Internal solitary waves in a variable medium

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Received February 1 2006

Key words solitary waves, Korteweg-de Vries, internal waves.

In both the ocean and the atmosphere, the interaction of a density stratified flow with topography can generate large-amplitude, horizontally propagating internal solitary waves. Often these waves are observed in regions where the waveguide properties vary in the direction of propagation. In this article we consider nonlinear evolution equations of the Korteweg-de Vries type, with variable coefficients, and use these models to review the properties of slowly-varying periodic and solitary waves.

1 Introduction

Solitary waves are nonlinear waves of quasi-permanent form, first observed by Russell (1844) in a now famous report on his observations of a free surface solitary wave in a canal, and his subsequent experiments. Theoretical work by Boussinesq (1871) and Rayleigh (1876) later established a theoretical model, and then Korteweg and de Vries (1895) derived the well-known equation which now bears their names. But it was not until the second half of the twentieth century that it was realised that the Korteweg-de Vries equation was a valid model for solitary waves in a wide variety of physical contexts. Of principal concern here are the large-amplitude internal solitary waves which propagate in density-stratified fluids such as the ocean and atmosphere (see, e.g., Apel (1995), Grimshaw (2001), Holloway et al (2001) and Rottmann and Grimshaw (2001)). They owe their existence to a balance between nonlinear wave-steepening effects and linear wave dispersion, and hence can be effectively modeled by nonlinear evolution equations of the Korteweg-de Vries (KdV) type.

Many studies based on KdV-type models have used equations with constant coefficents. However, particularly in the oceanic case, the waves are propagating on a background whose properties vary in the wave propagation direction. In this situation, an appropriate model equation is the variable-coefficient Korteweg-de Vries (vKdV) equation

$$\eta_\tau + c\eta_\chi + \frac{cQ_\chi}{2Q}\eta + \mu\eta\eta_\chi + \lambda\eta_{\chi\chi\chi} = 0\,, \tag{1}$$

Here $\eta(\chi,\tau)$ is the amplitude of the wave, and χ,τ are space and time variables respectively. The coefficient c is the relevant linear long wave speed, and Q is the linear magnification factor, defined so that $Q\eta^2$ is the wave action flux; the coefficients μ and λ of the nonlinear and

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dispersive terms are determined by the waveguide properties of the specific physical system being considered. In a variable medium, each of these is a function of χ. The vKdV equation was derived by Johnson(1973) for water waves and by Grimshaw (1981) for internal waves (for a recent review, see Grimshaw 2001). The derivation assumes the usual KdV balance that the amplitude η has the same order as the dispersion, measured by $\partial^2/\partial\chi^2$, and in addition assumes that the waveguide properties (i.e. the coefficients c, Q, μ, λ) vary slowly so that Q_χ/Q for instance is of the same order as the dispersion. In this scenario, the first two terms in (1) are the dominant terms, and it is useful to make the transformation

$$A = \sqrt{Q}\,\eta\,, \quad t = \int^\chi \frac{d\chi}{c}\,, \quad x = t - \tau\,. \tag{2}$$

Substitution into (1) yields, to the same order of approximation as in the derivation of (1),

$$A_t + \alpha A A_x + \delta A_{xxx} = 0 \tag{3}$$

$$\alpha = \frac{\mu}{c\sqrt{Q}}\,, \quad \delta = \frac{\lambda}{c^3}\,. \tag{4}$$

Here the coefficients α, β are functions of t alone. Note that although t is a variable along the spatial path of the wave, we shall subsequently refer to it as the “time”. Similarly, although x is a temporal variable, in a reference frame moving with speed c, we shall subsequently refer to it as a ”space” variable.

In this paper, we shall review the theory of slowly-varying periodic and solitary waves based on the variable-coefficient Korteweg-de Vries equation (3) in Section 2. Because internal solitary waves are often of large amplitudes, it is sometimes useful to include a cubic nonlinear term in (1) and (3), which then becomes (see the review by Grimshaw 2001),

$$A_t + \alpha A A_x + \beta A^2 A_x + \delta A_{xxx} = 0\,. \tag{5}$$

In Section 3, we describe the slowly-varying solitary wave solutions of this extended vKdV equation (5), and in particular examine the behaviour at certain critical points where either α or β vanish.

2 Slowly-varying waves in the Korteweg-de Vries equation

2.1 Periodic waves

We now suppose that the coefficients α, β are slowly varying, and write

$$\alpha = \alpha(T)\,, \quad \delta = \delta(T)\,, \quad T = \epsilon t\,, \quad \epsilon << 1\,. \tag{6}$$

next we seek a standard multi-scale expansion for a modulated periodic wave, namely

$$A = A_0(\theta, T) + \epsilon A_1(\theta, T) + \cdots\,, \tag{7}$$

$$\theta = k(x - \frac{1}{\epsilon}\int^T V(T)\,dT)\,. \tag{8}$$

Here it is assumed that A is periodic in θ with a fixed period of 2π. Substitution into (3) yields at the leading orders

$$-VA_{0\theta} + \alpha A_0 A_{0\theta} + \delta k^2 A_{0\theta\theta\theta} = 0\,, \tag{9}$$

$$-VA_{1\theta} + \alpha(A_0 A_1)_\theta + \delta k^2 A_{1\theta\theta\theta} = -\frac{1}{k} A_{0T}\,. \tag{10}$$

Each of these is essentially an ordinary differential equation with θ as the independent variable, and with T as a parameter.

The solution of (9) is the well-known cnoidal wave

$$A_0 = a\{b(m) + \text{cn}^2(\gamma\theta; m)\} + d\,, \tag{11}$$

$$\text{where} \quad b = \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,, \quad \alpha a = 12 m \delta \gamma^2 k^2\,, \tag{12}$$

$$\text{and} \quad V = \alpha d + \frac{\alpha a}{3}\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\}. \tag{13}$$

Here $cn(x;m)$ is the Jacobian elliptic function of modulus $m, 0 < m < 1$, $K(m), E(m)$ are the elliptic integrals of the first and second, The amplitude is a, the mean value of A over one period is d, while the spatial period is $2K(m)/\gamma k$. But since A_0 is 2π-periodic in θ we see that $\gamma = K(m)/\pi$. As the modulus $m \to 1$, this becomes a solitary wave, since then $b \to 0$ and $\text{cn}^2(x) \to \text{sech}^2(x)$; in this limit $\gamma \to \infty, k \to 0$ with $\gamma k = K$ held fixed. As $m \to 0, \gamma \to 1/2$, and it reduces to sinusoidal waves of small amplitude $a \sim m$ and wavenumber k. The cnoidal wave (11) contains three free parameters, which here depend on the slow variable T; we take these to be the amplitude a, the mean level d and the modulus m, so that equations (12, 13) then determine k, V respectively.

The task now is to determine how A_0 depends on T. There are two principal methods used to achieve this. One is the so-called Whitham averaging method, where one seeks three appropriate conservation laws for the vKdV equation (3), inserts the cnoidal wave into these laws, and then averages over the phase θ (see Whitham 1974). It is important tha, in addition, one should also use the law for conservation of waves, namely

$$k_T + \omega_X = 0\,, \tag{14}$$

where here $X = \epsilon x$ and $\omega = kV$. But in the present case, there is no X-dependence, and so this readily yields the result that k is a constant. For the vKdV equation, it is convenient to take the remaining two conservations laws as those for "mass" and "momentum",

$$\frac{\partial}{\partial t}\int_0^{2\pi} A d\theta = 0\,, \tag{15}$$

$$\frac{\partial}{\partial t}\int_0^{2\pi} A^2 d\theta = 0\,. \tag{16}$$

Each of these is readily established from (3). Note that although we shall call these the laws for conservation of mass and momentum, the integrands do not necessarily correspond to the corresponding physical entities. Indeed, (16) is in fact the law for conservation of wave action flux. Substitution of (11) into (15) readily shows that d is a constant. Hence the remaining variable, namely the wave amplitude a, can now be found by substituting (11) into (16). The result is

$$a^2\{\frac{1}{2\pi}\int_0^{2\pi} \text{cn}^4(\gamma\theta;m)\,d\theta \;-\; b(m)^2\} \;=\; \text{constant}\,. \tag{17}$$

$$\text{or} \quad \frac{a^2}{m^2}\{(2-3m)(1-m)+\frac{(4m-2)E(m)}{K(m)}-3m^2b(m)^2\}=\text{constant}\,. \tag{18}$$

Substitution of the expressions (12) into (18) finally yields the expression for the variation of the modulus m,

$$F(m) \equiv K(m)^2\{(4-2m)E(m)K(m)-3E(m)^2-(1-m)K(m)^2\} = \text{constant}\,\frac{\alpha^2}{\delta^2}\,. \tag{19}$$

This expression was obtained by Ostrovsky and Pelinovsky (1970, 1975) and Miles (1979) for the special case of surface water waves, and is plotted in Figure 1, which shows that $F(m)$ is a monotonically increasing function of m. It follows that as α/δ increases so does m. Two limiting situations are of interest. First, if the nonlinear coefficient α decreases towards zero, then so does the modulus m where it can be shown that $F(m) \sim m^2$ as $m \to 0$; it follows that the modulus $m \sim \alpha$, but remarkably the amplitude a is finite in this limit. On the other hand, if the dispersive coefficient $\delta \to 0$, which is often the case when internal waves move in to shallow water, then $m \to 1$ and the waves become more like solitary waves.

An alternative to the Whitham averaging procedure, is to continue the asymptotic expansion to the next order, and invoke the condition that A_1 is a periodic function of θ. Indeed, it is implicit in the Whitham averaging procedure that the higher-order terms in the expansion have this property. Although it can be shown that the presence of a suitable underlying Lagrangian usually ensures that this is so (see, for instance, Whitham 1974), we shall nevertheless verify it directly here for the first-order term. This is given by (10) in which the right-hand side is now a known periodic function of θ, given by (11). A necessary and sufficient condition for A_1 to be periodic in θ is that the right-hand side of (10) should be orthogonal to the periodic solutions of the adjoint to the homogeneous operator on the left-hand side. This adjoint is

$$-VA_{1\theta} + \alpha A_0 A_{1\theta} + \delta k^2 A_{1\theta\theta\theta} \;=\; 0\,.\,. \tag{20}$$

It is readily seen that two solutions of (20) are $1, A_0$, both of which are periodic. A third solution can be found by the variation-of-parameters method, but is not periodic. Hence there are two orthogonality conditons, the first showing that d is a constant, while the second condition recovers the momentum conservation law (16). These are then supplemented by the equation for conservation of waves, which as before yields that k is a constant.

2.2 Solitary waves

The results obtained above for a slowly-varying periodic wave cannot be extrapolated to the case of a slowly-varying solitary wave, as the limits $m \to 1$ and $\epsilon \to 0$ do not commute. In physical terms, the basis for the validity of the slowly-varying periodic wave is that the local period (i.e. $1/kV$) should be much less than the scale of the variable medium (i.e. $1/\epsilon$). The

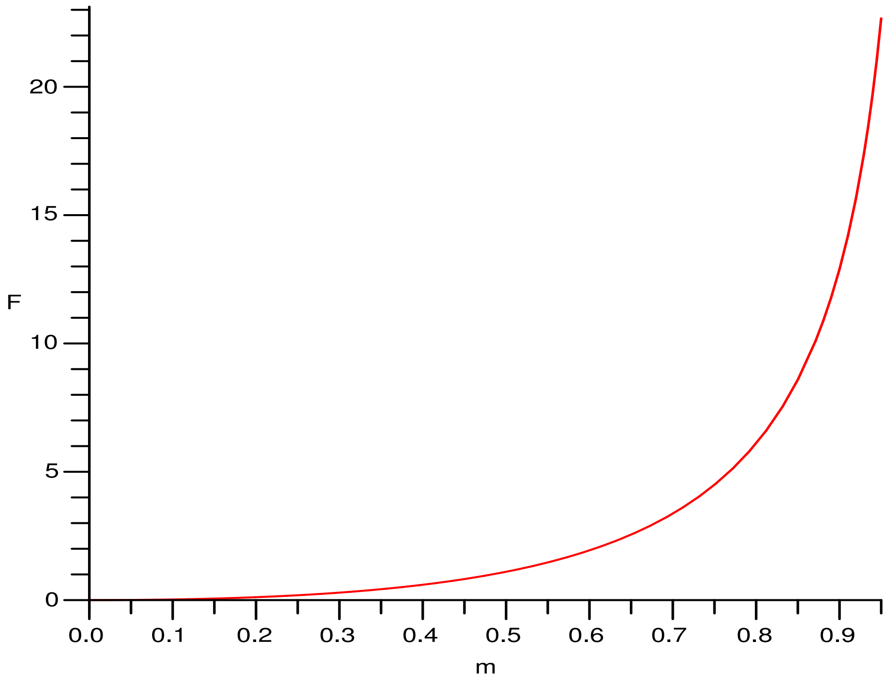

Fig. 1 A plot of $F(m)$ (19) versus m.

limit $m \to 1$ in (11, 12, 13) requires that $\gamma \to \infty, k \to 0$ with $\gamma k = K$ held constant, and so the period technically becomes much larger than the scale of the medium. A new concept of slowly-varying is needed, which in physical terms is that the half-width (i.e. the width of the wave at the level of one half of the maximum amplitude) should be much less than $1/\epsilon$. Technically we proceed as above and invoke a multi-scale asymptotic expansion of the form (6. 7). but now replace (8) with

$$\phi = x - \frac{1}{\epsilon}\int^{T} V(T)\,dT\,. \tag{21}$$

A is not now required to be periodic in ϕ. Instead we consider the domain $-\infty < \phi < \infty$, and require that A remain bounded in the limits $\phi \to \pm\infty$. We can suppose without loss of generality that $\delta > 0$, since the alternative case is recovered by replacing A, x with $-A, -x$ respectively. Then, small-amplitude waves will propagate in the negative x-direction, and we can suppose that $A \to 0$ as $\phi \to \infty$. However, it will transpire that we cannot impose this boundary condition as $\phi \to -\infty$.

The counterpart of (9, 10) is

$$-VA_{0\phi} + \alpha A_0 A_{0\phi} + \delta A_{0\phi\phi\phi} = 0\,, \tag{22}$$

$$-VA_{1\phi} + \alpha (A_0 A_1)_{\phi} + \delta A_{1\phi\phi\phi} = -A_{0T}\,. \tag{23}$$

But now the solution for A_0 is taken to be the solitary wave (obtained from (11, 12, 13) by the limit $m \to 1$ as described above),

$$A = a\text{sech}^2(K\phi)\,, \tag{24}$$

$$\text{where} \quad V = \frac{\alpha a}{3} = 4\delta K^2\,. \tag{25}$$

A background term d can be added as in (11), but is readily shown to be a constant, and can then be removed by a Galilean transformation. At the next order, we seek a solution of (23) for A_1 which is bounded as $\phi \to \pm\infty$, and in fact $A_1 \to 0$ as $\phi \to \infty$. As before, the adjoint equation to (23) is

$$-VA_{1\phi} + \alpha A_0 A_{1\phi} + \delta A_{1\phi\phi\phi} = 0\,. \tag{26}$$

Two solutions are $1, A_0$; while both are bonded, only the second solution satisfies the condtion that $A_1 \to 0$ as $\phi \to \infty$. A third solution can be constructed using the variation-of-parameters method, but it is unbounded as $\phi \to \pm\infty$. Hence only one othogonality condition can be imposed, namely that the right-hand side of (23) is orthogonal to A_0, which leads to

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty} A_0^2\,d\phi = 0\,. \tag{27}$$

As the solitary wave (24) has just one free parameter (e.g. the amplitude a), this equation suffices to determine its variation. Substituting (24, 25) into (27) leads to the law

$$a^3 = \text{constant}\frac{\alpha}{\delta}\,. \tag{28}$$

We now recall that the vKdV equation possesses two conservation laws

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} A dx = 0\,, \tag{29}$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} A^2 dx = 0\,, \tag{30}$$

for mass and momentum respectively; compare (15, 16) for the case of periodic waves. The condition (27) is easily recognized as the leading order expression for conservation of momentum (30). But since this completely defines the slowly-varying solitary wave, we now see that this cannot simultaneously conserve total mass. This is apparent when one examines the solution of (23) for A_1, from which it is readily shown that although $A_1 \to 0$ as $\phi \to \infty$, $A_1 \to H_1$ as $\phi \to -\infty$ where

$$VH_1 = -\frac{\partial}{\partial T}\int_{-\infty}^{\infty} A_0\,dx\,, \tag{31}$$

$$\text{or} \quad H_1 = \frac{1}{3\alpha K}\frac{a_T}{a}\,. \tag{32}$$

This non-uniformity in the slowly-varying solitary wave has been recognized for some time, see, for instance, Grimshaw and Mitsudera (1993) and the references therein. The remedy is the construction of a trailing shelf A_s of small amplitude $O(\epsilon)$ but long length-scale $O(1/\epsilon)$, which thus has $O(1)$ mass, but $O(\epsilon)$ momentum. It resides behind the solitary wave, and to leading order has a value independent of T, so that $A_s = \epsilon A_s(X)$ where $X = \epsilon x$ for $X < \Phi(T) = \int^T V(T)\,dT$; it is determined by its value at the location $X = \Phi(T)$ of the solitary wave, namely $A_s(\Phi(T)) = H_(T)$ (32). At higher orders in ϵ the shelf itself will evolve and may generate secondary solitary waves (El and Grimshaw, 2002). It may readily be verified that the slowly-varying solitary wave and the trailing shelf together satisfy conservation of mass.

The expression (28) shows that the amplitude increases (decreases) as α/δ increases/decreases. Then, from (32) we see that a slowly-varying solitary wave of increasing (decreasing) amplitude, will generate a trailing shelf of the same (opposite) polarity (recall that the sign of α determines the polarity of the solitary wave. A particular case of interest is when the nonlinear coefficient α passes through zero, while δ stays finite. Suppose this occurs at $T = 0$, where, without loss of generality, we may suppose that α passes from positive to negative values as T increases. Initially the solitary wave is located in $T < 0$ and has positive polarity. Then, near the transition point, the amplitude of the wave decreases to zero as $a \sim \alpha^{1/3}$, while $K \sim \alpha^{2/3}$; the momentum of the solitary wave is of course conserved (at least to leading order), the mass of the solitary wave increases as $1/\alpha^{1/3}$, its speed decreases as $\alpha^{4/3}$, and the amplitude H_1 of the trailing shelf just behind the solitary wave grows as $-1/\alpha^{8/3}$; the total mass of the trailing shelf grows as $-1/\alpha^{1/3}$, in balance with that of the solitary wave, while the total mass remains a positive constant. Thus the solitary wave itself is destroyed as the wave attempts to pass through the critical point $\alpha = 0$. The structure of the solution beyond this critical point has been examined numerically by Grimshaw et al (1998), who showed that, in essence, the shelf passes through the critical point as a negative disturbance, which then being in an environment with $\alpha < 0$, can generate a train of solitary waves of negative polarity, riding on a positive pedestal. Of course, these conclusions may need to be modified when the cubic nonlinear term in (5) is taken into account near the critical point (Grimshaw et al, 1999), and this issue is taken up in the next section.

3 Slowly-varying solitary waves in the extended Korteweg-de Vries equation

3.1 Periodic waves

Although the main focus in this section is on solitary waves, we shall briefly describe the analogous theory for periodic waves. We again use the multi-scale asymptotic expansion (6, 7, 8), and substitute this into (5). The leading order term is now given by

$$A_0 = D(T) + V(\theta, T)\,, \quad V = \frac{H}{1 + B\mathrm{cn}^2\gamma\theta}\,, \tag{33}$$

$$\text{where} \quad \hat{\alpha}H = 12\delta\gamma^2k^2(\frac{3m}{B} + 4m - 2 - B(1-m))\,, \tag{34}$$

$$\beta H^2 = 24\delta\gamma^2k^2(1 - m - \frac{m}{B})(B+1)\,, \tag{35}$$

$$\hat{V} = 4\delta\gamma^2k^2(\frac{3m}{B} + 2m - 1)\,, \tag{36}$$

$$\text{and} \quad \hat{\alpha} = \alpha + 2\beta D\,, \hat{V} = V - \alpha D - \beta D^2\,. \tag{37}$$

The spatial period is again $2K(m)/\gamma k$, and so, since A_0 is 2π-periodic in θ we see that $\gamma = K(m)/\pi$. The amplitude of the wave is $a = HB/1 + B$. When $\beta < 0$ there is a single family with $0 < B < m/(1-m)$, while if $\beta > 0$ there are two families, $-1 < B < 0, m/(1-m) < B < \infty$. The KdV case $\beta = 0$ is recovered by putting $H = a/B$, $D = d - a/B$ and taking the limit $B \to 0$ with a, d, γ fixed. As for the KdV case discussed in section 2.1, the periodic solution contains three free parameters, which here we take to be m, B, D. The relations (34, 35. 36) then determine k, H, V. Note that the limit $B \to m/1-m$ corresponds to the limit $H, k \to 0$ with $H \sim k^2$ and m fixed, $0 < m < 1$. On the other hand, the limits $B \to -1, B \to \infty$ produce singular solutions.

The determination of how A_0 depends on T, that is, how to determine H, B, D as functions of T follows the same procedure described in section 2.1. Thus k is a constant, and then we use the conservation laws (15, 16). The first determines D by the requirement that the mean level of A_0 be a constant, say d, and the second can then be regarded as determining either H or B. Thus, we get

$$D + \frac{1}{2\pi}\int_0^{2\pi} V(\theta)\, d\theta \;=\; d\,. \tag{38}$$

$$\frac{1}{2\pi}\int_0^{2\pi} V(\theta)^2\, d\theta \;=\; (D-d)^2 + \text{constant}\,. \tag{39}$$

The integrals in (38, 39) can now be evaluated in terms of elliptic integrals,

$$\frac{1}{2\pi}\int_0^{2\pi} V(\theta)\, d\theta \;=\; M(B,m) \;=\; \frac{1}{(1+B)K(m)}\Pi(\frac{B}{1+B}, m)\,, \tag{40}$$

$$\frac{1}{2\pi}\int_0^{2\pi} V(\theta)^2\, d\theta \;=\; \frac{\partial}{\partial B}(BM(B,m))\,. \tag{41}$$

$$= \frac{(C_1\Pi(B/(1+B)) + C_2K(m) + C_3E(m))}{(2(1+B)^2(m - B(1-m))}\,, \tag{42}$$

where $C_1 = 3m + (4m-2)B - (1-m)B^2\,,$
$C_2 = -m + (1-2m)B + (1-m)B^2\,,\; C_3 = -B(1+B)\,.$

Here $\Pi(n,m)$ is the complete elliptic integral of the third kind. The requirement that k is a constant, leads to a relationship between m and B, found by eliminating H from (34, 35). Then the relations (38, 39) provide explicit expressions linking the wave parameters B (or m) and D with the environmental parameters α, β, δ. However, these expressions are quite complicated to unravel, and we shall not this matter any further here.

3.2 Solitary waves

As for the vKdV equation (3) we use the same multi-scale asymptotic expansion used in section 2.2, that is, (6, 7) with (21). The leading term is the solitary wave, which can be obtained from (33) in the limit $m \to 1$ (noting that the paramters H, B change their meaning in the process), or directly from (5),

$$A_0 = \frac{H}{1 + B\cosh K\phi}\,, \tag{43}$$

$$\text{where} \quad V = \frac{\alpha H}{6} = \delta K^2\,, \tag{44}$$

$$\text{and} \quad B^2 = 1 + \frac{6\delta\beta K^2}{\alpha^2}\,. \tag{45}$$

As before, a background term d can be added as in (11), but is readily shown to be a constant, and can then be removed by a Galilean transformation. The amplitude is $a = H/1 + B$. The family of solutions (43) depend on a single parameter, which can conveniently be taken as B, and are displayed in Figure 2. As before, we take $\delta > 0$ without loss of generality. Then, for $\beta < 0$ there is just one branch of solutions, with $0 < B < 1$; they range from small-amplitude solitary waves of KdV-type with the familiar "sech^2"-profile when $B \to 1$, to a limiting wave of amplitude $-\alpha/\beta$ as $B \to 0$; this limiting wave is characterized by a flat top, and are sometimes called "table-top" waves. For $\beta > 0$ there are two branches; one has $1 < B < \infty$ and ranges from small-amplitude KdV-type waves when $B \to 1$, to arbitrarily large waves with a "sech"-profile as $B \to \infty$. The other branch has has the opposite polarity, exists for $-\infty < B < -1$, and ranges from arbitrarily large waves with a "sech"-profile to a limiting algebraic solitary wave of amplitude $-2\alpha/\beta$. Solitary waves with smaller momentum cannot exist, and from the point of view of the associated spectral problem are replaced by breathers (see, for instance, Clarke et al 2000, Grimshaw et al 1999, Pelinovsky and Grimshaw 1997).

We now follow the same procedure described in section 2.2. That is, the determination of how the key parameter B of (43) varies with T is found either by considering the next-order term in the expansion, or equivalently by using the conservation law (30) for momentum, which can easily be shown to also hold for the variable-coefficient extended KdV equation (5). The outcome is that (27) holds for the solitary wave (43) and so we get that

$$\frac{H^2}{K}\int_{-\infty}^{\infty} \frac{du}{(1 + B\cosh u)^2} = \text{constant}\,, \tag{46}$$

$$\text{or} \quad G(B) = \text{constant}\left|\frac{\beta^3}{\delta\alpha^2}\right|^{1/2}, \tag{47}$$

$$\text{where} \quad G(B) = |B^2 - 1|^{3/2}\int_{-\infty}^{\infty} \frac{du}{(1 + B\cosh u)^2}\,. \tag{48}$$

The integral term in $G(B)$ can be explicitly evaluated, and so we finally get

$$B^2 > 1: \quad G(B) = 2(B^2 - 1)^{1/2} \mp 4\text{arctan}\sqrt{\frac{B-1}{B+1}}\,, \tag{49}$$

$$0 < B < 1: \quad G(B) = 4\text{arctanh}\sqrt{\frac{1-B}{1+B}} - 2(1 - B^2)^{1/2}\,. \tag{50}$$

The alternative signs in (49) correspond to the cases $B > 1$ or $B < -1$. Expressions of these forms have ben considered by Egorov (1993) for water waves, and Grimshaw et al (1999, 2004) for internal waves.

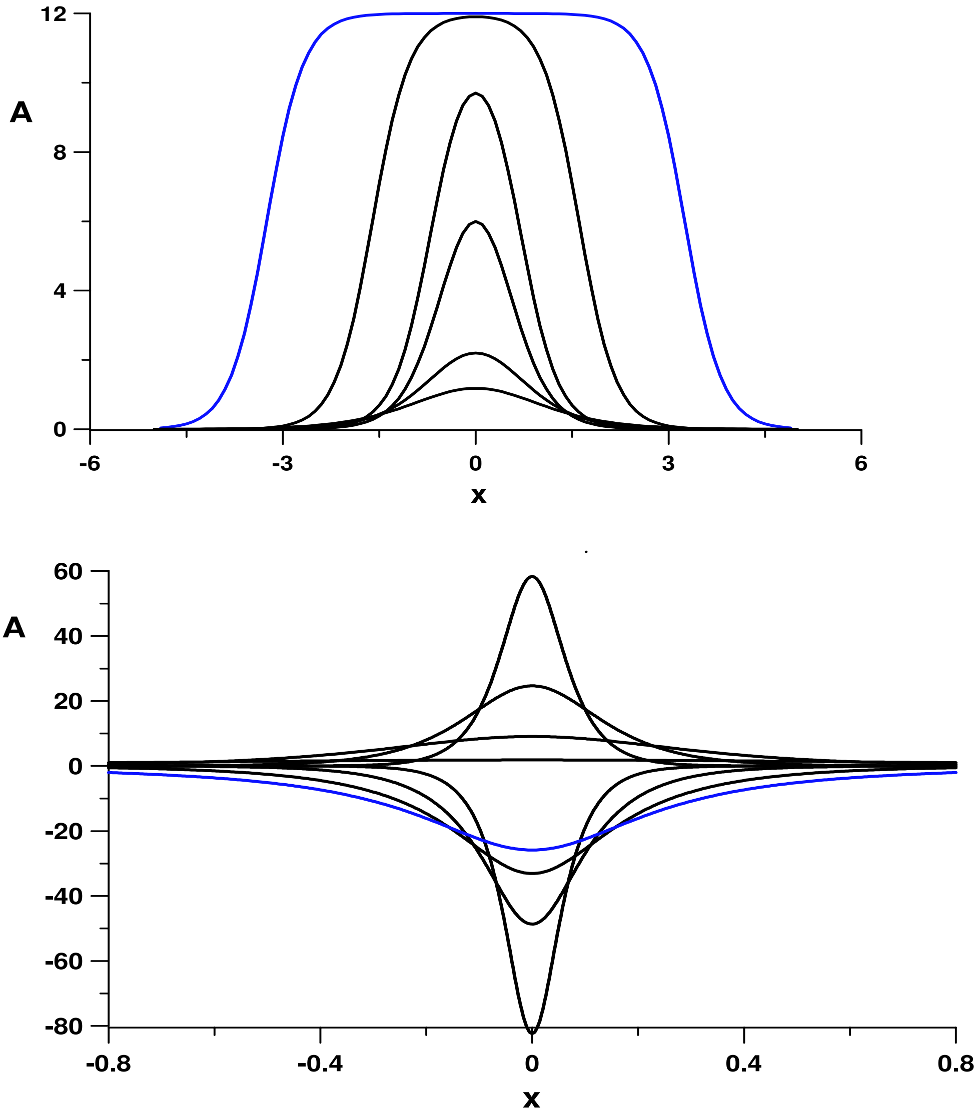

Fig. 2 The solitary wave family (43). The upper panel is for $\beta < 0$ and the lower panel is for $\beta > 0$; in both panels $\alpha > 0, \delta > 0$.

Next, just as for the vKdV case discussed in section 2.2, the slowly-varying solitary wave is accompanied by a trailing shelf, in order to conserve total mass. As before, to leading order it is given by $A_s = A_s(X)$ where $X = \epsilon x$ for $X < \Phi(T) = \int^T V(T)\, dT$, and is determined by its value at the location $X = \Phi(T)$ of the solitary wave, namely $A_s(\Phi(T)) = H_1(T)$, where

$$VH_1(T) = -\frac{\partial M}{\partial T}\,, \tag{51}$$

$$\text{where} \quad M(T) = \int_{-\infty}^{\infty} A_0 d\phi\,, \tag{52}$$

is the mass of the solitary wave. Substitution of (43, 44) into (52), yields

$$B^2 > 1: \quad M = \pm\left|\frac{6\delta}{\beta}\right|^{1/2} 4\mathrm{arctan}\sqrt{\frac{B-1}{B+1}}\,, \tag{53}$$

$$0 < B < 1: \quad M = \pm\left|\frac{6\delta}{\beta}\right|^{1/2} 4\mathrm{arctanh}\sqrt{\frac{1-B}{1+B}}\,. \tag{54}$$

Here the alternative signs in (53) and (54) correspond to the cases $\alpha B > 0$ or $\alpha B < 0$.

The expression (46) provides an explicit formula for the dependence of B on the environmental parameters α, β, δ. It is readily shown that $G(B)$ is a monotonically increasing function of B for $1 < B < \infty$, and is a monotonically decreasing function of B for $-\infty < B < -1$ and for $0 < B < 1$. In general, as $|\beta^3/\delta\alpha^2| \to \infty$, then so does $G(B)$; we infer that then, if $\beta < 0$ so that $0 < B < 1$, $B \to 0$ and the wave approaches the limiting "table-top" shape. On the other hand if $\beta > 0$ and $1 < B < \infty$ then $B \to \infty$ and the wave shape approaches the "sech"-profile, while if $-\infty < B < 1$, then $B \to \infty$ and the wave approaches the limiting algebraic solitary wave. The behaviour of the wave amplitude in these limits depends on the behaviour of each of the parameters α, β, δ.

We will now return to the special case of interest when α passes through zero, while δ stays finite. This was considered in section 2.2 when the cubic nonlinear term in the vKdV equation (5) is omitted, and now we reconsider this limit when β stays finite. First, let us suppose that $\beta < 0, 0 < B < 1$. Then as $\alpha \to 0$, we see from (46) and (50) that $B \to 0$ with $B \sim 2\exp(-1/2|\alpha|)$. Thus the approach to the limiting "table-top" wave is quite rapid. From (44. 45) we see that in this limit, $K \sim |\alpha|$ and the amplitude $a \sim |\alpha|$. Curiously, this is more rapid destruction of the solitary wave than for the case when $\beta = 0$. At the same time, the mass M (54) of the solitary wave grows as $|\alpha|$. The overall scenario after α has passed through zero is similar to that described above for the vKdV equation (3) and has been discussed by Grimshaw et al (1999). Essentially the trailing shelf passes through the critical point as a disturbance of the opposite polarity to that of the original solitary wave, which then being in an environment with the opposite sign of α, can generate a train of solitary waves of the opposite polarity, riding on a pedestal.

Next, let us suppose that $\beta > 0$ so that $1 < B^2 < \infty$ There are the two sub-cases to consider, $B > 0$ or $B < 0$, when the the solitary wave has the same or opposite polarity to α. Then, as $\alpha \to 0, |B| \to \infty$ as $|B| \sim 1/|\alpha|$. It follows from (44. 45) that then $K \sim 1$, $h \sim 1/|\alpha|$ and $a \sim 1$. It follows that the wave adopts the "sech"-profile, but has *finite* amplitude, and so can pass through the critical point $\alpha = 0$ without destruction. Note that here the mass M (53) is finite.

Finally, we consider the situation when $\beta \to 0$. In this situation we see from (47) that $G \sim |\beta|^{3/2}$ and so $|B| \to 1$. There are three sub-cases to consider. First, suppose that initially $\beta < 0$ and so $0 < B < 1$. Then it follows from (50) that $1 - B \sim |\beta|$ and so the wave profile becomes the familiar KdV "sech^2"-shape. Also, it is readily shown from (44. 45) that $K \sim 1, a \sim 1, M \sim 1$ and so the wave can pass through the critical point $\beta = 0$ without destruction. However, after passage through the critical point, the wave has moved to a different solitary branch (see Figure 2), and this may change its ultimate fate. Second, suppose that initially $\beta > 0$ and $1 < B < \infty$. Then it follows from (49) that $B - 1 \sim \beta$ and so again the wave profile becomes the familiar KdV "sech^2"-shape, while $K, a, M \sim 1$. This is just the reverse of the first case and again the wave can pass through the critical point $\beta = 0$ without destruction. Third, suppose that initially $\beta > 0$ and $-1 > B > -\infty$. In this case it an be shown from (49) that $G(B)$ decreases from ∞ to a finite value of 2π as B increases from $-\infty$ to -1. Consequently the limit $\beta \to 0$ in (47) cannot be achieved. Instead as β decreases the limit $B = -1$ is reached, when the wave has become an algebraic solitary wave. Presumably a further decrease in β could generate breathers.

4 Discussion

In this paper we have reviewed the procedure for determining the behaviour of an internal solitary wave propagating in a variable medium. The discussion has been based on the variable coefficient KdV equation (3) and its extension to (5) which takes account of cubic as well as quadratic nonlinearity. The results have been put into context by a brief discussion of the corresponding theory for periodic waves; the essential difference between a solitary wave and a periodic wave is that while both deform to conserve momentum, the solitary wave by itself cannot simultaneously conserve mass and so generates a trailing shelf, whereas the periodic wave has two extra degrees of freedom and hence can also simultaneously conserve both mass and wavenumber. This difference is crucial when one examines the behaviour near critical points where one of the nonlinear coefficients in (3) or (5) passes through zero.

Application of the theory presented here is widespread for the variable coefficient KdV equation (3) and its validity has been confirmed by several numerical simulations. Essentially, the solitary wave will deform adiabatically (that is, conserving its momentum) as long as the background environment varies slowly relative to the solitary wave, and the wave does not not encounter a critical point where the nonlinear coefficient α passes through zero. The variable coefficient extended KdV equation (5) has only recently received similar attention, most notably by Grimshaw et al (2004) who used it to model oceanic internal solitary waves over three typical oceanic shelves. Their numerical simulations again demonstrated the validity of the slowly-varying solitary wave in the framework of (5), again provided that the background environment varies slowly relative to the solitary wave, and that the wave does not not encounter critical point where one of the nonlinear coefficients $\alpha.\beta$ pass through zero.

Finally, we note that our discussion of periodic waves has been for the special case when the parameters vary slowly with T only. While this is a valid technical assumption, and is made here to facilitate comparison with the corresponding theory for slowly-varying solitary waves, it is usually not a very practical assumption, as in effect it assumes that the periodic wave train has infinite length. A more realistic assumption is to allow the slowly varying periodic wave train to vary with both $X = \epsilon x$ and $T = \epsilon t$. This case can also be considered using the Whitham averaging procedure, and indeed, such space-time modulated periodic waves have been extensively studied for constant coefficient evolution equations (see, for instance, Kamchatnov 2000). The outcome is usually a set of nonlinear hyperbolic equations for the wave parameters, widely-known as the Whitham modulation equations. However, their counterpart for the present case of evolution equations with variable coefficients has only rarely been considered but see, for instance, Myint and Grimshaw (1994) or Kamchatnov (2004).

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