic evolution of an arbitrary decaying potential satisfying the condition (30). The spectrum of such a potential generally contains both discrete and continuous components. The discrete component is responsible for the appearance in the asymptotic solution of the chain of solitons ordered by the amplitudes and moving to the right. At the same time, the continuous component contributes to the linear dispersive wave propagating to the left. A simple *sufficient* condition for appearance of at least one bound state in the spectrum (i.e. of a soliton in the solution) is

$$\int_{-\infty}^{+\infty} u_0(x) dx > 0. \quad (53)$$

Thus, the long-time asymptotic outcome of the general KdV initial-value problem for decaying initial data can be represented in the form

$$u(x, t) \sim \sum_{n=1}^N 2\eta_n^2 \operatorname{sech}^2(\eta_n(x - 4\eta_n t - x_n)) + \text{linear radiation}, \quad (54)$$

where the soliton amplitudes $a_n = 2\eta_n^2$ and the initial phases x_n , as well as the parameters of the radiation component, are determined from the scattering data for the initial potential. The upper bound for the number N of solitons in the solution can be estimated by the formula

$$N \leq 1 + \int_{-\infty}^{+\infty} |x| |u_0(x)| dx. \quad (55)$$

Unfortunately, even the direct scattering problem can be solved explicitly only for very few potential forms. In most cases, one has to use numerical simulations or asymptotic estimates.

3.8 Semi-classical asymptotics in the IST method

One of the important cases where some explicit analytic results of rather general form become available, occurs when the initial potential is a ‘large-scale’ function. Then, for positive $u_0(x)$ the Schrödinger operator (26) has

a large number of bound states located close to each other so that the discrete spectrum can be characterized by a single continuous distribution function. In this case, an effective asymptotic description of the spectrum can be obtained with the use of the semi-classical Wentzel-Kramers-Brillouin (WKB) method (see Landau & Lifshitz 1977 for instance).

We consider the KdV equation (3) with the large-scale positive initial data

$$u(x, 0) = u_0(x/L) > 0, \quad L \gg 1. \quad (56)$$

For simplicity, we assume that initial function (56) has a form of a single positive bump satisfying an additional condition

$$\int_{-\infty}^{\infty} u_0^{1/2} dx \gg 1, \quad (57)$$

whose meaning will be clarified soon. An estimate following from Eq. (57) is $A^{1/2}L \gg 1$, where $A = \max(u_0)$. Assuming $A = \mathcal{O}(1)$ we introduce ‘slow’ variables $X = \epsilon x$, $T = \epsilon t$, where $\epsilon = 1/L \ll 1$ is a small parameter, into Eq. (3) to get the small-dispersion KdV equation:

$$u_T + 6uu_X + \epsilon^2 u_{XXX} = 0, \quad \epsilon \ll 1 \quad (58)$$

with the initial condition

$$u(X, 0) = u_0(X) \geq 0, \quad (59)$$

where

$$u_0(X) \text{ is } C^1, \quad \text{and} \quad \int_{-\infty}^{\infty} u_0^{1/2} dX = \mathcal{O}(1). \quad (60)$$

The associated Schrödinger equation (26) in the Lax pair assumes the form

$$-\epsilon^2 \phi_{XX} - u\phi = \lambda\phi. \quad (61)$$

We note that in quantum mechanics the role of ϵ in Eq. (61) is played by the Planck constant $\hbar$. According to the IST ideology, in order to construct the solution of the KdV equation (58) in the asymptotic limit $\epsilon \rightarrow 0$ we need to study the corresponding asymptotic behaviour of the scattering data for the initial potential $-u_0(X)$. The scattering data set S consists of the two groups of parameters (see Section 3.3): bound states $-\eta_n^2$ and norming coefficients β_n , $n = 1, \dots, N$ characterising discrete spectrum, and the reflection coefficient $R(k)$ characterising continuous spectrum.

The WKB analysis of the Schrödinger equation (61) yields that, for the potential $-u_0(X) \leq 0$ satisfying condition (57) the reflection coefficient is asymptotically zero,

$$\lim_{\epsilon \rightarrow 0} R(k) = 0, \quad (62)$$

while the bound states $\lambda_m = -\eta_m^2$, $m = 1, \dots, N$, $\eta_1 < \eta_2 < \dots < \eta_N$ are packed in the range of the potential, $0 < \eta^2 < A$, with the density (Weyl's law)

$$\phi(\eta) = \frac{1}{\pi\epsilon} \int_{X^-(\eta)}^{X^+(\eta)} \frac{\eta}{\sqrt{u_0(X) - \eta^2}} dX. \quad (63)$$

Here the limits of integration $X^-(\eta) < X^+(\eta)$ are defined by $u_0(X^\pm) = \eta^2$. The formula (63) follows from the differentiation with respect to η of the famous Bohr-Sommerfeld semi-classical quantization rule (see Landau & Lifshitz 1977),

$$\oint \sqrt{u_0(X) - \eta^2} dX = 2\pi\epsilon(n + \frac{1}{2}), \quad (64)$$

which gives the number n of bound states λ_k in the spectral interval $(-A, -\eta^2)$, so that $\phi(\eta) = |dn/d\eta|$. The integration in (64) is performed over the full period of the classical motion of the particle with the energy $-\eta^2$ in the potential well $-u_0(X)$. The maximum value of n is achieved when $\eta = 0$ and yields the total number of bound states

$$N \sim \frac{1}{\pi\epsilon} \int_{-\infty}^{+\infty} u_0^{1/2}(X) dX \gg 1. \quad (65)$$

The inequality in (65) is equivalent to the condition (57) and clarifies its physical meaning. The norming constants $\beta_n(0)$ of the scattering data in the semi-classical limit are given by formulae (see Lax, Levermore & Venakides 1994)

$$\beta_n = \beta(\eta_n), \quad \beta(\eta) = \exp\{\chi(\eta)/\epsilon\}, \quad (66)$$

where

$$\chi(\eta) = \eta X^+(\eta) + \int_{X^+(\eta)}^{\infty} \left(\eta - \sqrt{\eta^2 - u_0(X)} \right) dX. \quad (67)$$

Now we interpret the semi-classical scattering data (62) – (67) in terms of the solution $u(X, T)$ of the small-dispersion KdV equation (58). First of all, the relation (62) implies that the potential $-u_0(X)$ is *asymptotically*

reflectionless and, hence, the initial data $u_0(X)$ can be approximated by the N -soliton solution (48), (49),

$$u_0(X) \approx u_N(X/\epsilon) \quad \text{for } \epsilon \ll 1, \quad (68)$$

where $N[u_0] \sim \epsilon^{-1}$ is given by (65) and the discrete spectrum is defined by (63), (66), (67). Now one can use the known N -soliton dynamics (see Section 3.5) for the description of the evolution of an arbitrary initial potential satisfying the condition (57). This observation served as a starting point in the series of papers by Lax, Levermore and Venakides (see their review (1994) and references therein), where the singular zero-dispersion limit of the KdV equation has been introduced and thoroughly studied. While the description of multisolitons at finite T turns out to be quite complicated in the zero-dispersion limit, the asymptotic behaviour as $T \rightarrow \infty$ can be easily predicted using formula (50) which implies that the asymptotic as $T \rightarrow \infty$ outcome of the evolution will be a ‘soliton train’ consisting of N free solitons ordered by their amplitudes $a_n = 2\eta_n^2$, $n = 1, \dots, N$ and propagating on a zero background. The number of solitons in the train having the amplitude within the interval $(a, a + da)$ is $f(a)da$ where the soliton amplitude distribution function $f(a)$ follows from Weyl’s law (63):

$$f(a) = \frac{1}{8\pi\epsilon} \oint \frac{dX}{\sqrt{u_0(X) - a/2}}. \quad (69)$$

The formula (69) was obtained for the first time by Karpman (1967). It follows from the Karpman formula that the range of soliton amplitudes in the train is

$$0 < a < 2A, \quad (70)$$

which means that the tallest soliton has the amplitude twice as big as the amplitude of the initial perturbation, $a_{\max} = 2A$. The Karpman formula also allows one to determine the spatial distribution of solitons in the soliton train resulting from the initial perturbation $u_0(X)$. Indeed, as the speed of the soliton with the amplitude a moving on a zero background is $c_s = 2a$ (see (14)), its asymptotic position for $T \gg 1$ is $X \cong 2aT$, which implies that the solitons in the soliton train are spatially distributed according to a ‘triangle’ law

$$a \cong X/2T, \quad 0 < X/2T < 2A, \quad X, T \gg 1. \quad (71)$$

The number of waves in the interval $(X, X + dX)$ is determined from the balance relationship

$$kdX = f(a)da, \quad (72)$$

where $k(X, T)$ is the spatial density of solitons (the soliton train wavenumber). Then, using (71) we obtain

$$k(X, T) \cong \frac{1}{2T} f\left(\frac{X}{2T}\right). \quad (73)$$

Whereas the long-time multisoliton dynamics in the semi-classical limit is simple enough, the corresponding behaviour at finite times T is quite non-trivial and reveals some remarkable features. As the studies of Lax, Levermore and Venakides showed, there is certain critical time $T_b[u_0(X)]$, after which the multisoliton solution of the small-dispersion KdV equation (58), resulting from the bump-like initial data, asymptotically to the first order in ϵ manifests itself as a *cnoidal wave* (10) with X, T scaled as ϵ and with the parameters b_j depending on the unscaled (slow) variables X, T . Moreover, Lax and Levermore obtained the evolution equations for the moments $\overline{u}(X, T)$, $\overline{u^2}(X, T)$, $\overline{u^3}(X, T)$ which turned out to coincide with the *modulation equations* derived much earlier by Whitham (1965).

4 Whitham modulation equations

4.1 Whitham method

In 1960s, G. Whitham developed an asymptotic theory to treat the problems involving periodic travelling wave solutions (cnoidal waves in the KdV equation context) rather than individual solitons. It is clear that the cnoidal wave solution (10) as such, similarly to the plane monochromatic wave in linear wave theory, does not transfer any ‘information’ and does not solve any reasonable class of initial-value problems (except for the problem with the initial data in the form of a cnoidal wave!). However, one can try to construct a *modulated cnoidal wave*, a nonlinear analog of a linear wavepacket, which can presumably be an asymptotic outcome in some class of the nonlinear dispersive initial-value problems.

It is convenient to represent the periodic travelling wave solution (10) in an equivalent general form

$$u(x, t) = U_0(\tau; \mathbf{b}), \quad \tau = kx - \omega t - \tau_0, \quad \mathbf{b} = (b_1, b_2, b_3), \quad (74)$$

where the functions $k(\mathbf{b})$ and $\omega(\mathbf{b})$ are determined from (9), (6), (7) and the function $U_0(\tau)$ is defined by the ODE $k^2(U_0')^2 = -G(U_0)$ (an equivalent of (5)) and is 2π -periodic, $U_0(\tau + 2\pi; \mathbf{b}) = U_0(\tau; \mathbf{b})$; $\tau_0 \pmod{2\pi}$ being an arbitrary initial angular phase. We introduce a *slowly modulated* cnoidal

wave by letting the constants of integration b_j be functions of x and t on a large spatio-temporal scale, i.e. $b_j = b_j(X, T)$, where $X = \epsilon x$, $T = \epsilon t$, and $\epsilon \ll 1$ is a small parameter. Now Eq. (10) (or (74)) no longer is an exact solution of the KdV equation (3). One can, however, require that $U_0(\tau, \mathbf{b}(X, T))$ satisfies the KdV equation *approximately*, i.e. to first order in ϵ . This requirement leads to a set of restrictions for the slowly varying functions $b_j(X, T)$, which are called *modulation equations*.

The modulation equations can be obtained by using an extension of the well-known multiple-scale perturbation method (see Nayfeh 1981 for instance) to nonlinear partial differential equations (see Luke 1966; Grimshaw 1979; Dubrovin and Novikov 1989 and references therein).

We shall seek an asymptotic solution of the KdV equation in the form

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \dots, \quad (75)$$

where the leading term $u^{(0)}$ has the form (74) but with slowly varying parameters b_1, b_2, b_3 . To this end, we introduce an auxiliary phase function $S(X, T)$ and represent the terms $u^{(n)}$ of the decomposition (75) in the form

$$u^{(n)} = U_n(S(X, T)/\epsilon; \mathbf{b}(X, T)), \quad (76)$$

where $U_n(S(X, T))$ are 2π -periodic functions which depend smoothly on X, T . Then, for the leading term in (75) to have the form of the travelling wave (74), i.e. $u^{(0)} \rightarrow U_0(kx - \omega t; \mathbf{b})$ as $\epsilon \rightarrow 0$, one should require

$$S_X = k(\mathbf{b}(X, T)), \quad S_T = -\omega(\mathbf{b}(X, T)) \quad (77)$$

(this is readily established by substituting (74) and (75)–(77) into the KdV equation (3) and comparing the coefficients of the resulting ODEs appearing to the leading order in ϵ). The compatibility condition $S_{XT} = S_{TX}$ yields the so-called wave conservation law

$$k_T + \omega_X = 0, \quad (78)$$

which is one of the modulation equations. The remaining two are obtained by considering the next leading order equation in the asymptotic chain occurring after the substitution of the expansion (75) into the KdV equation (3) with the account of the form of the leading periodic term U_0 (74). As a result, collecting the terms $\mathcal{O}(\epsilon)$, one arrives at the ODE

$$-\omega(U_1)_\tau + 6k(U_0 U_1)_\tau + k^3(U_1)_{\tau\tau\tau} = -\frac{\partial}{\partial T} U_0 - 6U_0 \frac{\partial U_0}{\partial X}. \quad (79)$$

Since the right-hand side of equation (79) is a 2π -periodic function in τ , an unbounded growth of the solutions is expected due to resonances with the

eigenfunctions of the linear operator on the left-hand side. To eliminate this unbounded growth one should impose the orthogonality conditions,

$$\int_0^{2\pi} y_\alpha R d\tau = 0, \quad \alpha = 1, 2, \quad (80)$$

where $R(U_0, \partial_T U_0, \partial_X U_0)$ is the right-hand side of Eq. (79) and y_α are the periodic eigenmodes of the operator adjoint to the homogeneous operator on the left-hand side of Eq. (79). The adjoint equation has the form:

$$-\omega y_\tau + 6kU_0 y_\tau + k^3 y_{\tau\tau\tau} = 0. \quad (81)$$

One can see that there are indeed just two periodic solutions of the equation (81): $y_1 = 1$ and $y_2 = U_0$. Equations (78), (80) represent then the full set of the modulation equations for the three parameters $b_j(X, T)$.

There is a convenient alternative to the direct perturbation procedure outlined above. This alternative (but, of course, equivalent at the end) method was proposed by Whitham in 1965. The Whitham method of obtaining the modulation equations prescribes *averaging* any three of the KdV conservation laws $\partial_t P_j + \partial_x Q_j = 0$ (Eqs. (18) - (20), for instance) over the period L of the travelling wave (10) (or over 2π -interval if one uses the solution in the form (74)). The averaging is made according to (5) as

$$\overline{F}(b_1, b_2, b_3) = \frac{1}{L} \int_0^L F(u(\theta; b_1, b_2, b_3)) d\theta = \frac{k}{\pi} \int_{b_2}^{b_3} \frac{F(u)}{\sqrt{-G(u)}} du. \quad (82)$$

In particular, the mean value is calculated as

$$\bar{u} = b_1 + 2(b_3 - b_1)E(m)/K(m), \quad (83)$$

where $E(m)$ is the complete elliptic integral of the second kind. We now express the partial t - and x - derivatives as asymptotic expansions

$$\frac{\partial F}{\partial t} = \frac{dF}{d\theta} + \epsilon \frac{\partial F}{\partial T} + \mathcal{O}(\epsilon)^2, \quad \frac{\partial F}{\partial x} = \frac{dF}{d\theta} + \epsilon \frac{\partial F}{\partial X} + \mathcal{O}(\epsilon)^2. \quad (84)$$

Then, in view of periodicity of F in θ , the averages of the derivatives (84) are calculated as

$$\overline{\frac{\partial}{\partial t} F} = \epsilon \frac{\partial}{\partial T} \overline{F} + \mathcal{O}(\epsilon)^2, \quad \overline{\frac{\partial}{\partial x} F} = \epsilon \frac{\partial}{\partial X} \overline{F} + \mathcal{O}(\epsilon)^2. \quad (85)$$

Now, applying the averaging (82) to the conservation laws (18), (19) and (20) and passing to the limit as $\epsilon \rightarrow 0$, we arrive at the KdV modulation

system in a conservative form

$$\frac{\partial}{\partial T} \overline{P}_j(b_1, b_2, b_3) + \frac{\partial}{\partial X} \overline{Q}_j(b_1, b_2, b_3) = 0, \quad j = 1, 2, 3, \quad (86)$$

where $\overline{P}_j(\mathbf{b})$, $\overline{Q}_j(\mathbf{b})$ are expressed in terms of the complete elliptic integrals of the first and the second kind.

The equivalence of Whitham's method of averaging to the formal multiple-scale perturbation procedure is now rigorously established for a broad class of integrable equations including, of course, the KdV equation (see Dubrovin and Novikov (1989) and references therein). We also mention two nontrivial but easy to understand facts, which play important role in the modulation theory:

(i) all modulation systems obtained by averaging of *any three* independent conservation laws from the infinite set available for the KdV equation are equivalent to the basic system (86);

(ii) the wave conservation equation (78), which was obtained here using the multiple-scale expansions and which does not correspond to any particular averaged KdV conservation law, is consistent with the system of three averaged conservation laws (86), and thus, can be used instead of any of them (see Whitham 1965).

It was discovered in Whitham (1965) that, upon introducing symmetric combinations

$$r_1 = \frac{b_1 + b_2}{2}, \quad r_2 = \frac{b_1 + b_3}{2}, \quad r_3 = \frac{b_2 + b_3}{2}, \quad (87)$$

$r_3 \geq r_2 \geq r_1$, the system (86) assumes the diagonal (Riemann) form

$$\frac{\partial r_j}{\partial T} + V_j(r_1, r_2, r_3) \frac{\partial r_j}{\partial X} = 0, \quad j = 1, 2, 3, \quad (88)$$

where the characteristic velocities $V_3 \geq V_2 \geq V_1$ are expressed as certain combinations of the complete elliptic integrals of the first and the second kind. No summation over the repeated indices is assumed in (88). The variables r_j are called *Riemann invariants*. It is known very well (see Whitham 1974 for instance) that Riemann invariants can always be found for the systems consisting of two quasilinear equations, but for the systems of three or more equations they generally do not exist. The remarkable fact of the existence of the Riemann invariants for the KdV modulation system (86) is connected with the preservation of integrability under the averaging. We will discuss this issue in the next section.

Although the direct derivation of the system (88) from the conservative system (86) is a rather laborious task (see Kamchatnov 2000, for instance), one can easily obtain explicit expressions for the characteristic velocities by taking advantage of the established existence of the diagonal form (88) and the known dependence of the cnoidal wave parameters (7) - (9) on the Riemann invariants r_j via Eq. (87). For that, we notice that for the wave number conservation law (78) to be consistent with the diagonal system (88) the following relationships must hold (Gurevich, Krylov & El 1991, 1992; see also Kamchatnov 2000)

$$V_j = \frac{\partial(kc)}{\partial r_j} / \frac{\partial k}{\partial r_j}, \quad j = 1, 2, 3, \quad (89)$$

where k , m and c specified by Eqs. (7) - (9) are expressed in terms of the Riemann invariants (87) as

$$k = \frac{\pi(r_3 - r_1)^{1/2}}{K(m)}, \quad m = \frac{r_2 - r_1}{r_3 - r_1}, \quad c = 2(r_1 + r_2 + r_3). \quad (90)$$

Indeed, introducing in (78) the Riemann invariants explicitly and using (88) we obtain

$$\sum_{j=1}^3 \left\{ \frac{\partial \omega}{\partial r_j} - V_j \frac{\partial k}{\partial r_j} \right\} \frac{\partial r_j}{\partial X} = 0. \quad (91)$$

Since the derivatives $\partial_X r_j$ are independent and generally do not vanish, we readily arrive at the ‘potential’ representation (89), which can be viewed as a generalisation to a nonlinear case of the group velocity notion defined in linear wave theory as $c_g = \partial \omega / \partial k$. Substituting (90) into (89) we obtain explicit expressions for the characteristic velocities in terms of the complete elliptic integrals,

$$V_1 = 2(r_1 + r_2 + r_3) - 4(r_2 - r_1) \frac{K(m)}{K(m) - E(m)}, \quad (92)$$

$$V_2 = 2(r_1 + r_2 + r_3) - 4(r_2 - r_1) \frac{(1 - m)K(m)}{E(m) - (1 - m)K(m)}, \quad (93)$$

$$V_3 = 2(r_1 + r_2 + r_3) + 4(r_3 - r_1) \frac{(1 - m)K(m)}{E(m)}. \quad (94)$$

We now study the Whitham equations in two distinguished asymptotic limits: linear ($m \rightarrow 0$) and soliton ($m \rightarrow 1$). For that, we write down

the relevant asymptotic expansions of the complete elliptic integrals (see Abramowitz & Stegun 1965):

$$m \ll 1: K(m) = \frac{\pi}{2} \left(1 + \frac{m}{4} + \frac{9}{64} m^2 + \dots \right), \quad E(m) = \frac{\pi}{2} \left(1 - \frac{m}{4} - \frac{3}{64} m^2 + \dots \right); \quad (95)$$

$$(1-m) \ll 1: K(m) \approx \frac{1}{2} \ln \frac{16}{1-m}, \quad E(m) \approx 1 + \frac{1}{4} (1-m) \left(\ln \frac{16}{1-m} - 1 \right). \quad (96)$$

Using the expansions (95) we get in the harmonic limit $m \rightarrow 0$: $V_3 \rightarrow 6r_3$; $V_2 \rightarrow V_1 \rightarrow (12r_1 - 6r_3)$ so that the Whitham system reduces to

$$r_2 = r_1, \quad \frac{\partial r_3}{\partial T} + 6r_3 \frac{\partial r_3}{\partial X} = 0, \quad \frac{\partial r_1}{\partial T} + (12r_1 - r_3) \frac{\partial r_1}{\partial X} = 0. \quad (97)$$

In the soliton limit, $m \rightarrow 1$ and we have using (96): $V_2 \rightarrow V_3 \rightarrow (2r_1 + 4r_3)$; $V_1 \rightarrow 6r_1$ so that the Whitham system reduces to

$$r_2 = r_3, \quad \frac{\partial r_1}{\partial T} + 6r_1 \frac{\partial r_1}{\partial X} = 0, \quad \frac{\partial r_3}{\partial T} + (2r_1 + 4r_3) \frac{\partial r_3}{\partial X} = 0. \quad (98)$$

Thus, the Whitham system (88) admits two *exact* reductions to the systems of lower order via the limiting transitions $r_2 \rightarrow r_1$ (linear limit) and $r_2 \rightarrow r_3$ (soliton limit). In both limits one of the Whitham equations converts into the Hopf equation $r_T + 6rr_X = 0$, which coincides with the dispersionless limit of the KdV equation (3), while the remaining two merge into one for the Riemann invariant along a double characteristic. Such a special structure of the KdV-Whitham system (88) will enable us in Section 5 to formulate and solve some physically important boundary-value problem.

4.2 Integrability of the Whitham equations

Since the Whitham system (88) has been obtained by the averaging of the completely integrable KdV equation (3) one can expect that it possesses, apart from the existence of the Riemann invariants, some properties allowing for its exact integration. Indeed, as studies of Tsarev (1985), Krichever (1988) and Dubrovin and Novikov (1989) showed, the integrability is inherited under the Whitham averaging and the Whitham system for the KdV equation is integrable via the so-called *generalised hodograph transform*.

First, one can observe that the Riemann form (88) implies that any $r_j = \text{constant}$ is an exact solution of the Whitham equations. We now consider a reduction of the Whitham system when one of the Riemann invariants, say r_3 , is constant, $r_3 = r_{30}$. Then for the remaining two $r_{1,2}(X, T)$

one has a 2×2 system, which can be solved using the classical hodograph transform provided $r_{1X} \neq 0, r_{2X} \neq 0$ (see Whitham 1974 for instance). This is achieved through the change of variables $(r_1, r_2) \mapsto (X, T)$. The resulting (hodograph) system for $X(r_1, r_2)$ and $T(r_1, r_2)$ consists of two *linear* equations:

$$\partial_1 X - V_1(r_1, r_2, r_{30})\partial_1 T = 0, \quad \partial_2 X - V_2(r_1, r_2, r_{30})\partial_2 T = 0, \quad (99)$$

where $\partial_j \equiv \partial/\partial r_j$. Next, we introduce in (99) a substitution

$$W_1(r_1, r_2) = X - V_1 T, \quad W_2(r_1, r_2) = X - V_2 T \quad (100)$$

to cast it into the form of a symmetric system for $W_{1,2}$:

$$\frac{\partial_1 W_2}{W_1 - W_2} = \frac{\partial_1 V_2}{V_1 - V_2}; \quad \frac{\partial_2 W_1}{W_2 - W_1} = \frac{\partial_2 V_1}{V_2 - V_1}. \quad (101)$$

Now, any solution of the linear system (101) will generate, via (100), a *local* solution $\{r_1(X, T), r_2(X, T), r_{30}\}$ of the Whitham system. One can see that analogous systems can be obtained for any two pairs of the Riemann invariants provided the third invariant is constant. In 1985 Tsarev showed that even if *all three* Riemann invariants vary, any smooth non-constant solution of the Whitham equations (88) can be obtained from the algebraic system

$$X - V_j(r_1, r_2, r_3)T = W_j(r_1, r_2, r_3), \quad i = 1, 2, 3, \quad (102)$$

where the functions W_j are found from the over-determined system of linear PDEs,

$$\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}, \quad i, j = 1, 2, 3, \quad i \neq j. \quad (103)$$

Now, the condition of integrability of the nonlinear diagonal system (88) is reduced to the condition of consistency for the over-determined linear system (103), which has the form (see Tsarev (1985) or Dubrovin & Novikov (1989))

$$\partial_i \left(\frac{\partial_j V_k}{V_j - V_k} \right) = \partial_j \left(\frac{\partial_i V_k}{V_i - V_k} \right), \quad i \neq j, \quad i \neq k, \quad j \neq k. \quad (104)$$

It is not difficult to show that the characteristic velocities (92) – (94) satisfy relationships (104) and thus, the KdV-Whitham system (88) is integrable. The construction (102) – (104) is known as the generalised hodograph transform.

4.3 Whitham equations and spectral problem

There exists a deep connection between the Whitham equations (88) and the spectral problem associated with the original KdV equation. This connection has been discovered and thoroughly studied in the paper of Flaschka, Forest and McLaughlin (FFM)(1979). Let us consider the cnoidal wave solution (10), taken with negative sign, as a potential in the linear Schrödinger equation (26) in the associated spectral problem. It is well known that the spectrum of the periodic Schrödinger operator generally consists of an *infinite number* of disjoint intervals called *bands*. Correspondingly, the ‘forbidden’ zones between the bands are called *gaps*. The unique property of the cnoidal wave solution (10) is that its spectrum contains only *one* finite band. To be exact, the spectral set for the potential $-u_{cn}(x)$ is $S = \{\lambda : \lambda \in [\lambda_1, \lambda_2] \cup [\lambda_3, \infty)\}$. This fact had been known long before the creation of the soliton theory in connection with so-called *Lamé* potentials. The soliton studies showed that the cnoidal wave solutions of the KdV equation represent the simplest case of potentials belonging to a general class of so-called *finite-gap potentials* discovered by Novikov (1974) and Lax (1975). These finite-gap potentials can be expressed in terms of the Riemann theta-functions and give rise to *multiphase* almost periodic solutions of the KdV equation.

It is clear that the cnoidal wave solution can be parametrised by three spectral parameters $\lambda_1, \lambda_2, \lambda_3$ instead of the roots of the polynomial, b_1, b_2, b_3 (see Eq. (87)). The remarkable general fact established by FFM is that the Riemann invariants of the Whitham system (88) coincide with the endpoints of the spectral bands of finite-gap potential. In particular, for the single-gap solution (the cnoidal wave), $r_1 = \lambda_1, r_2 = \lambda_2, r_3 = \lambda_3$. Thus, the spectral problem provides one with the most convenient set of modulation parameters (the Riemann invariants) and, therefore, the Whitham equations (88) describe slow evolution of the *spectrum* of multi-phase KdV solutions. The general theory of finite-gap integration and the spectral theory of the Whitham equations are quite technical. However, in the case of the single-phase waves, which is the most important from the viewpoint of fluid dynamics applications, a simple universal method has been developed by Kamchatnov (2000) enabling one to construct periodic solutions and the Whitham equations directly in Riemann invariants for a broad class of integrable nonlinear dispersive wave equations.

5 Undular bores

5.1 Formation and structure of an undular bore

Let the initial data $u(x, 0) = u_0(x)$ for the KdV equation (3) have the form of a smooth step with a single inflection point at the origin:

$$u_0(-\infty) = u^-, \quad u_0(+\infty) = u^+, \quad u'_0(x) < 0, \quad u''_0(0) = 0. \quad (105)$$

Let $\epsilon = |u'_0(0)|/(u^- - u^+) \ll 1$, i.e. the characteristic width of the transition region is much larger than the characteristic wavelength, which is unity in the KdV equation (3). The qualitative picture of the KdV evolution of such a large-scale step transition is as follows. During the initial stage of the evolution, $|u_x| \sim \epsilon$, $|u_{xxx}| \sim \epsilon^3$, hence $|u_{xxx}| \ll |uu_x|$ and one can neglect the dispersive term in the KdV equation. The evolution at this stage is approximately described by the dispersionless (classical) limit of the KdV equation,

$$u \approx r(x, t) : \quad r_t + 6rr_x = 0, \quad r(x, 0) = u_0(x), \quad (106)$$

which is the Hopf equation for a simple wave (see Landau & Lifshitz 1987 or Whitham 1974). The evolution (106) leads to a gradient catastrophe, which occurs at the inflection point at a certain *breaking time* $t \rightarrow t_b$, $x \rightarrow x_b$: $r_x \rightarrow -\infty$, $r_{xx} \rightarrow 0$. In classical (dissipative) hydrodynamics the wave breaking leads to the formation of a *shock*, which can be asymptotically represented as a discontinuity where intense dissipation occurs and the flow parameters undergo a rapid change (see Whitham 1974). Instead, in dispersive hydrodynamics the resolution of the breaking singularity occurs through the generation of nonlinear waves. Indeed, for $t > t_b$ (without loss of generality we can put $t_b = 0$), one can no longer neglect the KdV dispersive term u_{xxx} in the vicinity of the breaking point and, as a result, the regularization of the singularity happens through the generation of nonlinear oscillations confined to a finite, albeit expanding, space region. This oscillatory structure represents a dispersive analog of a shock wave and is often called *an undular bore*. Originally, undular bore is a name for a natural phenomenon occurring on some rivers (River Severn in England and River Dordogne in France are among the best known) and representing a wave-like transition between two basic flows with different depth. Although the mathematical modelling of such shallow-water undular bores requires taking into account weak dissipation (Benjamin & Lighthill 1954; Johnson 1970; Whitham 1974), which stabilises the expansion of the oscillatory zone, it is

now customary to use the term ‘undular bore’ for any wave-like transition between two different smooth flows in solutions of nonlinear dispersive systems. The significance of the study of purely conservative, unsteady undular bores is twofold: they can be viewed as an initial stage of the development of the undular bores with small dissipation to a steady state and also, importantly, they represent a universal mechanism of the soliton generation out of non-oscillatory initial or boundary conditions in conservative systems. In fact, the study of purely dispersive undular bores has stimulated a number of important discoveries in modern nonlinear wave theory.

An undular bore solution to the KdV equation has the distinctive spatial structure, which has been first observed in numerical simulations. Near the leading edge of the undular bore the oscillations appear to be close to successive solitons while in the vicinity of the trailing edge they are nearly linear (see Fig. 1). Using knowledge of this qualitative structure, Gurevich and Pitaevskii (GP) (1974) made an assumption that the undular bore (it is called a ‘collisionless shock wave’ in the original GP paper) represents a *modulated single-phase solution* of the KdV equation. More precisely, the undular bore is constructed cnoidal wave solution (10) where the parameters b_1, b_2, b_3 (with an account of relations (87)) evolve according to the Whitham equations (88). We note that this whole asymptotic construction has been recovered in a later rigorous theory of the singular zero-dispersion limit of the KdV equation by Lax, Levermore and Venakides (see Section 4) but in the GP approach it is a plausible assumption.

Figure 1: Qualitative structure of the undular bore evolving from the initial step (dashed line).

5.2 Gurevich-Pitaevskii problem: general formulation

In the GP approach, the accent is shifted from finding the exact solution of the KdV equation (3) with the initial conditions (105) to finding the corresponding exact solution of the associated Whitham system (88). Hence, the first task is to translate the initial data $u_0(x)$ of the KdV equation into some initial or boundary conditions for the Whitham system. The Whitham equations (88), unlike the KdV equation itself, are of hydrodynamic type, i.e. they don't contain higher order derivatives. This, similarly to classical ideal hydrodynamics, results in nonexistence of the global solution for the general initial-value problem (see Landau & Lifshitz 1987 or Whitham 1974 for instance). Indeed, due to non-linearity of the Whitham equations, the modulation parameters r_1, r_2, r_3 would develop infinite derivatives on their profiles in finite time, which would make the whole Whitham system invalid as it is based on the assumption of the *slowly* modulated cnoidal wave. Therefore, the Whitham equations should be supplied with the boundary conditions ensuring the existence of the global solution. It is physically natural to require the continuous matching of the mean flow $\bar{u}(x, t)$ in the undular bore with the smooth flow $u(x, t)$ outside the undular bore at some free boundaries. Also, it follows from the (assumed) structure of the undular bore that the matching of the undular bore with the non-oscillating external flow must occur at the points of the linear ($m \rightarrow 0$) degeneration of the undular bore at the trailing edge and soliton ($m \rightarrow 1$) degeneration at the leading edge. The mean value given by formula (83) can be readily expressed in terms of the Riemann invariants r_j using the relationships (87). Then using asymptotic expansions (95), (96) of the complete elliptic integrals in the linear and soliton limits we obtain

$$\bar{u}|_{m=0} = r_3, \quad \bar{u}|_{m=1} = r_1. \quad (107)$$

Now the problem of the continuous matching of the mean flow can be formulated in the following, mathematically accurate, way. From here on we will not introduce slow variables $X = \epsilon x, T = \epsilon t$ in the modulation equations (88) explicitly, instead, we assume that the small parameter ϵ naturally arises *in the solution* of the KdV equation as the ratio of the characteristic wavelength to the width of the oscillations zone. Let the upper (x, t) half-plane be split into three domains : $\{x \in \mathbb{R}, t > 0\} = \{(-\infty, x^-(t)) \cup [x^-(t), x^+(t)] \cup (x^+(t), +\infty)\}$ (see Fig. 2), in which the solution is governed by different equations: outside the interval $[x^-(t), x^+(t)]$ it is governed by the Hopf equation (106) while within the interval $[x^-(t), x^+(t)]$ the dynamics

Figure 2: Splitting of the (x, t) - plane in the Gurevich – Pitaevskii problem.

is described by the Whitham equations (88) so that the following matching conditions are satisfied:

$$\begin{aligned} x = x^-(t) : \quad & r_2 = r_1, \quad r_3 = r, \\ x = x^+(t) : \quad & r_2 = r_3, \quad r_1 = r, \end{aligned} \quad (108)$$

where $r(x, t)$ is the solution of the Hopf equation (106) and the boundaries $x^\pm(t)$ are unknown at the onset. One can see that conditions (108) are consistent with the limiting structure of the Whitham equations given by Eqs. (97), (98) and thus, the lines $x^\pm(t)$ represent free boundaries. It follows from Eq. (108) and the limiting properties of the Whitham velocities described in Section 4.1. that these boundaries are defined by the multiple characteristics of the Whitham system for $m = 0$ ($x = x^-(t)$) and $m = 1$ ($x = x^+(t)$) and are found from the ordinary differential equations

$$dx^-/dt = (12r_1 - 6r_3)|_{x=x^-}, \quad dx^+/dt = (2r_1 + 4r_3)|_{x=x^+} \quad (109)$$

defined on the solution $\{r_j(x, t)\}$ of the GP problem.

5.3 Decay of an initial discontinuity

As an important example, where a simple representation of the undular bore can be obtained, we consider the decay of an initial discontinuity problem.

We take the initial data in the form of a sharp step,

$$t = 0 : \quad u = A > 0 \quad \text{if} \quad x < 0, \quad u = 0 \quad \text{for} \quad x > 0, \quad (110)$$

which implies immediate formation of an undular bore. We now assume the modulation description of the undular bore and make use of the GP problem formulation. First we observe that, since both initial data (110) and the modulation equations (88) are invariant with respect to the linear transformation $x \rightarrow Cx$, $t \rightarrow Ct$, $C = \text{constant}$, the modulation variables must be functions of a self-similar variable $s = x/t$ alone, $r_j = r_j(s)$. Thus, the Whitham system (88) reduces to the system of ODEs:

$$\frac{dr_j}{ds}(V_j - s) = 0, \quad j = 1, 2, 3. \quad (111)$$

The GP matching conditions (108) then assume the form

$$\begin{aligned} s = s^- : \quad & r_2 = r_1, \quad r_3 = A, \\ s = s^+ : \quad & r_2 = r_3, \quad r_1 = 0, \end{aligned} \quad (112)$$

where $s^\pm$ are the (unknown) speeds of the undular bore edges, $x^\pm = s^\pm t$. The boundary-value problem (111), (112) has the solution in the form of a centred simple wave in which all but one Riemann invariants are constant:

$$r_1 = 0, \quad r_3 = A, \quad V_2(0, r_2, A) = s, \quad (113)$$

or, explicitly, using the expression (93) for $V_2(r_1, r_2, r_3)$,

$$2A \left\{ 1 + m - \frac{2(1-m)m}{E(m)/K(m) - (1-m)} \right\} = \frac{x}{t}, \quad (114)$$

where $m = (r_2 - r_1)/(r_3 - r_1) = r_2/A$. The obtained solution for the Riemann invariants is schematically shown in Fig. 3. It provides the required modulation of the cnoidal wave (10) in the undular bore transition. Since the solution (114) represents a characteristic fan it never breaks for $t > 0$ and therefore is global. The speeds $s^\pm$ of the trailing and leading edges of the undular bore are found by putting $m = 0$ and $m = 1$ respectively in the solution (114):

$$s^- = s(0) = -6A, \quad s^+ = s(1) = 4A. \quad (115)$$

Thus, the undular bore is confined to an expanding zone $-6At \leq x \leq 4At$. The amplitude of the lead soliton is simply $a^+ = 2(r_3 - r_1) = 2A$. This

Figure 3: Riemann invariant behaviour in the self-similar undular bore

agrees with the semi-classical IST result (70), which gives the amplitude of the greatest soliton evolving out of the large-scale initial perturbation with the amplitude $u_{0\max} = A$. Indeed, the solution (114) of the decay of a step problem can be viewed as an intermediate asymptotics for $1 \ll t \ll l$ in the problem of the evolution of a spatially extended rectangular profile of the width $l \gg 1$: $u_0(x) = A$ if $x \in [-l, 0]$ and $u_0(x) = 0$ otherwise. It can also be readily inferred from (114) that the phase velocity $c = 2(r_1 + r_2 + r_3) = 2A(1 + m) > V_2(0, r_2, A)$ if $m < 1$ and $c = V_2(0, r_2, A)$ for $m = 1$. Thus, any individual crest within the wavetrain moves towards the leading edge of the undular bore, i.e. for any crest $m \rightarrow 1$ as $t \rightarrow \infty$. In this sense, the undular bore evolves into a soliton train.

Using asymptotic expansions (95), (96) for the complete elliptic integrals we obtain the asymptotic behaviour of the modulus m near the undular bore boundaries. Near the trailing edge we have

$$m \simeq (s - s^-)/9 \ll 1, \quad (116)$$

which also describes the amplitude variations since $m = a/A$ for the solution under study. Near the leading edge we get with logarithmic accuracy:

$$1 - m \simeq (s^+ - s)/2 \ln(1/(s^+ - s)) \ll 1, \quad (117)$$

which, in particular, yields the asymptotic behaviour

$$\bar{u} \simeq 12k, \quad k \simeq 2\pi / \ln(1/(s^+ - s)) \quad (118)$$

for the mean value and the wavenumber respectively.

It should be stressed that, although formula (114) represents an exact solution of the Whitham equations, the full undular bore solution (i.e. the travelling wave (10) modulated by (114)) is an *asymptotic solution* of the KdV equation as the Whitham method itself is based on the perturbation theory. As a result, the location of the undular bore is determined up to typical wavelength (indeed, the initial phase x_0 in (10) is lost after the Whitham averaging) and thus, the accuracy ϵ of the obtained undular bore description can be estimated as the ratio of the typical soliton width, which is $\mathcal{O}(1)$, to the width of the oscillations zone, $l \cong 10At$, i.e. $\epsilon \sim t^{-1}$. The obtained undular bore description is thus asymptotically accurate as $t \rightarrow \infty$.

If the constant A in the initial conditions (105) is negative, $A < 0$, the initial step does not break and the undular bore is not generated. Instead, the asymptotic solution of the KdV equation in this case is a rarefaction wave,

$$u = 0 \quad \text{if } x > 0, \quad u = x/6t \quad \text{if } 6At < x < 0, \quad u = 0 \quad \text{if } x < 6At. \quad (119)$$

The solution (119) contains weak discontinuities at $x = 0$ and $x = 6At < 0$. These discontinuities are resolved in the full solution of the KdV equation with the small-amplitude linear wavetrains which smooth out with time (see Gurevich & Pitaevskii (1974) for further details).

Fornberg and Whitham (1978) compared the modulation solution of Gurevich and Pitaevskii with the full numerical solution of the KdV equation with the step initial conditions and found a very good agreement between the two. More recently, Apel (2003) utilised the GP solution for the modelling the internal waves undular bores in the Strait of Gibraltar in the Mediterranean Sea.

5.4 General solution of the Gurevich-Pitaevskii problem

In the case when the initial data is not the step function, the similarity solution (114) is not applicable and to construct the appropriate solution of the GP problem one needs a more general solution of the Whitham equations, in which two or all three Riemann invariants vary. Such solutions can be constructed using the generalised hodograph transform described in Section 4.2. However, although the resulting hodograph equations (103) are linear, they still are too complicated to be treated directly. It was shown by Gurevich, Krylov and El (1991, 1992) that the scalar substitution (cf. (89))

$$W_i = \frac{\partial_i(kf)}{\partial_i k}, \quad i = 1, 2, 3, \quad (120)$$

where $k(r_1, r_2, r_3)$ is given by (90) and $f(r_1, r_2, r_3)$ is unknown function, – is compatible with the system (103) and reduces it to the system of classical Euler-Poisson-Darboux equations

$$2(r_i - r_j)\partial_{ij}^2 f = \partial_i f - \partial_j f, \quad i, j = 1, 2, 3, \quad i \neq j. \quad (121)$$

It is not difficult to show that the over-determined system (121) is consistent and has general solution (Eisenhart 1919)

$$f = \sum_{i=1}^3 \int \frac{\phi_i(\tau) d\tau}{\sqrt{(r_3 - \tau)(r_2 - \tau)(\tau - r_1)}}, \quad (122)$$

where $\phi_i(\tau)$ are arbitrary (generally complex) functions. Next, by applying the GP matching conditions (108) to Eqs. (102), (120), (122) the unknown functions ϕ_i can be expressed in terms of the linear Abel transforms of the monotone parts of the KdV initial profile $u_0(x)$. Then, for monotonically decreasing initial data $u(x, 0) = u_0(x)$, $u'_0(x) < 0$ with a single breaking point at $u = 0$ the resulting solution for the function $f(\mathbf{r})$ assumes the form (Gurevich, Krylov & El 1992)

$$f = \frac{1}{\pi(r_3 - r_2)^{1/2}} \int_{r_2}^{r_3} \frac{W(\tau)}{\sqrt{\tau - r_1}} K(z) d\tau + \frac{1}{\pi(r_2 - r_1)^{1/2}} \int_{r_1}^{r_2} \frac{W(\tau)}{\sqrt{r_3 - \tau}} K(z^{-1}) d\tau, \quad (123)$$

where

$$z = \left[\frac{(r_2 - r_1)(r_3 - \tau)}{(r_3 - r_2)(\tau - r_1)} \right]^{1/2} \quad (124)$$

and $W(u) = u_0^{-1}(x)$ is the inverse function of the initial profile. The sought dependence $r_j(x, t)$ in the undular bore is now found by the substitution of the solution (123) into formulae (120), (102) and resolving them for $r_j(x, t)$. It was shown in Krylov, Khodorovskii and El (1992) that, for decaying initial functions $u_0(x)$ the long-time asymptotics of the solution of the GP problem agrees with the semi-classical Karpman formula (69) thus providing another connection of the Whitham theory with the IST method.

6 Propagation of KdV soliton through a variable environment

In many physical situations the properties of the medium vary in space. The weakly nonlinear wave propagation in such media is described by the KdV equation (1) with the variable coefficients $\alpha(t)$, $\beta(t)$ (see Johnson (1997) or

Grimshaw (2001) for instance). One should note that in the modelling of the variable environment effects, the variables x, t in the KdV equation (1) are not necessarily the same physical space and time co-ordinates as in the traditional interpretation of the KdV equation (3). However, for convenience we shall retain the same x, t - notations here. Assuming $\alpha \neq 0$ we introduce the new variables

$$t' = \int_0^t \alpha(\hat{t}) d\hat{t}, \quad \lambda(t') = \frac{\beta}{\alpha}, \quad (125)$$

so that, on omitting the superscript for t , equation (1) becomes

$$u_t + 6uu_x + \lambda(t)u_{xxx} = 0. \quad (126)$$

The physical problems modelled by (126) include shallow-water waves moving over an uneven bottom, internal gravity waves in lakes of varying cross-section, long waves in rotating fluids contained in cylindrical tubes and many others (see, for instance, Johnson (1997) or Grimshaw (2001) and references therein).

We shall assume the following physically reasonable behaviour for the variable coefficient $\lambda(t)$: let $\lambda(t)$ be constant, say 1 for $t < 0$ then changes smoothly until some $t = t_1$ and then again is a (different) constant λ_1 . Suppose that a soliton solution of the constant-coefficient KdV equation (3) (which incidentally is a solution of Eq. (126) for $t < 0$) is moving through the medium so that it reaches the point $x = 0$ at $t = 0$. It is clear that for $t > 0$ it is no longer an exact solution of Eq. (126) and some wave modification must occur. Generally, equation (126) is not integrable by the IST method so the problem should be solved numerically. There are, however, two distinct limiting cases which can be treated analytically.

If the medium properties change rapidly, i.e. $0 < t_1 \ll 1$ then the known soliton waveform (13) can be used as an initial condition for the KdV equation (126) with $\lambda = \lambda_1 = \text{constant}$ (see Johnson 1997). This problem can be solved by the IST method and the outcome is that for $\lambda_1 > 0$ the initial soliton *fissions* into N solitons and some radiation (see Section 3.7). If $\lambda_1 < 0$ then the initial soliton (13) completely transforms into the linear radiation (see Section 3.6).

An opposite situation occurs when the medium properties vary slowly so that $\sigma \sim |\lambda_t| \ll 1$, i.e. $t_1 \sim \sigma^{-1} \gg 1$. In this case one can use an adiabatic approximation to describe the solitary wave variations to leading order. Thus, we assume that λ is slowly varying so that

$$\lambda = \lambda(T), \quad T = \sigma t, \quad \sigma \ll 1. \quad (127)$$

Then, the slowly-varying solitary wave asymptotic expansion is given by

$$u = u_0 + \sigma u_1 + \dots, \quad (128)$$

where the leading term is given by

$$u_0 = a \operatorname{sech}^2\left\{\gamma\left(x - \frac{\Phi(T)}{\sigma}\right)\right\}, \quad (129)$$

so that

$$d\Phi/dT = c = 2a = 4\lambda\gamma^2. \quad (130)$$

The variations of the amplitude a , the inverse half-width parameter γ and the speed c with the slow time variable T are determined by noticing that the variable-coefficient KdV equation (126) possesses the momentum conservation law

$$\int_{-\infty}^{\infty} u^2 dx = \text{constant}. \quad (131)$$

Substitution of (129) into (131) readily shows that

$$\frac{\gamma}{\gamma_0} = \left(\frac{\lambda_0}{\lambda}\right)^{2/3}, \quad (132)$$

where the subscript ‘0’ indicates quantities evaluated at $T = 0$ say, i.e. $\lambda_0 = 1$.

It follows from (129), (130) and (132) that the slowly-varying solitary wave, u_0 is now completely determined. However, the variable-coefficient KdV equation (126) also has a conservation law for the ‘mass’

$$\int_{-\infty}^{\infty} u dx = \text{constant}, \quad (133)$$

which is not satisfied by the leading order adiabatic expression (129). The situation can be remedied by taking into account the next term in the asymptotic expansion (128) and allowing $\int_{-\infty}^{\infty} u_1(x) dx$ be $\mathcal{O}(\sigma^{-1})$.

More precisely, conservation of mass is assured by the generation of a trailing shelf u_s , such that $u = u_0 + u_s$ where u_s typically has an amplitude $\mathcal{O}(\sigma)$ and is supported on the interval $0 < x < \Phi(T)/\sigma$ (see Newell 1981, Ch.3 and the references therein; or Grimshaw & Mitsudera 1993). Thus, the

shelf stretches over a zone of $\mathcal{O}(\sigma^{-1})$, and hence carries $\mathcal{O}(1)$ mass. The law (133) for conservation of mass then shows that

$$\int_{-\infty}^{\Phi/\sigma} u_s dx + \int_{-\infty}^{\infty} u_0 dx = \text{constant}. \quad (134)$$

The second term on the left-hand side of (134) is readily found to be

$$\frac{2a}{\gamma} = 4\lambda\gamma = 4\gamma_0\lambda^{1/3}, \quad (135)$$

on using (129), (130) and (132) in turn. Next, we assume that $u_s = \sigma q(X, T)$, where

$$X = \sigma x, \quad T = \sigma t. \quad (136)$$

Since the spatial overlap between u_0 and u_s is small compared with the shelf width, one can assume $u \approx u_s$ for $0 < x < \Phi(T)/\sigma$ and, therefore, $u_s(x, t)$ must satisfy (126). Then we have

$$q_T + 6\sigma q q_X + \sigma^2 \lambda(T) q_{XX} = 0. \quad (137)$$

For sake of definiteness we shall assume that $\lambda > 0$. Next, on differentiating (134) with respect to T , we find that to leading order in σ ,

$$q = -\frac{2}{\gamma_0} \lambda_T \lambda^{-1/3} \equiv Q(T) \quad \text{at} \quad X = \Phi(T). \quad (138)$$

Now, (137), (138) together with (130) and (132) present a completely formulated boundary-value problem.

We assume that $\lambda(T)$ is a smooth function and, in particular, has a continuous second derivative so that $Q(T)$ has a continuous first derivative. Then at least in some finite neighbourhood of the curve $X = \Phi(T)$ one can neglect the dispersive term in the KdV equation (137) and approximately describe the evolution by the Hopf equation

$$q_T + \sigma q q_X = 0 \quad (139)$$

with the same boundary condition (138). The solution to the boundary-value problem (139), (138) is readily found using the characteristics:

$$q = Q(T_0), \quad X - \Phi(T_0) = \sigma Q(T_0)(T - T_0), \quad (140)$$

where T_0 is a parameter along the initial curve $\Phi(T)$. The expression (140) remains valid until neighbouring characteristics intersect and a breaking of

the $q(x)$ -profile begins. This occurs when

$$\Phi'(T_0) + \sigma Q'(T_0)(T - T_0) - \sigma Q(T_0) = 0. \quad (141)$$

Since $\Phi' = c > 0$ and $\sigma \ll 1$, it follows that the singularity forms in finite time only if $Q'(T_0) < 0$ at least for some values of T_0 . Let T_b be the minimum value, as T_0 varies, such that (141) is satisfied. Then the breaking

Figure 4: Formation and evolution of a soliton trailing shelf undular bore
a) KdV soliton at $t = 0$; b) formation of a constant-amplitude $a \sim \sigma$ elevation shelf at $t \sim \sigma^{-1}$; c) generation of the undular bore in the shelf at $t \sim \sigma^{-2}$

singularity forms first at T_b and the corresponding value X_b is determined from Eq. (140). It can be easily seen from Eq. (141) that $T_b = \mathcal{O}(\sigma^{-1})$ provided $Q'(T_0) = \mathcal{O}(1)$.

It is clear that in the vicinity of the breaking point (X_b, T_b) , the Hopf equation (139) no longer describes the evolution of q adequately and the full KdV equation (137) should be considered. As a result, the singularity is regularized by an unsteady undular bore (see Fig. 4). One can see that the undular bore forms at times $T \sim T_b \sim \sigma^{-1}$, which are much greater than the time $T_1 = \sigma t_1 = \mathcal{O}(1)$ for the change of the variable coefficient $\lambda(T)$ from 1 to λ_1 . Thus, the trailing shelf undular bore essentially forms and evolves in the region where $\lambda(T) = \lambda_1 > 0$ is *constant* and, as a result, the problem of the long-time trailing shelf evolution reduces to solving the familiar constant-coefficient KdV equation

$$q_T + 6\sigma q q_X + \sigma^2 \lambda_1 q_{XXX} = 0. \quad (142)$$

The initial data for Eq. (142), $q_0(X) = q(X, 0)$ is obtained from the boundary condition (138), specified at the initial curve $\Phi(T)$, by its projection, along the characteristics of the Hopf equation (139), onto X -axis. The sign of $q_0(X)$ depends on the sign of λ_T in (138) (i.e. on whether λ_1 is greater or less than unity). If $\lambda_1 < 1$, then $q_0(X) > 0$. As the variations of the function $Q(T)$ in (138) are determined by the variations of the function $\lambda(T)$, the characteristic spatial scale l of this equivalent initial condition $q_0(X)$ is $\mathcal{O}(1)$. On the other hand, the typical wavelength of the travelling wave solutions of Eq. (142) is $\Delta X \approx \sigma^{1/2} \ll l$. Simple rescaling to the standard form (3) shows that the condition (57) of applicability of the semi-classical asymptotics in the IST method is satisfied. Therefore, if $\lambda_1 < 1$ the trailing shelf eventually decomposes into a large number of small-amplitude solitons and one can use Karpman's formula (69) to obtain the amplitude distribution in the soliton train. Alternatively, if $\lambda_1 > 1$, then $q_0(X) < 0$ and the trailing shelf converts as $t \rightarrow \infty$ into a linear wavepacket, again, via an intermediate stage of an undular bore.

The undular bore stage of the trailing shelf evolution has been studied in detail in El & Grimshaw (2002) where it has been argued that, for the variable environment with $\lambda_T < 0$, the solitons generated in the trailing shelf undular bore can be identified with the secondary solitons in the fissioning scenario. Thus, both extreme cases of the soliton propagation through a variable environment are now reconciled by showing that the trailing shelf contains the seeds for the generation of secondary solitons, albeit on a long-time scale.

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