Fully localised solitary-wave solutions of the three-dimensional gravity-capillary water-wave problem

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Abstract

A model equation derived by B. B. Kadomtsev & V. I. Petviashvili (1970) suggests that the hydrodynamic problem for three-dimensional water waves with strong surface-tension effects admits a *fully localised solitary wave* which decays to the undisturbed state of the water in every horizontal spatial direction. This prediction is rigorously confirmed for the full water-wave problem in the present paper. The theory is variational in nature. A simple but mathematically unfavourable variational principle for fully localised solitary waves is reduced to a locally equivalent variational principle with significantly better mathematical properties using a generalisation of the Lyapunov-Schmidt reduction procedure. A non-trivial critical point of the reduced functional is detected using the direct methods of the calculus of variations.

1 Introduction

1.1 The main result

The classical *three-dimensional gravity-capillary water wave problem* concerns the irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid motion is described by the Euler equations in a domain bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = h + \rho(x, z, t)\}$, where h denotes the depth of the water in its undisturbed state and the function ρ depends upon the two horizontal spatial directions x, z and time t. *Steady waves* are water waves which are uniformly translating in a distinguished horizontal direction without change of shape; without loss of generality we assume that the waves propagate in the x-direction with speed c and continue to write x as an abbreviation for $x - ct$. In terms of an Eulerian velocity potential $\phi(x, y, z, t)$ the mathematical problem for steady waves is to solve the equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad 0 < y < 1 + \rho, \tag{1}$$

$$\phi_y = 0 \qquad \text{on } y = 0, \tag{2}$$

$$\phi_y = \rho_x\phi_x + \rho_z\phi_z - \rho_x \qquad \text{on } y = 1 + \rho \tag{3}$$

and

$$\begin{aligned} -\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha\rho \\ - \beta\left[\frac{\rho_x}{\sqrt{1 + \rho_x^2 + \rho_z^2}}\right]_x - \beta\left[\frac{\rho_z}{\sqrt{1 + \rho_x^2 + \rho_z^2}}\right]_z = 0 \quad \text{on } y = 1 + \rho \end{aligned} \tag{4}$$

(see Stoker [35]), in which we have introduced dimensionless variables. The equations involve two physical parameters $\alpha := gh/c^2$ and $\beta := \sigma/hc^2$, where g and σ are respectively the acceleration due to gravity and the coefficient of surface tension.

The steady water-wave problem (1)–(4) is a free boundary-value problem with nonlinear boundary conditions, and in this respect its solution poses considerable mathematical difficulties. At a formal level these difficulties may be overcome by replacing the above equations by a simpler model equation based upon certain approximations. One of the more widely used model equations is the KP-I equation

$$\partial_{xx}\left(u_{xx} - u - \frac{3}{2}u^2\right) - u_{zz} = 0, \tag{5}$$

in which u depends upon two unbounded spatial directions x and z. This equation was derived formally by Kadomtsev & Petviashvili [21] as a long-wave approximation for solutions of the steady gravity-capillary water-wave problem (1)–(4) in which

$$\beta > 1/3, \qquad \alpha = 1 + \varepsilon, \quad 0 < \varepsilon \ll 1; \tag{6}$$

the variable u is supposed to approximate the free surface of the water via the formula

$$\rho(x, z) = \varepsilon\, u\left(\frac{\varepsilon^{1/2}x}{2(\beta - 1/3)^{1/2}}, \varepsilon z\right) + \mathcal{O}(\varepsilon^2).$$

The KP-I equation (5) admits the the explicit solution

$$u(x,z) = -8\frac{3-x^2+z^2}{(3+x^2+z^2)^2} \tag{7}$$

which defines a *fully localised solitary wave*, that is a wave which decays to zero at large distances in both spatial directions (Ablowitz & Segur [1]); this wave is sketched in Figure 1. In the present paper we confirm the prediction made by the KP-I equation by proving that the steady water-wave problem (1)–(4) has a fully localised solitary-wave solution in the parameter regime (6). Our result contrasts with a recent theorem by Craig [11], who showed that in the absence of surface tension there are no fully localised solitary waves with $\rho \geq 0$.

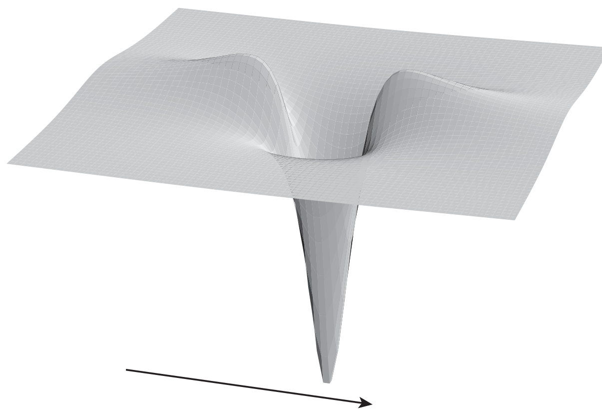

Figure 1: A fully localised solitary wave; the arrow shows the direction of wave propagation.

1.2 Variational methods

The key to our existence theory for fully localised solitary waves is the observation that the hydrodynamic problem (1)–(4) in the parameter regime (6) follows from the formal variational principle

$$\delta\Bigg\{\int_{\mathbb{R}^2}\Bigg(\int_0^{1+\rho}\Bigg(-\phi_x+\frac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2)\Bigg)\,\mathrm{d}y \\ +\frac{1}{2}(1+\varepsilon)\rho^2+\beta(\sqrt{1+\rho_x^2+\rho_z^2}-1)\Bigg)\,\mathrm{d}x\,\mathrm{d}z\Bigg\}=0, \tag{8}$$

where the variation is taken in (ρ, ϕ) (see Luke [28]). A more satisfactory version of this variational principle is obtained using the transformation

$$y = \tilde{y}(1+\rho(x,z)), \qquad \phi(x,y,z) = \Phi(x,\tilde{y},z),$$

which maps the variable fluid domain $\mathcal{D}_\rho = \{(x,y,z) : (x,z) \in \mathbb{R}^2, \tilde{y} \in (0, 1+\rho(x,z))\}$ bijectively into the fixed strip $\Sigma = \{(x,\tilde{y},z) : (x,z) \in \mathbb{R}^2, \tilde{y} \in (0,1)\}$, and it is also appropriate to introduce the scaled variables

$$(\tilde{\rho}(\tilde{x},\tilde{z}), \tilde{\Phi}(\tilde{x},y,\tilde{z})) = (\varepsilon^{-1}\rho(x,z), \varepsilon^{-\frac{1}{2}}\Phi(x,y,z)), \qquad (\tilde{x},\tilde{z}) = (\varepsilon^{\frac{1}{2}}x, \varepsilon z) \tag{9}$$

associated with the KP scaling limit. The hydrodynamic problem (1)–(4) is transformed into the equation

$$(1+\varepsilon)\rho - \beta\varepsilon\rho_{xx} - \beta\varepsilon^2\rho_{zz} = \Phi_x|_{y=1} + \varepsilon^{-1}N_1(\rho,\Phi) \tag{10}$$

and the boundary-value problem

$$-\varepsilon\Phi_{xx} - \varepsilon^2\Phi_{zz} - \Phi_{yy} = \varepsilon^{-\frac{1}{2}}N_2(\rho,\Phi), \qquad 0<y<1, \tag{11}$$

$$\varepsilon\rho_x + \Phi_y = \varepsilon^{-\frac{1}{2}}N_3(\rho,\Phi) \qquad \text{on } y=1, \tag{12}$$

$$\Phi_y = 0 \qquad \text{on } y=0, \tag{13}$$

while the functional in the above variational principle is transformed into

$$\mathcal{V}(\rho,\Phi) = \int_{\mathbb{R}^2}\left\{\int_0^1 \left(\frac{\varepsilon}{2}\left[\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right]^2 + \frac{\Phi_y^2}{2(1+\varepsilon\rho)^2} + \frac{\varepsilon^2}{2}\left[\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right]^2\right)(1+\varepsilon\rho)\,\mathrm{d}y \right.$$
$$\left. + \frac{1}{2}\varepsilon(1+\varepsilon)\rho^2 + \beta\varepsilon^{-1}[\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}-1] + \varepsilon\int_0^1(\rho_x y\Phi_y - \rho\Phi_x)\,\mathrm{d}y\right\}\mathrm{d}x\,\mathrm{d}z;$$

here the tildes have been dropped for notational simplicity and explicit formulae for the nonlinear functions N_1, N_2, N_3 are given in Section 2. At a formal level it is readily confirmed that critical points of $\mathcal{V}$ correspond to weak solutions of (10)–(13). Our strategy is therefore to apply the direct methods of the calculus of variations to find critical points of $\mathcal{V}$ (defined upon a suitable function space) and develop a regularity theory which shows that the corresponding weak solutions of (10)–(13) are in fact strong solutions of these equations.

The calculus of variations offers a variety of results for studying functionals of the type

$$\mathcal{J}(u) = \int_{\mathcal{S}} J(u)\,\mathrm{d}x^n$$

which are defined on spatially extended domains $\mathcal{S}$ (that is subsets of $\mathbb{R}^n$ which are unbounded in one or more spatial directions). A problem of this kind is typically treated in two stages. Firstly one establishes the existence of a *Palais-Smale* sequence $\{u_m\}$ with the property that $\mathcal{J}(u_m)\to a$, $\mathcal{J}'(u_m)\to 0$ as $m\to\infty$ for some nonzero constant a, so that $\{u_m\}$ is a sequence of successively better approximations to a putative critical point $u\neq 0$ with $\mathcal{J}(u)=a$, $\mathcal{J}'(u)=0$. The second step is to study the convergence properties of $\{u_m\}$ (note that weaker results than the strong convergence of $\{u_m\}$ are sufficient to guarantee the existence of a nonzero critical point). The *concentration-compactness principle* of Lions [26, 27] is frequently helpful in this respect; it has been applied with great success to the following class of problems collectively known as 'the coercive, semilinear, locally compact case'. Suppose that $\mathcal{J}$ is a smooth functional on $\mathcal{X}(\mathcal{S})$, where $\mathcal{X}(U)$ is a Sobolev space of functions defined upon the spatial domain $U\subseteq\mathbb{R}^n$. Let us write

$$\mathcal{J}(u) = \mathcal{J}_2(u) + \mathcal{J}_{\mathrm{NL}}(u),$$

where $\mathcal{J}_2 : \mathcal{X}(\mathcal{S})\to\mathbb{R}$ is the quadratic part of $\mathcal{J}$, and suppose that $\mathcal{J}_{\mathrm{NL}}$ extends to a smooth functional $\mathcal{J}_{\mathrm{NL}} : \mathcal{Y}(\mathcal{S})\to\mathbb{R}$, where

(i) ('coerciveness') $\mathcal{J}_2$ is equivalent to the $\mathcal{X}(\mathcal{S})$-norm;

(ii) ('semilinearity') $\mathcal{Y}(\mathcal{S})$ is continuously embedded in $\mathcal{X}(\mathcal{S})$;

(iii) ('local compactness') $\mathcal{Y}(U)$ is compactly embedded in $\mathcal{X}(U)$ for every compact subset U of $\mathbb{R}^n$.

The use of concentration-compactness methods to find solitary-wave solutions of model equations for two-dimensional water waves was pioneered by Weinstein [39], who considered a variety of third-order equations. The method has been extended to many other equations arising in water-wave theory, including fifth-order models (Kichenassamy [22], Groves [14], Levandosky [24]), systems of model equations (Bona & Chen [4]) and model equations for three-dimensional water waves (de Bouard & Saut [13], Pego & Quintero [32]); all of these problems satisfy the coerciveness, semilinearity and local compactness conditions. Let us now examine the variational functional $\mathcal{V}$ associated with the full water-wave problem. A straightforward calculation shows that

$$\begin{aligned}\mathcal{V}_2(\rho,\Phi) = \int_{\mathbb{R}^2} \Bigg\{ \int_0^1 \left(\frac{\varepsilon}{2}\Phi_x^2 + \frac{\varepsilon^2}{2}\Phi_z^2 + \frac{1}{2}\Phi_y^2 + \varepsilon(\rho_x y \Phi_y - \rho\Phi_x)\right) \mathrm{d}y \\ + \frac{1}{2}\varepsilon(1+\varepsilon)\rho^2 + \frac{\beta}{2}\varepsilon^2\rho_x^2 + \frac{\beta}{2}\varepsilon^3\rho_z^2 \Bigg\} \,\mathrm{d}x\,\mathrm{d}z,\end{aligned}$$

$$\begin{aligned}\mathcal{V}_{\mathrm{NL}}(\rho,\Phi) = \int_{\mathbb{R}^2} \Bigg\{ \int_0^1 \Bigg(\frac{\varepsilon^2}{2}\rho\Phi_x^2 + \frac{1}{2}\varepsilon^3\rho\Phi_z^2 - \frac{\varepsilon\rho\Phi_y^2}{2(1+\varepsilon\rho)} + \frac{\varepsilon^3 y^2\Phi_y^2\rho_x^2}{2(1+\varepsilon\rho)} + \frac{\varepsilon^4 y^2 \Phi_y^2 \rho_z^2}{2(1+\varepsilon\rho)} \\ - \varepsilon^2 y \Phi_y \Phi_x \rho_x - \varepsilon^3 y \Phi_y \Phi_z \rho_z\Bigg) \mathrm{d}y \\ - \frac{\beta\varepsilon^{-1}(\varepsilon^3\rho_x^2 + \varepsilon^4\rho_z^2)^2}{2(\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}+1)^2} \Bigg\} \,\mathrm{d}x\,\mathrm{d}z,\end{aligned}$$

and it is readily confirmed that there are no function spaces $\mathcal{X}(\mathbb{R}^2 \times \Sigma)$, $\mathcal{Y}(\mathbb{R}^2 \times \Sigma)$ that meet the criteria set out above. (In particular, it is not possible to choose a function space for $\mathcal{V}_{\mathrm{NL}}$ which requires less regularity of its elements than that for $\mathcal{V}_2$; the problem is *quasilinear* rather than semilinear in this respect.) We therefore proceed by studying $\mathcal{V}$ in one of the widest possible Sobolev spaces upon which it defines a smooth functional, namely $[W^{1,2}(\mathbb{R}^2) \times U^{0,2}(\Sigma)] \cap [W^{1+\delta,p}(\mathbb{R}^2) \times U^{\delta,p}(\Sigma)]$ for $\delta \in (0,1)$ and $p \in (3/\delta, \infty)$, where

$$U^{s,p}(\Sigma) := \{\Phi : \|\Phi\|_{U^{s,p}(\Sigma)} := \|\Phi_x\|_{W^{s,p}(\Sigma)} + \|\Phi_y\|_{W^{s,p}(\Sigma)} + \|\Phi_z\|_{W^{s,p}(\Sigma)} < \infty\},$$

and using a reduction technique to show that the problem of finding critical points of $\mathcal{V}$ on this function space is locally equivalent to one of finding critical points of a reduced functional which falls into the coercive, semilinear, locally compact category.

Our reduction procedure is an extension of the variational Lyapunov-Schmidt reduction (e.g. see Mielke [30, pp. 62–63]). Consider the Euler-Lagrange equation

$$F(u) = 0 \tag{14}$$

associated with a variational functional $\mathcal{J} : \mathcal{X} \to \mathbb{R}$. Suppose that $\mathcal{X}$ admits a direct-sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, and write (14) as

$$F_1(u_1+u_2) = 0, \qquad F_2(u_1+u_2) = 0,$$

where $u_1 = Pu$, $u_2 = (I-P)u$, $F_1 = PF$, $F_2 = (I-P)F$ and $P : \mathcal{X} \to \mathcal{X}$ is the projection onto $\mathcal{X}_1$ along $\mathcal{X}_2$. The decomposition is constructed so that the equation for F_2 can be locally solved for u_2 as a function of u_1 using the implicit-function theorem; substituting $u_2 = u_2(u_1)$ into the equation for F_1, we obtain the *reduced equation for* u_1, namely

$$F_1(u_1 + u_2(u_1)) = 0. \tag{15}$$

The variational structure of (14) is inherited in a natural fashion by (15) provided that the quadratic part $\mathcal{J}_2$ of $\mathcal{J}$ can be expressed as a sum

$$\mathcal{J}_2(u_1 + u_2) = \mathcal{J}_2^1(u_1) + \mathcal{J}_2^2(u_2)$$

of separate quadratic forms for u_1 and u_2. The calculation

$$\begin{aligned}
&\mathrm{d}\mathcal{J}[(u_1 + u_2(u_1)](w_1) \\
&\quad = (\mathrm{d}\mathcal{J}_2^1[u_1] + \mathrm{d}\mathcal{J}_{\mathrm{NL}}[u_1 + u_2(u_1)])(w_1) \\
&\qquad\quad + (\mathrm{d}\mathcal{J}_2^2[u_2] + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[u_1 + u_2])(\mathrm{d}u_2[u_1])(w_1) \\
&\quad = (\mathrm{d}\mathcal{J}_2^1[u_1] + \mathrm{d}\mathcal{J}_{\mathrm{NL}}[u_1 + u_2(u_1)])(w_1),
\end{aligned}$$

in which the second equality follows by defining property of $u_2(u_1)$ as a solution of the equation for F_2, shows that (15) is the Euler-Lagrange equation for the reduced functional $\mathcal{J}(u_1+u_2(u_1))$.

The classical application of this theory is the scenario in which $\mathrm{d}F[0]$ is a (necessarily self-adjoint) Fredholm operator and $\mathcal{X}_1 = \ker \mathrm{d}F[0]$, $\mathcal{X}_2 = \mathrm{Im}\, \mathrm{d}F[0]$; in this framework the method is termed the *variational Lyapunov-Schmidt reduction* and is particularly useful when equation (14) is a system of partial differential equations, since they are reduced to a locally equivalent system of ordinary differential equations. This method has been applied to several problems involving wave phenomena, in particular by Moser [31] in his investigation of the resonant case of the Lyapunov centre theorem for periodic solutions of Hamiltonian systems, and by Craig & Nicholls [12] in their existence theory for doubly periodic three-dimensional water waves. In the present paper we use the theory in the more general framework given above to reduce our *quasilinear* system of partial differential equations to a locally equivalent *semilinear* partial differential equation which meets the criteria set out above for an application of the concentration-compactness method.

1.3 The reduction technique

A preliminary step is necessary before the reduction method can be applied to our water-wave problem, namely elimination of the variable ρ. To this end we solve equation (10) for ρ as a function of Φ and substitute $\rho = \rho(\Phi)$ into equations (11)–(13). Observing that (10) and (11)–(13) correspond to the Euler-Lagrange equations for $\mathcal{V}$ with respect to ρ and Φ, that is

$$\mathrm{d}_1\mathcal{V}[\rho, \Phi] = 0, \qquad \mathrm{d}_2\mathcal{V}[\rho, \Phi] = 0,$$

we find that the 'reduced' version of (11)–(13) with $\rho = \rho(\Phi)$ is the Euler-Lagrange equation for the functional $\mathcal{W} = \mathcal{V}(\rho(\Phi), \Phi)$, since

$$\begin{aligned}
\mathrm{d}\mathcal{W}[\Phi] &= \mathrm{d}_1\mathcal{V}[\rho(\Phi), \Phi](\mathrm{d}\rho[\Phi]) + \mathrm{d}_2\mathcal{V}[\rho(\Phi), \Phi] \\
&= \mathrm{d}_2\mathcal{V}[\rho(\Phi), \Phi],
\end{aligned}$$

in which the second line follows by the defining property of $\rho(\Phi)$ as a solution of the Euler-Lagrange equation for $\mathcal{V}$ with respect to ρ. This calculation shows that the elimination of ρ also qualifies as 'natural' with respect to the variational structure.

Taking Fourier transforms of the 'reduced' version of (11)–(13), we obtain the equations

$$-\hat{\Phi}_{yy} + q^2\hat{\Phi} = \hat{H}(\Phi), \qquad 0 < y < 1, \tag{16}$$

$$\hat{\Phi}_y = 0, \qquad y = 0, \tag{17}$$

$$\hat{\Phi}_y - \frac{\epsilon\mu^2\hat{\Phi}}{1+\epsilon+\beta q^2} = \hat{h}(\Phi), \qquad y = 1, \tag{18}$$

where (μ, k) is the independent variable associated with the Fourier transform in (x, z) and $q^2 = \epsilon\mu^2 + \epsilon^2 k^2$; the nonlinear functions H, h are defined by

$$H = \varepsilon^{-\frac{1}{2}} N_2(\rho(\Phi), \Phi), \qquad \hat{h} = \varepsilon^{-\frac{1}{2}} \hat{N}_3(\rho(\Phi), \Phi) - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi), \Phi).$$

Consider the equation

$$\begin{aligned}\frac{\varepsilon^2}{1+\varepsilon}&[-c_0\varepsilon(\partial_x^2 + \varepsilon\partial_z^2)^3 + (\beta - \tfrac{1}{3})(\partial_x^2 + \varepsilon\partial_z^2)^2 - (1+\varepsilon)\partial_z^2 - \partial_x^2]\Phi_1 \\ &= \int_0^1 H(\Phi_1 + \Phi_2)\,\mathrm{d}y + h(\Phi_1 + \Phi_2)\end{aligned} \tag{19}$$

for $\Phi_1 = \Phi_1(x, z)$ and the boundary-value problem

$$-\hat{\Phi}_{2yy} + q^2\hat{\Phi}_2 + \frac{q^2(1+\varepsilon)}{\varepsilon^2 QS}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \hat{H}(\Phi_1 + \Phi_2), \quad 0 < y < 1, \tag{20}$$

$$\hat{\Phi}_{2y} - \frac{\varepsilon\mu^2\hat{\Phi}_2}{1+\varepsilon+\beta q^2} + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 QS(1+\varepsilon+\beta q^2)}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \hat{h}(\Phi_1 + \Phi_2), \quad y = 1, \tag{21}$$

$$\hat{\Phi}_{2y} = 0, \qquad y = 0 \tag{22}$$

for $\Phi_2 = \Phi_2(x, y, z)$, where

$$Q = k^2(1+\varepsilon) + \mu^2 + (\beta - \tfrac{1}{3})\varepsilon^{-2}q^4 + c_0\varepsilon^{-2}q^6,$$

$$S = 1 - \frac{q^2(1+\varepsilon)}{\varepsilon^2 Q} + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}.$$

One can show that any solution (Φ_1, Φ_2) of this pair of equations yields a solution $\Phi = \Phi_1 + \Phi_2$ of (16)–(18), and conversely any solution Φ of (16)–(18) can be decomposed into a sum $\Phi = \Phi_1 + \Phi_2$, where (Φ_1, Φ_2) solve (19) and (20)–(22) (the functions Φ_1 and Φ_2 are calculated from the formulae obtained by replacing $\Phi_1 + \Phi_2$ by Φ on the right-hand sides of (19) and (20)–(22)). The boundary-value problem (16)–(18) is therefore equivalent to equation (19) and the boundary-value problem (20)–(22).

The left-hand side of (19) defines a formally self-adjoint operator acting upon $\Phi_1(x,z)$ which is associated with the quadratic form

$$\varepsilon^2 Q_1(\Phi_1) = \frac{\varepsilon^2}{2(1+\varepsilon)} \int_{\mathbb{R}^2} \{c_0(\varepsilon\Phi_{1xxx}^2 + 3\varepsilon^2\Phi_{1xxz}^2 + 3\varepsilon^3\Phi_{1xzz}^2 + \varepsilon^4\Phi_{1zzz}^2) \\ + (\beta - \tfrac{1}{3})(\Phi_{1xx}^2 + 2\varepsilon\Phi_{1xz}^2 + \varepsilon^2\Phi_{1zz}^2) + \Phi_{1x}^2 + (1+\varepsilon)\Phi_{1z}^2\} \,\mathrm{d}x\,\mathrm{d}z,$$

and similarly the left-hand side of the boundary-value problem (20)–(22) defines a formally self-adjoint operator acting upon $\Phi_2(x,y,z)$ which is associated with the quadratic form

$$Q_2(\Phi_2) = \frac{1}{2}\int_{\mathbb{R}^2} \left\{ \int_0^1 (|\hat{\Phi}_{2y}|^2 + q^2|\hat{\Phi}_2|^2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}|\hat{\Phi}_2|_{y=1}|^2 \right. \\ \left. + \frac{1+\varepsilon}{\varepsilon^2 QS}\left| q^2 \int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right|^2 \right\} \mathrm{d}\mu\,\mathrm{d}k;$$

furthermore, note that

$$\mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi](\Psi) = \int_{\mathbb{R}^2} \left\{ \int_0^1 H(\Phi)\Psi\,\mathrm{d}y + h(\Phi)\Psi|_{y=1} \right\} \mathrm{d}x\,\mathrm{d}z.$$

One concludes that (19) and (20)–(22) are the Euler-Lagrange equations for the functional $Q_1(\Phi_1) + Q_2(\Phi_2) + \mathcal{W}_{\mathrm{NL}}(\Phi_1 + \Phi_2)$ corresponding to Φ_1 and Φ_2.

We now have all the ingredients necessary to apply the variational reduction method described in Section 1.2. Solving (20)–(22) for Φ_2 as a function of Φ_1 and substituting $\Phi_2 = \Phi_2(\Phi_1)$ into (19), we obtain the reduced equation for Φ_1, namely

$$\frac{\varepsilon^2}{1+\varepsilon}[-c_0\varepsilon(\partial_x^2 + \varepsilon\partial_z^2)^3 + (\beta - \tfrac{1}{3})(\partial_x^2 + \varepsilon\partial_z^2)^2 - (1+\varepsilon)\partial_z^2 - \partial_x^2]\Phi_1 \\ = \int_0^1 H(\Phi_1 + \Phi_2(\Phi_1))\,\mathrm{d}y + h(\Phi_1 + \Phi_2(\Phi_1)),$$

which is the Euler-Lagrange equation for the functional

$$I(\Phi_1) = \varepsilon^2 Q_1(\Phi_1) + Q_2(\Phi_2(\Phi_1)) + \mathcal{W}_{\mathrm{NL}}(\Phi_1 + \Phi_2(\Phi_1)).$$

It is apparent from the above discussion that I_2 (which is $\varepsilon^2 Q_1$) involves higher derivatives of Φ_1 than I_{NL}, and carrying out the reduction procedure in appropriate function spaces (see below), one in fact finds that I_2 and I_{NL} define smooth functionals upon respectively

$$X = \{\Phi_1 : \|\Phi_1\|_X := Q_1(\Phi_1) < \infty\}$$

and $U^{0,2}(\mathbb{R}^2) \cap U^{\delta,p}(\mathbb{R}^2)$ for $\delta \in (0,1)$, $p \in (3/\delta, \infty)$, where

$$U^{s,p}(\mathbb{R}^2) = \{\Phi_1 : \|\Phi_1\|_{U^{s,p}(\mathbb{R}^2)} := \|\Phi_{1x}\|_{W^{s,p}(\mathbb{R}^2)} + \|\Phi_{1z}\|_{W^{s,p}(\mathbb{R}^2)} < \infty\}.$$

It is readily confirmed that X is continuously and locally compactly embedded in $U^{0,2}(\mathbb{R}^2) \cap U^{\delta,p}(\mathbb{R}^2)$; the functional I therefore falls into the ‘coercive, semilinear, locally compact’ category.

The above discussion is designed to describe the reduction procedure in an illustrative fashion; complete mathematical information is given in Section 2. Section 2.1 presents a full description of the reduction procedure itself, including an explanation of the decomposition of Φ into the sum $\Phi_1 + \Phi_2$ and precise definitions of weak and strong solutions of the original hydrodynamic problem (10)–(13), the equation (19) for Φ_1 and boundary-value problem (20)–(22) for Φ_2. It is essential to develop the reduction procedure in terms of weak solutions of the various equations since critical points of a variational functional in general correspond to weak solutions of the associated system of partial differential equations. Sections 2.2 and for 2.3 are concerned with the details of solving the weak forms of the equations to find ρ as a function of Φ and Φ_2 as a function of Φ_1. The conclusion of the analysis is that $\rho \in W^{1,2}(\mathbb{R}^2) \times W^{1+\delta,p}(\mathbb{R}^2)$ is a function of $\Phi \in U^{0,2}(\Sigma) \cap U^{\delta,p}(\Sigma)$ and that $\Phi_2 \in W^{1,2}(\Sigma) \cap W^{1+\delta,p}(\Sigma)$ is a function of $\Phi_1 \in U^{0,2}(\mathbb{R}^2) \cap U^{\delta,p}(\mathbb{R}^2)$ for sufficiently small values of $\delta \in (0,1)$ and sufficiently large values of $p \in (3/\delta, \infty)$. In Section 2.4 we develop a regularity theory by demonstrating that any weak solution of the reduced equation for Φ_1 (which by definition belongs to X) in fact lies in $\Phi_1 \in U^{0,2}(\mathbb{R}^2) \cap U^{1,p}(\mathbb{R}^2)$; this improved regularity is inherited by Φ_2 and ρ, which belong to respectively $W^{1,2}(\Sigma) \cap W^{2,p}(\Sigma)$ and $W^{1,2}(\mathbb{R}^2) \times W^{2,p}(\mathbb{R}^2)$. We thus obtain the final result any weak solution of the reduced equation for Φ_1 generates a strong solution of the water-wave equations (10)–(13).

The tasks of solving for ρ as a function of Φ and for Φ_2 as a function of Φ_1 are accomplished by re-formulating the equations for ρ and Φ_2 as integral equations (by taking the Fourier transform and using a Green's function); these integral problems define fixed-point problems in suitable Banach spaces. One solves the fixed-point problems using the contraction mapping principle, controlling the size of the Lipschitz constant using the bifurcation parameter ε introduced in equation (6). Recall that ε also plays the role of a scaling parameter (see equation (9)), and it is in fact necessary to work in correspondingly scaled versions of the Banach spaces mentioned above to confirm that the functions under consideration are contractions. The main issue here is the careful book-keeping required to control the ε-dependence of many constants.

Section 3 deals with the remaining part of the existence theory, namely the proof that the reduced equation for Φ_1 has a non-zero weak solution. The key step here is of course to establish that the reduced variational functional I has a nonzero critical point; critical points of I correspond to weak solutions of the reduced equation for Φ_1. Precise details of the variational structure of the reduced equation for Φ_1 are given in Section 3.1, and Section 3.2 presents the proof that I has a nonzero critical point using the method outlined in Section 1.2 above. We show that I is a functional of *mountain-pass type*, that is it has a strict local minimum at the origin and is negative at some non-zero element of X. The *mountain-pass lemma* (e.g. see Brezis & Nirenberg [5, p. 943]) yields the existence of a Palais-Smale sequence $\{\Phi_{1m}\}$ with $I(\Phi_{1m}) \to a$, $I'(\Phi_{1m}) \to 0$ as $m \to \infty$, where a is a nonzero constant (which may be interpreted geometrically as the minimum height attained by any path connecting the origin to another point at 'sea level'). The convergence properties of this Palais-Smale sequence are examined with the concentration-compactness principle according to the method given by Groves [14] in a study of solitary-wave solutions to a fifth-order model equation for water waves.

Two significant technical difficulties emerge in the analysis outlined above, and both involve the Fourier-multiplier operators used to convert our equations into fixed-point problems.

(i) The appearance of Fourier-multiplier operators in L^p-based spaces for $p \neq 2$ means that a more detailed study of their mapping properties is necessary than would be the case in L^2-based spaces (where straightforward results such as Parseval's theorem can be used to estimate their norms). Suitably scaled versions of the classical theorems of Mikhlin and Marcinkiewicz can be used to obtain estimates on the norms of Fourier-multiplier operators in $L^p(\mathbb{R}^2)$-based spaces; dealing with Fourier-multiplier operators in $L^p(\Sigma)$-based spaces however requires the use of deeper results from singular-integral theory (vector-valued versions of Mikhlin's and Marcinkiewicz's theorem are not available).

(ii) The appearance of non-local operators, namely the functional relationships $\Phi_2 = \Phi_2(\Phi_1)$, $\rho = \rho(\Phi)$, in the integrand of the functional $I : X \to \mathbb{R}$ introduces an additional difficulty in the critical-point theory. In applying the concentration-compactness principle to a functional $\mathcal{J} : \mathcal{X} \to \mathbb{R}^2$ it is necessary at one step to demonstrate that

$$\langle \mathcal{J}'(\Phi_{1m}^{(2)}), \Psi_1 \rangle \to 0,$$

where Ψ_1 is a function of compact support, $\{\Phi_{1m}^{(2)}\}$ is a sequence of functions whose supports are contained in $\mathbb{R}^2 \setminus B_{R_m}(0)$, and $\{R_m\}$ is a sequence of positive real numbers with the property that $R_m \to \infty$ as $m \to \infty$. The above limit is easily obtained when $\mathcal{J}$ is defined by an integrand containing only *local* operations such as differentiation, pointwise addition and pointwise multiplication, since in that case the integrand defining $\langle \mathcal{J}'(\Phi_{1m}^{(2)}), \Psi_1 \rangle$ is identically zero whenever R_m is larger than the radius of support of Ψ_1. This simple argument does not apply to the integrand defining I since it contains non-local functions. Fourier-multiplier operators again lie at the heart of this difficulty, since the non-local relationships $\Phi_2 = \Phi_2(\Phi_1)$, $\rho = \rho(\Phi)$ are constructed using them. In Section 3.2 we show that the proof of the above limit reduces to showing that each of our Fourier-multiplier operators $\mathcal{G}$ satisfies $\Psi_1 \mathcal{G}(\Phi_{1m}^{(2)}) \to 0$ in $W^{1+\delta,p}(\mathbb{R}^2)$ for sufficiently large values of p.

These technical difficulties are encountered in respectively Sections 2.2–2.4 and Section 3.2, where we merely state the required results concerning the Fourier-multiplier operators in question. Full proofs are presented in Section 4, which is entirely devoted to these issues.

1.4 Other variational existence theories for water waves

A number of existence theories for three-dimensional gravity-capillary water waves have recently been published, all of which are based upon variational principles equivalent to (8). There are also several existence theories for two-dimensional steady water waves which are variational in character (and many that are not). In this section we present a brief survey of the currently available variational results.

The present paper is the latest in a series of results justifying the use of the KP-I equation (5) as a model equation for solitary gravity-capillary water waves. This equation has several explicit solitary-wave solutions, namely the *line solitary wave*

$$u(x) = -\operatorname{sech}^2\left(\frac{x}{2}\right),$$

which decays exponentially to zero as $x \to \infty$ and does not depend upon the transverse spatial direction z, the family

$$u^\delta(x,z) = -\frac{4(1-\delta^2)}{4-\delta^2}\frac{1-\delta\cosh(a^\delta x)\cos(\omega^\delta z)}{(\cosh(a^\delta x)-\delta\cos(\omega^\delta z))^2}, \qquad a^\delta = \sqrt{\frac{1-\delta^2}{4-\delta^2}}, \quad \omega^\delta = \frac{\sqrt{3(1-\delta^2)}}{4-\delta^2},$$

where $\delta \in (0,1)$, of *periodically modulated solitary waves*, which decay exponentially to zero as $x \to \pm\infty$ and are periodic with frequency ω^δ in z (see Tajiri & Murakami [36]), and of course the fully localised solitary wave (7) which decays algebraically to zero as $|(x,z)| \to \infty$. (In fact the line and fully localised solitary waves correspond to the limiting cases u_0 and u_1 in the above formula.) It was shown by respectively Kirchgässner [23] (see also Amick & Kirchgässner [3] and Sachs [33]) and Groves, Haragus & Sun [17] that the steady water-wave problem has a line solitary-wave solution and a family of periodically modulated solitary-wave solutions in the KP-I parameter regime (6).

The existence theories of Kirchgässner and Groves, Haragus & Sun are based upon a method known as 'spatial dynamics'. This phrase refers to an approach where a system of partial differential equations governing a physical problem is formulated as a (typically ill-posed) evolutionary equation in which an unbounded spatial coordinate plays the role of the time-like variable. The steady water-wave problem has one bounded direction, namely the vertical direction; by contrast no restriction is placed upon the behaviour of the waves in horizontal directions, and so any horizontal coordinate qualifies as 'time-like'. One may therefore study the problem using spatial dynamics by formulating it as an evolutionary system whose time-like coordinate ξ is an arbitrary horizontal spatial direction and whose infinite-dimensional phase space consists of functions of the vertical coordinate and another, different horizontal coordinate Z, in which the behaviour of the waves is prescribed (e.g. they may be periodic in Z or decay to zero as $Z \to \pm\infty$). The spatial dynamics formulation is derived by considering the functional in the variational principle $\delta\mathcal{V} = 0$ as an action functional in which ξ is the time-like variable, (η, Φ) are the coordinates and (η_ξ, Φ_ξ) the corresponding velocities; the Legendre transform yields the required evolutionary equation in the form of an (infinite-dimensional) Hamiltonian evolutionary system. A wide variety of three-dimensional water waves has been found using this method by Groves & Mielke [18], Groves [15] (who studied waves aligned parallel with and perpendicular to their direction of propagation) and Groves & Haragus [16] (who studied waves with an arbitrary orientation). In these references solutions are found using a reduction technique which shows that the infinite-dimensional Hamiltonian system is locally equivalent to a Hamiltonian system with finitely many degrees of freedom, whose solution set can be analysed.

A different technique was used by Craig & Nicholls [12] in an existence theory for doubly periodic water waves. The starting point of their analysis is again the variational principle (8), but they overcome the difficulty posed by the variable domain D_ρ by introducing a new variable $\xi = \phi|_{y=1+\rho}$ and expressing the variational functional in terms of ρ and ξ. The resulting expression, which is still quasilinear in character, involves the nonlocal 'Dirichlet-Neumann' operator $G(\rho)$ defined by $G(\rho)\xi = \nabla\phi.(-\rho_x, -\rho_z, 1)|_{y=1+\rho}$, where the potential function ϕ is the harmonic extension of ξ into D_ρ with Neumann data at $y = 0$. Craig & Nicholls apply a version of the variational Lyapunov-Schmidt reduction discussed in Section 1.2 above to show that their variational principle is locally equivalent to a finite-dimensional variational principle and find critical points of their reduced functional using topological arguments.

The method of Craig & Nicholls, like the result in the present paper, relies upon a *reduction* method which converts a global variational principle into a more tractable local variational principle. An alternative method is to *extend* a variational principle to a more general problem to which the direct methods of the calculus of variations can be applied. Buffoni, Séré & Toland [10] have recently used this approach in a study of two-dimensional periodic steady waves on deep water in the absence of surface tension. These authors use a conformal mapping of the fluid domain to the lower half-plane together with complex-variable methods; the relevant version of the variational principle (8) is transformed into a variational principle whose functional depends upon the single variable w defined implicitly by $\eta(x + \mathcal{C}w(x)) = w(x)$, where $\mathcal{C}$ is the Hilbert transform. This quasilinear functional is made semilinear by the addition of a regularising term (with higher derivatives), and *a priori* estimates are used to confirm that the detected critical points of the regularised functional are actually critical points of the original. The method has been extended to gravity-capillary solitary water waves (in finite and infinite depth) by Buffoni [6, 7].

There are several further variational results in the literature concerning two-dimensional steady water waves. Hamiltonian spatial dynamics methods have been successfully applied to the problem for gravity-capillary waves by Buffoni, Groves & Toland [9] and Buffoni & Groves [8], who found a multitude of solitary-wave solutions to this problem. Finally, Turner [37] found periodic and solitary-wave solutions to the problem for gravity waves by applying the direct methods of the calculus of variations. Turner used semi-Lagrangian coordinates to map the fluid domain into a strip; the resulting quasilinear variational functional is handled by extending it to a tractable semilinear problem and using *a priori* estimates to return to the original setting.

1.5 The functional-analytic framework

In this section we define the scaled function spaces in which the subsequent theory is developed and state the fixed-point theorem used to solve nonlinear equations in these spaces. Here, and in the remainder of this paper, we use the symbol c to denote a general positive constant (which in particular does not depend upon ε).

Function spaces

In the following analysis we use four basic spaces for functions of two real variables, namely

(i) the Hilbert space $X = \{u : |\!|\!|u|\!|\!| < \infty\}$, where

$$\langle\!\langle\!\langle u, v\rangle\!\rangle\!\rangle = \int_{\mathbb{R}^2} \Big\{ c_0(\varepsilon u_{xxx}v_{xxx} + 3\varepsilon^2 u_{xxz}v_{xxz} + 3\varepsilon^3 u_{xzz}v_{xzz} + \varepsilon^4 u_{zzz}v_{zzz}) + (\beta - \tfrac{1}{3})(u_{xx}v_{xx} + 2\varepsilon u_{xz}v_{xz} + \varepsilon^2 u_{zz}v_{zz}) + u_x v_x + (1+\varepsilon)u_z v_z \Big\}\,\mathrm{d}x\,\mathrm{d}z \qquad (23)$$

and $c_0 = \beta/2 - 2\alpha/15$;

(ii) the Banach space $W_\varepsilon^{\delta,p}(\mathbb{R}^2) = \{u : \|u\|_{\delta,p,\varepsilon} < \infty\}$, where

$$\|u\|_{\delta,p,\varepsilon} = \|\mathcal{F}^{-1}[(1+\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}u]\|_p,$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote respectively the Fourier and inverse Fourier transforms, (μ, k) is the independent variable associated with the Fourier transform in (x, z) and $\|\cdot\|_p$ is the $L^p(\mathbb{R}^2)$-norm;

(iii) the Banach space $V_\varepsilon^{\delta,p}(\mathbb{R}^2) = \{u : |u|_{\delta,p,\varepsilon} < \infty\}$, where

$$|u|_{\delta,p,\varepsilon} = \|\mathcal{F}^{-1}[(1 + \varepsilon^{\frac{1}{2}}(\mu^2 + \varepsilon k^2)^{\frac{1}{2}+\frac{\delta}{2}})\mathcal{F}u]\|_p;$$

(iv) the Banach space $U_\varepsilon^{\delta,p}(\mathbb{R}^2) = \{u : \|u\|_{U_\varepsilon^{\delta,p}} < \infty\}$, where

$$\|u\|_{U_\varepsilon^{\delta,p}} = \|u_x\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|u_z\|_{\delta,p,\varepsilon}.$$

The spaces $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ are scaled versions of the standard Sobolev spaces $W^{\delta,p}(\mathbb{R}^2)$ and $W^{1+\delta,p}(\mathbb{R}^2)$ defined using the Bessel potential (see Adams & Fournier [2, §7.63]); similarly X and $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ are scaled versions of familiar spaces in which only the derivatives of functions play a role. Both the scaling and the choice of coefficients c_0 and $\beta - 1/3$ used in the definition of X are dictated by the hydrodynamic problem (see Section 3.1); on the other hand the scalings used in the other spaces are chosen in view of their compatibility with X and usefulness in fixed-point arguments for solving nonlinear equations.

The following proposition states some of the basic properties of the above function spaces. Parts (i)–(iv) are proved by applying straightforward scaling arguments to well-known properties of the standard function spaces from which they are constructed, parts (v) and (vi) follow by scaling the results given by Mazya [29, §7.1.2], and part (vii) is obtained using the method described by Wang, Ablowitz & Segur [38, Lemma 1].

Proposition 1.1

(i) The function spaces $W_\varepsilon^{\delta_2,p}(\mathbb{R}^2)$ and $V_\varepsilon^{\delta_2,p}(\mathbb{R}^2)$ are continuously embedded in respectively $W_\varepsilon^{\delta_1,p}(\mathbb{R}^2)$ and $V_\varepsilon^{\delta_1,p}(\mathbb{R}^2)$ whenever $\delta_1 \leq \delta_2$; in particular we have the embedding inequalities

$$\|u\|_{\delta_1,p,\varepsilon} \leq \|u\|_{\delta_2,p,\varepsilon}, \quad |u|_{\delta_1,p,\varepsilon} \leq |u|_{\delta_2,p,\varepsilon}, \qquad \delta_1 \leq \delta_2.$$

(ii) The space $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is a Banach algebra and continuously embedded in $C_{\mathrm{b}}(\mathbb{R}^2)$ whenever $\delta > 2/p$; in particular we have the inequalities

$$\|uv\|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2p}}\|u\|_{\delta,p,\varepsilon}\|v\|_{\delta,p,\varepsilon}, \quad \|u\|_\infty \leq c\varepsilon^{-\frac{1}{2p}}\|u\|_{\delta,p,\varepsilon}, \qquad \delta > 2/p.$$

(iii) The inequality

$$\|u\|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{\delta}{2}}|u|_{\delta,p,\varepsilon}$$

holds for each $\delta \geq 0$.

(iv) The space X is continuously embedded in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ for $\delta \in [0, 1]$ and we have the embedding inequality

$$\|u\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{\frac{1}{2p}-\frac{1}{4}-\frac{\delta}{2}}|\!|\!|u|\!|\!|, \qquad \delta \in [0, 1]. \tag{24}$$

(v) The $W^{\delta,p}_\varepsilon(\mathbb{R}^2)$ norm may be replaced by the equivalent norm

$$\|u\|_{\delta,p,\varepsilon} = \|u\|_p + \|\mathcal{F}^{-1}[(\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}u]\|_p.$$

(vi) The $V^{\delta,p}_\varepsilon(\mathbb{R}^2)$ norm may be replaced by the equivalent norm

$$|u|_{\delta,p,\varepsilon} = \|u\|_p + \varepsilon^{\frac{1}{2}}\|\mathcal{F}^{-1}[(\mu^2+\varepsilon k^2)^{\frac{1}{2}+\frac{\delta}{2}}\mathcal{F}u]\|_p.$$

(vii) The sharper embedding inequality

$$\|u\|_{U^{1,p}_\varepsilon} \leq c|||u||| \tag{25}$$

holds whenever $p \in (2,6)$.

It is also necessary to consider functions $u = u(x,z)$ defined upon an open subset S of $\mathbb{R}^2$ (with smooth boundary); for this purpose we use the space X_S, whose norm is defined by formula (23) with the range of integration replaced by S, the space $W^{\delta,p}_\varepsilon(S)$, which is defined by interpolation (see below), and the space $U^{\delta,p}_\varepsilon(S)$, which is obtained from $W^{\delta,p}_\varepsilon(S)$ in the same way that $U^{\delta,p}_\varepsilon(\mathbb{R}^2)$ is obtained from $W^{\delta,p}_\varepsilon(\mathbb{R}^2)$. The function space $W^{\delta,p}_\varepsilon(S)$ defined by an interpolation procedure according to the formulae

$$W^{\delta,p}_\varepsilon(S) = \{u : \|u\|_{\delta,p,\varepsilon} < \infty\}, \quad \|u\|_{s,p,\varepsilon} = \sum_{i+k=0}^{s} \varepsilon^{\frac{k}{2}}\|\partial_x^i\partial_z^k u\|_p$$

for $s = 0,1,2,\ldots$ and

$$W^{\delta,p}_\varepsilon(S) = [W^{\lfloor\delta\rfloor,p}_\varepsilon(S), W^{\lceil\delta\rceil,p}_\varepsilon(S)]_{\delta-\lfloor\delta\rfloor}$$

for arbitrary $\delta \geq 0$, in which $\|\cdot\|_p$ is the $L^p(S)$-norm, the symbols $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ refer to the 'floor' and 'ceiling' of a positive real number and the interpolation is carried out in the sense of Lions & Magenes [25] (see also Adams & Fournier [2, §7.57]). Of course this procedure can also be used to define the space $W^{\delta,p}_\varepsilon(\mathbb{R}^2)$ itself, and in fact leads to a space which coincides with that constructed using the Fourier transform (see Adams & Fournier [2, §§7.50–7.66]). The following proposition states the key properties of X_S, $W^{\delta,p}_\varepsilon(S)$ and $U^{\delta,p}_\varepsilon(S)$; note that it is the compactness of certain embeddings rather than the size of embedding constants which is of most interest here.

Proposition 1.2 *Suppose that S is an open subset of $\mathbb{R}^2$ with smooth boundary.*

(i) The space $W^{\delta,p}_\varepsilon(S)$ is a Banach algebra and continuously embedded in $C_\mathrm{b}(S)$ whenever $\delta > 2/p$.

(ii) The space X_S is continuously embedded in $U^{\delta,p}_\varepsilon(S)$ for $\delta \in [0,1]$. The embedding is compact whenever S is bounded.

We also consider functions of three variables $(x,y,z) \in \Sigma$, where Σ is the strip $\{(x,y,z) : (x,z) \in \mathbb{R}^2, y \in (0,1)\}$, using the function space $W_\varepsilon^{\delta,p}(\Sigma)$ defined by an interpolation procedure according to the the formulae

$$W_\varepsilon^{\delta,p}(\Sigma) = \{u : \|u\|_{\delta,p,\varepsilon} < \infty\}, \quad \|u\|_{s,p,\varepsilon} = \sum_{i+j+k=0}^{s} \varepsilon^{\frac{k}{2}} \|\partial_x^i \partial_y^j \partial_z^k u\|_p$$

for $s = 0, 1, 2, \ldots$ and

$$W_\varepsilon^{\delta,p}(\Sigma) = [W_\varepsilon^{\lfloor\delta\rfloor,p}(\Sigma), W_\varepsilon^{\lceil\delta\rceil,p}(\Sigma)]_{\delta-\lfloor\delta\rfloor}$$

for arbitrary $\delta \geq 0$, in which $\|\cdot\|_p$ is the $L^p(\Sigma)$-norm. The space $U_\varepsilon^{\delta,p}(\Sigma) = \{u : \|u\|_{U_\varepsilon^{\delta,p}} < \infty\}$ is derived from $W^{\delta,p}(\Sigma)$ in the usual fashion, so that

$$\|u\|_{U_\varepsilon^{\delta,p}} = \|u_x\|_{\delta,p,\varepsilon} + \|u_y\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}} \|u_z\|_{\delta,p,\varepsilon}.$$

The following properties of $W_\varepsilon^{\delta,p}(\Sigma)$ are readily deduced from the fact that it is a scaled version of the standard interpolation space $W^{\delta,p}(\Sigma)$.

Proposition 1.3

(i) The space $W_\varepsilon^{\delta_2,p}(\Sigma)$ is continuously embedded in $W_\varepsilon^{\delta_1,p}(\Sigma)$ whenever $\delta_1 \leq \delta_2$; in particular we have the embedding inequality

$$\|u\|_{\delta_1,p,\varepsilon} \leq \|u\|_{\delta_2,p,\varepsilon}, \qquad \delta_1 \leq \delta_2.$$

(ii) The space $W_\varepsilon^{\delta,p}(\Sigma)$ is a Banach algebra and continuously embedded in $C_\mathrm{b}(\Sigma)$ whenever $\delta > 3/p$; in particular we have the inequalities

$$\|uv\|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2p}} \|u\|_{\delta,p,\varepsilon} \|v\|_{\delta,p,\varepsilon}, \quad \|u\|_\infty \leq c\varepsilon^{-\frac{1}{2p}} \|u\|_{\delta,p,\varepsilon}, \qquad \delta > 3/p.$$

Finally, we state some elementary properties of operators which arise naturally when passing between functions defined on $\mathbb{R}^2$ and functions defined on Σ.

Proposition 1.4

(i) The mapping

$$u \mapsto \int_0^1 u(\cdot, y)\, \mathrm{d}y$$

defines a bounded linear operator $W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{\delta,p}(\mathbb{R}^2)$.

(ii) The natural extension of $u : \mathbb{R}^2 \to \mathbb{R}$ to $u : \Sigma \to \mathbb{R}$ defines a bounded linear operator $W_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{\delta,p}(\Sigma)$.

(iii) The trace mapping $u \mapsto u|_S$ defines a bounded linear operator $W_\varepsilon^{1,2}(\Sigma) \to W_\varepsilon^{1/2,2}(\mathbb{R}^2)$ and $W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{\delta-1/p,p}(\mathbb{R}^2)$ for $p > 2$.

The norms of the linear operators listed above are all independent of ε.

A fixed-point theorem

A large part of the theory in this paper is taken up with solving fixed-point problems, and for this purpose we use the following fixed-point theorem, which is a straightforward extension of a standard argument in nonlinear analysis.

Theorem 1.5 *Let $\mathcal{X}$, $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be Banach spaces, X, $Y_1, \ldots, Y_n$ be closed subsets of respectively $\mathcal{X}$, $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ which contain the origin and $\mathcal{F} : X \times Y_1 \times \ldots \times Y_n \to \mathcal{X}$ be a smooth function. Suppose there exists a function $r : Y_1 \times \ldots \times Y_n \to [0, \infty)$ such that*

$$\|\mathcal{F}(0, y)\| \leq r/2, \qquad \|\mathrm{d}_1\mathcal{F}[x, y]\| \leq 1/2$$

for each $x \in \bar{B}_r(0) \subset X$ and each $y \in Y_1 \times \ldots \times Y_n$.

Under these hypotheses there exists for each $y \in Y_1 \times \ldots \times Y_n$ a unique solution $x = x(y)$ of the fixed-point equation

$$x = \mathcal{F}(x, y)$$

satisfying $x(y) \in \bar{B}_r(0)$. Moreover $x(y)$ is a smooth function of $y \in Y_1 \times \ldots \times Y_n$ and in particular we have the estimates

$$\|\mathrm{d}_i x[y_1, \ldots, y_n]\| \leq 2\|\mathrm{d}_{i+1}\mathcal{F}[x(y), y_1, \ldots, y_n]\|, \qquad i = 1, \ldots, n$$

for its first derivatives.

2 Reduction to a single pseudodifferential equation

2.1 Overview of the reduction method

We begin by introducing the transformation

$$y = \tilde{y}(1 + \rho(x, z)), \qquad \phi(x, y, z) = \Phi(x, \tilde{y}, z),$$

which maps the variable fluid domain $\mathcal{D}_\rho = \{(x, y, z) : (x, z) \in \mathbb{R}^2, \tilde{y} \in (0, \rho(x, z))\}$ bijectively into the fixed strip $\Sigma = \{(x, \tilde{y}, z) : (x, z) \in \mathbb{R}^2, \tilde{y} \in (0, 1)\}$, and the scaled variables

$$(\tilde{\rho}(\tilde{x}, \tilde{z}), \tilde{\Phi}(\tilde{x}, y, \tilde{z})) = (\varepsilon^{-1}\rho(x, z), \varepsilon^{-\frac{1}{2}}\Phi(x, y, z)), \qquad (\tilde{x}, \tilde{z}) = (\varepsilon^{\frac{1}{2}}x, \varepsilon z)$$

associated with the KP scaling limit. The hydrodynamic problem (1)–(4) is transformed into the equation

$$(1 + \varepsilon)\rho - \beta\varepsilon\rho_{xx} - \beta\varepsilon^2\rho_{zz} = \Phi_x|_{y=1} + \varepsilon^{-1}N_1(\rho, \Phi) \tag{26}$$

and the boundary-value problem

$$-\varepsilon\Phi_{xx} - \varepsilon^2\Phi_{zz} - \Phi_{yy} = \varepsilon^{-\frac{1}{2}}N_2(\rho, \Phi), \qquad 0 < y < 1, \tag{27}$$

$$\varepsilon\rho_x + \Phi_y = \varepsilon^{-\frac{1}{2}}N_3(\rho, \Phi) \qquad \text{on } y = 1, \tag{28}$$

$$\Phi_y = 0 \qquad \text{on } y = 0, \tag{29}$$

in which the tildes have been dropped for notational simplicity and the nonlinearities N_1, N_2, N_3 are given by the formulae

$$\begin{aligned}
&N_1(\rho,\Phi) = \\
&\quad \beta\varepsilon^2\left[\frac{\rho_x}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}}\right]_x - \beta\varepsilon^2\rho_{xx} + \beta\varepsilon^3\left[\frac{\rho_z}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}}\right]_z - \beta\varepsilon^3\rho_{zz} \\
&\quad -\int_0^1 \left\{\varepsilon^2\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)^2 + \varepsilon^3\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)^2\right. \\
&\qquad + \varepsilon^2\left(\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)y\Phi_y\right)_x + \varepsilon^3\left(\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)y\Phi_y\right)_z \\
&\qquad \left. + \varepsilon^{\frac{3}{2}}\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)\frac{y\Phi_y}{1+\varepsilon\rho} + \varepsilon^2\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)\frac{y\Phi_y}{1+\varepsilon\rho} + \frac{\varepsilon\Phi_y^2}{2(1+\varepsilon\rho)^2}\right\}\mathrm{d}y,
\end{aligned}$$

$$\begin{aligned}
&N_2(\rho,\Phi) = \\
&\quad \varepsilon^{\frac{5}{2}}(\rho\Phi_x)_x + \varepsilon^{\frac{7}{2}}(\rho\Phi_z)_z - \varepsilon^{\frac{5}{2}}(y\Phi_y\rho_x)_x - \varepsilon^{\frac{7}{2}}(y\Phi_y\rho_z)_z \\
&\quad - \varepsilon^{\frac{5}{2}}\left(\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right)y\rho_x\right)_y - \varepsilon^{\frac{7}{2}}\left(\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right)y\rho_z\right)_y + \frac{\varepsilon^{\frac{3}{2}}\rho\Phi_{yy}}{1+\varepsilon\rho},
\end{aligned}$$

$$\begin{aligned}
&N_3(\rho,\Phi) = \\
&\quad \left[\varepsilon^{\frac{5}{2}}\rho_x\left(\Phi_x - \frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right) + \varepsilon^{\frac{7}{2}}\rho_z\left(\Phi_z - \frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right) - \frac{\varepsilon^{\frac{3}{2}}\rho\Phi_y}{1+\varepsilon\rho}\right]_{y=1}.
\end{aligned}$$

The goal of this paper is to find solutions (ρ,Φ) of the scaled equations (26)–(29) which lie in $[W^{1,2}(\mathbb{R}^2)\times U^{0,2}(\Sigma)]\cap[W^{2,p}(\mathbb{R}^2)\times U^{1,p}(\Sigma)]$ for all sufficiently large values of $p>2$; the trace $\Phi_x|_{y=1}$ and nonlinearities N_1, N_2, N_3 are well defined and smooth (in a neighbourhood of the origin) in these function spaces. We refer to such solutions as *strong solutions* of (26)–(29). Our strategy is to seek *weak solutions* of these equations which lie in the larger function space $[W^{1,2}(\mathbb{R}^2)\times U^{0,2}(\Sigma)]\cap[W^{1+\delta,p}(\mathbb{R}^2)\times U^{\delta,p}(\Sigma)]$ for sufficiently small values of $\delta\in(0,1)$ and establish a regularity result that weak solutions are in fact strong solutions. We always choose δ and p with $\delta>3/p$ so that the weak forms of the nonlinearities are well defined and smooth. It is moreover necessary to work in scaled versions of these function spaces in order to solve certain fixed-point equations, and we therefore henceforth employ the spaces $[V_\varepsilon^{0,2}(\mathbb{R}^2)\times U_\varepsilon^{0,2}(\Sigma)]\cap[V_\varepsilon^{1,p}(\mathbb{R}^2)\times U_\varepsilon^{1,p}(\Sigma)]$ and $[V_\varepsilon^{0,2}(\mathbb{R}^2)\times U_\varepsilon^{0,2}(\Sigma)]\cap[V_\varepsilon^{\delta,p}(\mathbb{R}^2)\times U_\varepsilon^{\delta,p}(\Sigma)]$ for strong and weak solutions.

Definition 2.1 *A* <u>*weak solution*</u> *of (26)–(29) is a pair (ρ,Φ) of functions which lie in $[V_\varepsilon^{0,2}(\mathbb{R}^2)\times U_\varepsilon^{0,2}(\Sigma)]\cap[V_\varepsilon^{\delta,p}(\mathbb{R}^2)\times U_\varepsilon^{\delta,p}(\Sigma)]$ and satisfy*

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left\{(1+\varepsilon)\rho\omega + \beta\varepsilon^2\rho_x\omega_x + \beta\varepsilon^4\rho_z\omega_z\,\mathrm{d}x\,\mathrm{d}z\right. \\
&\qquad\qquad = -\int_{\mathbb{R}^2}\int_0^1(\omega_x y\Phi_y - \omega\Phi_x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \int_{\mathbb{R}^2}\varepsilon^{-1}N_1(\rho,\Phi)\omega\,\mathrm{d}x\,\mathrm{d}z, \qquad (30)
\end{aligned}$$

$$\int_{\mathbb{R}^2} \left\{ \int_0^1 (\Phi_y \Psi_y + \varepsilon \Phi_x \Psi_x + \varepsilon^2 \Phi_z \Psi_z)\, \mathrm{d}y + \rho_x \Psi|_{y=1} \right\} \mathrm{d}x\, \mathrm{d}z$$
$$= \int_{\mathbb{R}^2} \int_0^1 \varepsilon^{-\frac{1}{2}} N_4(\rho, \Phi) \Psi \, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z + \int_{\mathbb{R}^2} \int_0^1 \varepsilon^{-\frac{1}{2}} N_5(\rho, \Phi) \hat{\Psi}_y \, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z \quad (31)$$

for all $(\omega, \Psi) \in V_\varepsilon^{0,2}(\mathbb{R}^2) \times W_\varepsilon^{1,2}(\Sigma)$ (or any dense subset thereof). Here

$$N_4(\rho, \Phi) = \varepsilon^{\frac{5}{2}} (\rho \Phi_x)_x + \varepsilon^{\frac{7}{2}} (\rho \Phi_z)_z - \varepsilon^{\frac{5}{2}} (y \Phi_y \rho_x)_x - \varepsilon^{\frac{7}{2}} (y \Phi_y \rho_z)_z,$$
$$N_5(\rho, \Phi) = \varepsilon^{\frac{5}{2}} \left(\Phi_x - \frac{\varepsilon y \rho_x \Phi_y}{1 + \varepsilon \rho} \right) y \rho_x + \varepsilon^{\frac{7}{2}} \left(\Phi_z - \frac{\varepsilon y \rho_z \Phi_y}{1 + \varepsilon \rho} \right) y \rho_z - \frac{\varepsilon^{\frac{3}{2}} \rho \Phi_y}{1 + \varepsilon \rho}$$

and the 'outer' derivatives with respect to x and z in N_4 and N_1 are transferred to respectively Ψ and ω by an integration by parts.

Let us now outline the strategy we use to find weak solutions of the scaled water-wave problem. We begin by fixing Φ and examining the equation for ρ. The first step here is to take the Fourier transform of the strong form (26) of the equation for ρ, so that

$$\hat{\rho} = \frac{1}{1 + \varepsilon + \beta q^2} \left(\mathrm{i}\mu \int_0^1 y \hat{\Phi}_y \, \mathrm{d}y + \int_0^1 \hat{\Phi}_x \, \mathrm{d}y + \varepsilon^{-1} \hat{N}_1(\rho, \Phi) \right), \quad (32)$$

where $q^2 = \varepsilon \mu^2 + \varepsilon^2 k^2$ and we have used the identity

$$\Phi|_{y=1} = \int_0^1 y \Phi_y \, \mathrm{d}y + \int_0^1 \Phi \, \mathrm{d}y.$$

Inspecting this equation, one finds that it is well defined for (ρ, Φ) in the larger function class $[V_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,2}(\Sigma)] \cap [V_\varepsilon^{\delta,p}(\mathbb{R}^2) \times U_\varepsilon^{\delta,p}(\Sigma)]$, and in this setting we refer to it as the *integral form of the equation for ρ*. We can also obtain a weak form of the equation for ρ by multiplying the strong form by a test function and integrating by parts.

Definition 2.2 *Suppose that $\Phi \in U_\varepsilon^{0,2}(\Sigma) \cap U_\varepsilon^{\delta,p}(\Sigma)$. A weak solution of the equation for ρ is a function $\rho^\star \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ which satisfies*

$$\int_{\mathbb{R}^2} \left\{ (1 + \varepsilon) \rho^\star \omega + \beta \varepsilon^2 \rho_x^\star \omega_x + \beta \varepsilon^4 \rho_z^\star \omega_z \, \mathrm{d}x\, \mathrm{d}z \right.$$
$$= - \int_{\mathbb{R}^2} \int_0^1 (\omega_x y \Phi_y - \omega \Phi_x)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z + \int_{\mathbb{R}^2} \varepsilon^{-1} N_1(\rho^\star, \Phi) \omega \, \mathrm{d}x\, \mathrm{d}z,$$

for all $\omega \in V_\varepsilon^{0,2}(\mathbb{R}^2)$ (or any dense subset thereof); here the 'outer' derivatives with respect to x and z in N_1 are transferred to ω by an integration by parts.

The weak and integral forms of the equation for ρ are in fact equivalent.

Proposition 2.3 *Suppose that $\Phi \in U_\varepsilon^{0,2}(\Sigma) \cap U_\varepsilon^{\delta,p}(\Sigma)$. A function $\rho^\star \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ solves the integral form of the equation for ρ if and only if it is a weak solution of the equation for ρ.*

The boundary-value problem for Φ yields integral and weak formulations in an analogous fashion. Taking the Fourier transform of the strong from (27)–(29) of the equations for Φ and using (32) to eliminate ρ from the linear part of the equations, we obtain the boundary-value problem

$$\begin{aligned} &-\hat{\Phi}_{yy}+q^2\hat{\Phi}=\varepsilon^{-\frac{1}{2}}\hat{N}_2(\rho,\Phi), && 0<y<1,\\ &\hat{\Phi}_y-\frac{\varepsilon\mu^2\hat{\Phi}}{1+\varepsilon+\beta q^2}=\frac{-\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)+\varepsilon^{-\frac{1}{2}}N_3(\rho,\Phi) && \text{on } y=1,\\ &\Phi_y=0 && \text{on } y=0. \end{aligned}$$

This boundary-value problem can be recast as the single equation

$$\hat{\Phi}=-\int_0^1 G\varepsilon^{-\frac{1}{2}}\hat{N}_2(\rho,\Phi)\,\mathrm{d}\xi-G|_{\xi=1}\left(\frac{-\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)+\varepsilon^{-\frac{1}{2}}\hat{N}_3(\rho,\Phi)\right),$$

in which the Green's function $G(y,\xi)$ is given by

$$G(y,\xi)=\begin{cases} \dfrac{\cosh qy}{\cosh q}\dfrac{(1+\varepsilon+\beta q^2)\cosh q(1-\xi)+(\varepsilon\mu^2/q)\sinh q(\xi-1)}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}, & 0<y<\xi<1,\\[2ex] \dfrac{\cosh q\xi}{\cosh q}\dfrac{(1+\varepsilon+\beta q^2)\cosh q(1-y)+(\varepsilon\mu^2/q)\sinh q(y-1)}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}, & 0<\xi<y<1, \end{cases}$$

and an integration by parts yields the alternative representation

$$\hat{\Phi}=-\int_0^1 G\varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}\xi-\int_0^1 G_\xi\varepsilon^{-\frac{1}{2}}\hat{N}_5(\rho,\Phi)\,\mathrm{d}\xi+\frac{\mathrm{i}\mu G|_{\xi=1}}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi). \tag{33}$$

Equation (33) is well defined for $(\rho,\Phi)\in[V_\varepsilon^{0,2}(\mathbb{R}^2)\times U_\varepsilon^{0,2}(\Sigma)]\cap[V_\varepsilon^{\delta,p}(\mathbb{R}^2)\times U_\varepsilon^{\delta,p}(\Sigma)]$, and in this setting we refer to it as the *integral form of the equation for* Φ.

The appropriate weak form of the equation for Φ is found by multiplying the above boundary problem by a test function and integrating by parts.

Definition 2.4 *Suppose that* $\rho\in V_\varepsilon^{0,2}(\mathbb{R}^2)\cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$. *A* <u>*weak solution*</u> *of the problem for* Φ *is a function* $\Phi^\star\in U_\varepsilon^{0,2}(\Sigma)\cap U_\varepsilon^{\delta,p}(\Sigma)$ *which satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^2}\left\{\int_0^1(\hat{\Phi}^\star_y\bar{\hat{\Psi}}_y+q^2\hat{\Phi}^\star\bar{\hat{\Psi}})\,\mathrm{d}y-\frac{1}{1+\varepsilon+\beta q^2}\left(\varepsilon\mu^2\int_0^1 y\hat{\Phi}_y\,\mathrm{d}y-\mathrm{i}\mu\int_0^1\hat{\Phi}_x\,\mathrm{d}y\right)\bar{\hat{\Psi}}|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k\\ &\quad=\int_{\mathbb{R}^2}\left\{\int_0^1(\varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi^\star)\bar{\hat{\Psi}}+\varepsilon^{-\frac{1}{2}}\hat{N}_5(\rho,\Phi^\star)\bar{\hat{\Psi}}_y)\,\mathrm{d}y-\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi^\star)\bar{\hat{\Psi}}|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k \end{aligned}$$

for all $\Psi\in W_\varepsilon^{1,2}(\Sigma)$ *(or any dense subset thereof); the 'outer' derivatives with respect to* x *and* z *in* N_4 *and* N_1 *are transferred to respectively* Ψ *and* ω *by an integration by parts.*

The next proposition shows that it is sufficient to consider the integral form of the equation for Φ when seeking weak solutions.

Proposition 2.5 *Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$. A solution $\Phi^\star \in U_\varepsilon^{0,2}(\Sigma) \cap U_\varepsilon^{\delta,p}(\Sigma)$ of the integral form of the problem for Φ is a weak solution of the problem for Φ.*

Proof. With slightly more generality we consider the problem posed by the above equations in which N_5 is an arbitrary function in $L^2(\Sigma)$, N_4 is an arbitrary function of the form

$$\hat{N}_4 = \mathrm{i}\mu \hat{N}_4^1 + \mathrm{i}\varepsilon^{\frac{1}{2}} k \hat{N}_4^2, \qquad N_4^1, N_4^2 \in L^2(\Sigma)$$

and N_1 is an arbitrary function of the form

$$\hat{N}_1 = \hat{N}_1^1 + \mathrm{i}\mu \hat{N}_1^2 + \mathrm{i}\varepsilon^{\frac{1}{2}} k \hat{N}_1^3, \qquad N_1^1, N_1^2, N_1^3 \in L^2(\mathbb{R}^2).$$

Fix $\Psi \in W_\varepsilon^{1,2}(\Sigma)$ and observe that any solution of (33) satisfies

$$\begin{aligned}
&\int_0^1 (\hat{\Phi}_y^\star \bar{\hat{\Psi}}_y + q^2 \hat{\Phi}^\star \bar{\hat{\Psi}})\,\mathrm{d}y - \frac{1}{1+\varepsilon+\beta q^2}\left(\varepsilon\mu^2 \int_0^1 y\hat{\Phi}_y\,\mathrm{d}y - \mathrm{i}\mu\int_0^1 \hat{\Phi}_x\,\mathrm{d}y\right)\bar{\hat{\Psi}}|_{y=1} \\
&\quad = -\int_0^1\!\!\int_0^1 \frac{G_y}{\varepsilon^{1/2}}\hat{N}_4\,\mathrm{d}\xi\,\bar{\hat{\Psi}}_y\,\mathrm{d}y - \int_0^1\!\!\int_0^1 \frac{q^2 G}{\varepsilon^{1/2}}\hat{N}_4\,\mathrm{d}\xi\,\bar{\hat{\Psi}}\,\mathrm{d}y + \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}\int_0^1 \frac{G}{\varepsilon^{1/2}}\hat{N}_4\,\mathrm{d}\xi\,\bar{\hat{\Psi}}|_{y=1} \\
&\qquad - \int_0^1\!\!\int_0^1 \frac{G_{y\xi}}{\varepsilon^{1/2}}\hat{N}_5\,\mathrm{d}\xi\,\bar{\hat{\Psi}}_y\,\mathrm{d}y - \int_0^1\!\!\int_0^1 \frac{q^2 G_\xi}{\varepsilon^{1/2}}\hat{N}_5\,\mathrm{d}\xi\,\bar{\hat{\Psi}}\,\mathrm{d}y + \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}\int_0^1 \frac{G_\xi}{\varepsilon^{1/2}}\hat{N}_5\,\mathrm{d}\xi\,\bar{\hat{\Psi}}|_{y=1} \\
&\qquad + \int_0^1 \frac{\mathrm{i}\mu G_y|_{\xi=1}}{1+\varepsilon+\beta q^2}\hat{N}_1\bar{\hat{\Psi}}_y\,\mathrm{d}y + \int_0^1 \frac{\mathrm{i}\mu q^2 G|_{\xi=1}}{1+\varepsilon+\beta q^2}\hat{N}_1\bar{\hat{\Psi}}\,\mathrm{d}y - \frac{\varepsilon\mathrm{i}\mu^3 G|_{\xi=1}}{(1+\varepsilon+\beta q^2)^2}\hat{N}_1\bar{\hat{\Psi}}|_{y=1}.
\end{aligned}$$

Suppose first that N_5 belongs to the dense subset $W_0^{1,2}(\Sigma)$ of $L^2(\Sigma)$. A straightforward calculation using integration by parts and the properties of the Green's function G shows that the first, second and third lines on the right-hand side of the above expression are equal to respectively

$$\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4\bar{\hat{\Psi}}\,\mathrm{d}y, \qquad \int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_5\bar{\hat{\Psi}}_y\,\mathrm{d}y, \qquad -\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1\bar{\hat{\Psi}}|_{y=1};$$

the extra regularity of N_5 is required to obtain the second equality. Integrating with respect to (μ, k) over $\mathbb{R}^2$, we find that $\Phi^\star$ is a weak solution of the equation for Φ.

It remains to confirm that

$$\begin{aligned}
&-\int_{\mathbb{R}^2}\int_0^1\!\!\int_0^1 \frac{G_{y\xi}}{\varepsilon^{1/2}}\hat{N}_5\,\mathrm{d}\xi\,\bar{\hat{\Psi}}_y\,\mathrm{d}y\,\mathrm{d}\mu\,\mathrm{d}k - \int_{\mathbb{R}^2}\int_0^1\!\!\int_0^1 \frac{q^2 G_\xi}{\varepsilon^{1/2}}\hat{N}_5\,\mathrm{d}\xi\,\bar{\hat{\Psi}}\,\mathrm{d}y\,\mathrm{d}\mu\,\mathrm{d}k \\
&\quad + \int_{\mathbb{R}^2}\frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}\int_0^1 \frac{G_\xi}{\varepsilon^{1/2}}\hat{N}_5\,\mathrm{d}\xi\,\bar{\hat{\Psi}}|_{y=1}\,\mathrm{d}\mu\,\mathrm{d}k - \int_{\mathbb{R}^2}\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_5\bar{\hat{\Psi}}_y\,\mathrm{d}y\,\mathrm{d}\mu\,\mathrm{d}k = 0
\end{aligned}$$

for a general function $N_5 \in L^2(\Sigma)$. Using the results presented in Lemma 2.15 below, we find that the left-hand side of this equation defines a continuous function $L^2(\Sigma) \to \mathbb{R}$ of N_5, and since it vanishes for $N_5 \in W_0^{1,2}(\Sigma)$ a standard density argument asserts that it also vanishes for each $N_5 \in L^2(\Sigma)$. □

The next step is to decompose the Green's function into a singular and a smooth part using the formula

$$G = -\frac{1+\varepsilon}{\varepsilon^2 Q} + \varepsilon^{-2}G_1,$$

where

$$Q = k^2(1+\varepsilon) + \mu^2 + (\beta - \tfrac{1}{3})\varepsilon^{-2}q^4 + c_0\varepsilon^{-2}q^6,$$

and to define functions $\Phi_1(x,z)$ and $\Phi_2(x,y,z)$ by replacing G with respectively its first and second component in the integral form of the equation for Φ, so that

$$\hat{\Phi}_1 = \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}\xi - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right), \tag{34}$$

$$\hat{\Phi}_2 = -\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_5(\rho,\Phi)\,\mathrm{d}\xi + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_1(\rho,\Phi). \tag{35}$$

It is a straightforward matter to confirm that equations (34), (35) are equivalent to equation (33).

Proposition 2.6

(i) Any solution of the integral form (33) of the equation for Φ can be expressed as the sum $\Phi = \Phi_1 + \Phi_2$, where Φ_1, Φ_2 solve (34), (35).

(ii) Suppose conversely that Φ_1, Φ_2 satisfy equations (34), (35) with $\Phi = \Phi_1 + \Phi_2$. The function Φ satisfies equation (33).

In keeping with this proposition, we henceforth abandon the integral form of the equation for Φ and work instead with (34), (35) with $\Phi = \Phi_1 + \Phi_2$ on their right-hand sides; these equations are the *integral forms of the equations for Φ_1 and Φ_2*. Equation (34) is valid for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\Phi_1 \in X$, $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$, while equation (35) is valid for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\Phi_1 \in U_\varepsilon^{1,2}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$. Notice the difference in the regularity requirements for Φ_1 here; in fact membership of $U_\varepsilon^{1,2}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is implied by membership of X, and this fact plays a key role in the existence theory presented in Section 3.2 below. It is convenient to place a further requirement upon Φ_1 in relation to the integral form of the problem for Φ_2, namely that it should also lie in $U_\varepsilon^{0,4}(\mathbb{R}^2)$ (which is again a subset of X). This restriction allows one to obtain better estimates for the Φ_2 equation in the subsequent existence theory; we also apply it in the requirements for a weak solution of the equation for Φ_2.

Strong and weak forms of the equations for Φ_1 and Φ_2 are derived in the usual fashion. The strong form of the equation for Φ_1 is clearly

$$\begin{aligned}&\frac{\varepsilon^2}{1+\varepsilon}[-c_0\varepsilon(\partial_x^2+\varepsilon\partial_z^2)^3 + (\beta-\tfrac{1}{3})(\partial_x^2+\varepsilon\partial_z^2)^2 - (1+\varepsilon)\partial_z^2 - \partial_x^2]\Phi_1\\ &\qquad = \int_0^1 \varepsilon^{-\frac{1}{2}} N_4(\rho,\Phi)\,\mathrm{d}\xi - \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right]\end{aligned}$$

and is well defined for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{1,p}(\mathbb{R}^2)$, $\Phi_1 \in U_\varepsilon^{5,p}(\mathbb{R}^2)$, $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{2,p}(\Sigma)$, while the strong form of the equation for Φ_2 is calculated by substituting

$$\Phi = \Phi_2 + \mathcal{F}^{-1}\left[\frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}\xi - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right)\right]$$

into the strong form of the equation for Φ; one finds that

$$-\hat{\Phi}_{2yy} + q^2\hat{\Phi}_2 = \varepsilon^{-\frac{1}{2}}\hat{N}_2(\rho,\Phi) - \frac{q^2(1+\varepsilon)}{\varepsilon^2 Q}\left[\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right], \qquad 0<y<1, \tag{36}$$

$$\hat{\Phi}_{2y} - \frac{\varepsilon\mu^2\hat{\Phi}_2}{1+\varepsilon+\beta q^2} = \varepsilon^{-\frac{1}{2}}\hat{N}_3(\rho,\Phi) - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi) + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}\left[\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right], \qquad y=1, \tag{37}$$

$$\hat{\Phi}_{2y} = 0, \qquad y=0, \tag{38}$$

and this boundary-value problem is well defined for $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2)\cap V_\varepsilon^{2,p}(\mathbb{R}^2)$, $\Phi_1 \in U_\varepsilon^{1,2}(\mathbb{R}^2)\cap U_\varepsilon^{1,p}(\mathbb{R}^2)$ and $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma)\cap W_\varepsilon^{2,p}(\Sigma)$.

Definition 2.7

(i) Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2)\cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma)\cap W_\varepsilon^{1+\delta,p}(\Sigma)$. A weak solution of the equation for Φ_1 is a function $\Phi_1^\star \in X$ which satisfies

$$\langle\!\langle\!\langle \Phi_1^\star, \Psi_1 \rangle\!\rangle\!\rangle = \frac{1+\varepsilon}{\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}N_4(\rho,\Phi)\,\mathrm{d}\xi - \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right]\right)\bar{\hat{\Psi}}_1\,\mathrm{d}x\,\mathrm{d}z$$

for all $\Psi_1 \in X$ (or any dense subset thereof); here $\Phi = \Phi_1^\star + \Phi_2$ and the 'outer' derivatives with respect to x and z in N_4 and N_1 are transferred to Ψ_2 by an integration by parts.

(ii) Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2)\cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $\Phi_1 \in U_\varepsilon^{0,2}(\mathbb{R}^2)\cap U_\varepsilon^{0,4}(\mathbb{R}^2)\cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$. A weak solution of the problem for Φ_2 is a function $\Phi_2^\star \in W_\varepsilon^{1,2}(\Sigma)\cap W_\varepsilon^{1+\delta,p}(\Sigma)$ which satisfies

$$\int_{\mathbb{R}^2}\Bigg\{\int_0^1(\hat{\Phi}_{2y}^\star\bar{\hat{\Psi}}_y + q^2\hat{\Phi}_2^\star\bar{\hat{\Psi}})\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2^\star|_{y=1}\bar{\hat{\Psi}}|_{y=1}}{1+\varepsilon+\beta q^2} + \left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\right) \times \left(\frac{(1+\varepsilon)q^2}{\varepsilon^2 Q}\int_0^1 \bar{\hat{\Psi}}_2\,\mathrm{d}y - \frac{(1+\varepsilon)\varepsilon\mu^2\bar{\hat{\Psi}}_2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}\right)\Bigg\}\,\mathrm{d}\mu\,\mathrm{d}k =$$

$$\int_{\mathbb{R}^2}\left\{\int_0^1(\varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho,\Phi)\bar{\hat{\Psi}}_2 + \varepsilon^{-\frac{1}{2}}\hat{N}_5(\rho,\Phi)\bar{\hat{\Psi}}_{2y})\,\mathrm{d}y - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho,\Phi)\bar{\hat{\Psi}}_2|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k$$

for all $\Psi_2 \in W_\varepsilon^{1,2}(\Sigma)$ (or any dense subset thereof); here $\Phi = \Phi_1 + \Phi_2^\star$ and the 'outer' derivatives with respect to x and z in N_4 and N_1 are transferred to Ψ_1 by an integration by parts.

The next result is obtained using the arguments given in Propositions 2.3 and 2.5.

Proposition 2.8

(i) Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $\Phi_2 \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$*. A function* $\Phi_1^\star \in X$ *solves the integral form of the equation for* Φ_1 *if and only if it is a weak solution of the equation for* Φ_1*.*

(ii) Suppose that $\rho \in V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $\Phi_1 \in U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$*. A solution* $\Phi_2^\star \in W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$ *of the integral form of the problem for* Φ_2 *is a weak solution of the problem for* Φ_2*.*

We now proceed in a fashion reminiscent of the classical Lyapunov-Schmidt reduction. In this method a problem is treated by writing it as a pair of coupled equations for two unknowns X and Y; one of the equations is solved to yield the functional relationship $Y = Y(X)$, and inserting this function into the other equation one obtains the 'reduced equation' for X. We use this two-step approach for our water-wave problem in the following manner. Firstly we apply fixed-point principles to solve the integral forms of the equations for ρ and Φ_2 for ρ, Φ_2 as functions of Φ_1 and secondly we substitute the solutions $\rho = \rho(\Phi_1)$, $\Phi_2 = \Phi_2(\Phi_1)$ into the integral form of the equation for Φ_1 to obtain a reduced equation for Φ_1. The result of the first step is stated in the following theorem, whose proof is given in Sections 2.2 and 2.3 below.

Theorem 2.9 *Suppose that* Φ_1 *belongs to* $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *with* $\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-1/4-\Delta}$*. For sufficiently small values of* δ *and sufficiently large values of* p *(with* $\delta > 3/p$*) the integral forms of the equations for* ρ *and* Φ_2 *admit unique solutions* $\rho = \rho(\Phi_1)$ *in* $V_\varepsilon^{0,2}(\mathbb{R}^2) \cap V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ *and* $\Phi_2 \in \Phi_2(\Phi_1)$ *in* $W_\varepsilon^{1,2}(\Sigma) \cap W_\varepsilon^{1+\delta,p}(\Sigma)$ *that satisfy*

$$\begin{aligned}
|\rho|_{\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}(\|\Phi_{1x}\|_{\delta,p,\varepsilon} + P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),\\
\|\Phi_2\|_{1+\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}),\\
\|\Phi_{2y}\|_{\delta,p,\varepsilon} &\leq c\varepsilon^{\frac{1}{2}-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}),
\end{aligned}$$

$$\begin{aligned}
|\rho|_{0,2,\varepsilon} &\leq c(\|\Phi_{1x}\|_2 + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,4}}^2 + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),\\
\|\Phi_2\|_{1,2,\varepsilon} &\leq c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,4}}^2 + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})),\\
\|\Phi_{2y}\|_2 &\leq c(\varepsilon\|\Phi_1\|_{U_\varepsilon^{0,4}}^2 + \varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}));
\end{aligned}$$

the functions ρ *and* Φ_2 *depend smoothly upon* Φ_1 *in the topology defined by these function spaces. The symbols* Δ *and* P_n *denote respectively a quantity which is* $O(\delta + 1/p)$ *and a polynomial which has unit positive coefficients and no monomials of degree less than* n*.*

Substituting $\rho = \rho(\Phi_1)$ and $\Phi_2 = \Phi_2(\Phi_1)$ into the integral form of the equation for Φ_1 we obtain the (integral form of the) reduced equation

$$\hat{\Phi}_1 = \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))\,\mathrm{d}\xi - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))\right)$$

for the single variable $\Phi_1 \in X$; the nonlinearities are well defined since X is continuously embedded in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $U_\varepsilon^{0,2}(\mathbb{R}^2)$ and $U_\varepsilon^{0,4}(\mathbb{R}^2)$ (see Proposition 1.1(iv)). The above analysis shows that any solution $\Phi_1^\star$ of this equation generates a weak solution (ρ, Φ) of the original hydrodynamic problem, where $\rho = \rho(\Phi_1^\star)$ and $\Phi = \Phi_1^\star + \Phi_2^\star(\Phi_1^\star)$. We now study the derivational aspects of this equation in detail; the details of the procedure used to solve the equations for ρ and Φ are given in Sections 2.2 and 2.3, while Section 2.4 presents a regularity theory which assures that any solution of the integral form of the reduced equation for Φ_1 in fact defines a strong solution of the hydrodynamic problem.

2.2 Elimination of the variable ρ

In this section we show how the integral form of the equation for ρ can be solved for ρ as a function of Φ. Anticipating the later stages of our analysis, we suppose that Φ admits a decomposition of the type

$$\Phi(x, y, z) = \Phi_1(x, z) + \Phi_2(x, y, z)$$

and consider the integral form of the equation for ρ in the form

$$\hat{\rho} = \frac{1}{1 + \varepsilon + \beta q^2}\left(\hat{\Phi}_{1x} + \mathrm{i}\mu \int_0^1 y\hat{\Psi}\,\mathrm{d}y + \int_0^1 \hat{\Phi}_{2x}\,\mathrm{d}y + \varepsilon^{-1}\hat{N}_1(\rho, \Phi_1 + \Phi_2)\right).$$

The new variable Ψ is identified with Φ_{2y} later; we introduce it here since it plays a significant role in the solution of the equation for Φ_2 in Section 2.3 below.

Let us therefore write the integral form of the equation for ρ as

$$\rho = \mathcal{F}_1(\rho, \Psi, \Phi_1, \Phi_2) \tag{39}$$

and solve this fixed-point problem for ρ as a function of Φ_1, Φ_2 and Ψ. For this purpose we need precise estimates on the norms of the Fourier-multiplier operators appearing in (39); the requisite information is given in the following lemma, whose proof is deferred to Section 4.

Lemma 2.10 *The following statements hold for each $\delta \in [0, 1]$ and $p \in (1, \infty)$.*

(i) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_1(u) = \mathcal{F}^{-1}\left[\frac{1}{1 + \varepsilon + \beta q^2}\mathcal{F}[u]\right]$$

belongs to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$|\mathcal{G}_1(u)|_{\delta,p,\varepsilon} \leq c\|u\|_{\delta,p,\varepsilon}.$$

(ii) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_2(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1 + \varepsilon + \beta q^2}\mathcal{F}[u]\right]$$

belongs to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$|\mathcal{G}_2(u)|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|u\|_{\delta,p,\varepsilon}.$$

(iii) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_3(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$|\mathcal{G}_3(u)|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|u\|_{\delta,p,\varepsilon}.$$

We now solve the fixed-point problem (39) by applying our basic fixed-point theorem (Theorem 1.5); the technique developed for this purpose in the following result involves showing that $\mathcal{F}_1$ is a contraction whose Lipschitz constant is bounded by a positive power of ϵ. We henceforth adopt the notation introduced in Theorem 2.9 that Δ is a quantity which is bounded by $c(\delta+1/p)$; it is always supposed to be as small as required for the result in question by taking δ sufficiently small and p sufficiently large while maintaining the relationship $\delta > 3/p$.

Theorem 2.11 *Suppose that*

$$\|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}. \tag{40}$$

Equation (39) has a unique solution $\rho = \rho(\Psi, \Phi_1, \Phi_2)$ which satisfies the estimate

$$|\rho|_{\delta,p,\varepsilon} \leq c(\|\Phi_x\|_{\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{\delta,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2). \tag{41}$$

Moreover ρ is a smooth function of (Ψ, Φ_1, Φ_2) with respect to the $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $W_\varepsilon^{\delta,p}(\Sigma) \times U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma)$ topologies and in particular its first derivatives with respect to Ψ and Φ_2 satisfy the estimates

$$|\rho_\Psi \tilde{\Psi}|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}, \qquad |\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}.$$

Proof. This result is established by applying Theorem 1.5 with $\mathcal{X} = V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{Y}_1 = W_\varepsilon^{\delta,p}(\Sigma)$, $\mathcal{Y}_2 = U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{Y}_3 = W_\varepsilon^{1+\delta,p}(\Sigma)$ and X, Y_1, Y_2, Y_3 closed origin-centred balls of radius $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$, $\mathcal{O}(\varepsilon^{\frac{1}{2}-\Delta})$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$, $\mathcal{O}(\varepsilon^{-\Delta})$. According to this theorem, we have to verify that

$$|\mathcal{F}_1(0,\Psi,\Phi_1,\Phi_2)|_{\delta,p,\varepsilon} \leq c(\|\Phi_x\|_{\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{\delta,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2) \tag{42}$$

and that

$$|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2]|_{V_\varepsilon^{\delta,p}(\mathbb{R}^2)\to V_\varepsilon^{\delta,p}(\mathbb{R}^2)} \leq \frac{1}{2} \tag{43}$$

whenever (40) and (41) hold.

To verify (42) note that

$$N_1(0,\Phi) = \\ -\int_0^1 \left\{\frac{\varepsilon^2}{2}\Phi_x^2 + \frac{\varepsilon^3}{2}\Phi_z^2 + \varepsilon^2(\Phi_x y\Phi_y)_x + \varepsilon^3(\Phi_z y\Phi_y)_z + \varepsilon^{\frac{3}{2}}\Phi_x y\Phi_y + \varepsilon^2\Phi_z y\Phi_y + \frac{\varepsilon}{2}\Phi_y^2\right\}\mathrm{d}y,$$

whence

$$
\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\hat{N}_1(0,\Phi)\right]\right|_{\delta,p,\varepsilon}\\
&\leq \quad \left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\left\{\frac{\varepsilon}{2}\Phi_x^2+\frac{\varepsilon^2}{2}\Phi_z^2+\varepsilon^{\frac{1}{2}}\Phi_x y\Phi_y+\varepsilon\Phi_z y\Phi_y+\frac{1}{2}\Phi_y^2\right\}\mathrm{d}y\right]\right]\right|_{\delta,p,\varepsilon}\\
&\qquad\qquad +\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\varepsilon\Phi_x y\Phi_y\,\mathrm{d}y\right]\right]\right|_{\delta,p,\varepsilon}\\
&\qquad\qquad +\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\varepsilon^{\frac{3}{2}}\Phi_z y\Phi_z\,\mathrm{d}y\right]\right]\right|_{\delta,p,\varepsilon}\\
&\quad\leq c(\varepsilon\|\Phi_x^2\|_{\delta,p,\varepsilon}+\varepsilon^2\|\Phi_z^2\|_{\delta,p,\varepsilon}+\varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon\|\Phi_z y\Phi_z\|_{\delta,p,\varepsilon}+\|\Phi_y^2\|_{\delta,p,\varepsilon}\\
&\qquad\qquad +\varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon\|\Phi_z y\Phi_z\|_{\delta,p,\varepsilon})\\
&\leq \quad c(\varepsilon^{1-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}^2+\varepsilon^{2-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}^2+\varepsilon^{-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}^2\\
&\qquad\qquad +\varepsilon^{\frac{1}{2}-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_y\|_{\delta,p,\varepsilon}+\varepsilon^{1-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}\|\Phi_y\|_{\delta,p,\varepsilon})\\
&\leq \quad c\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}+\|\Phi_y\|_{\delta,p,\varepsilon})^2,
\end{aligned}
$$

in which Lemma 2.10 and the properties of our function spaces have been used. We similarly find that

$$
\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x}+\mathrm{i}\mu\int_0^1 y\hat{\Psi}\,\mathrm{d}y+\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right)\right]\right|_{\delta,p,\varepsilon}\\
&\qquad\leq c(\|\Phi_{1x}\|_{\delta,p,\varepsilon}+\|\Phi_{2x}\|_{\delta,p,\varepsilon}+\varepsilon^{-1/2}\|\Psi\|_{\delta,p,\varepsilon}),
\end{aligned}
$$

and the estimate (42) follows directly from the above calculations.

The next step is to estimate

$$
\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{\delta,p,\varepsilon},
$$

where we note that

$$
\begin{aligned}
&\partial_1 N_1(\rho,\Phi)\tilde{\rho}=\\
&\quad\beta\varepsilon^2\left[\frac{-\tilde{\rho}_x(\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}(1+\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2})}\right]_x-\left[\frac{\beta\varepsilon^2\rho_x(\varepsilon^3\rho_x\tilde{\rho}_x+\varepsilon^4\rho_z\tilde{\rho}_z)}{(1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)^{3/2}}\right]_x\\
&\quad+\beta\varepsilon^3\left[\frac{-\tilde{\rho}_z(\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}(1+\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2})}\right]_z-\left[\frac{\beta\varepsilon^3\rho_z(\varepsilon^3\rho_x\tilde{\rho}_x+\varepsilon^4\rho_z\tilde{\rho}_z)}{(1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)^{3/2}}\right]_z\\
&\quad-\int_0^1\left\{\varepsilon^2\left(\Phi_x-\frac{\varepsilon y\Phi_y\rho_x}{1+\varepsilon\rho}\right)\left(\frac{-\varepsilon y\Phi_y\tilde{\rho}_x}{1+\varepsilon\rho}+\frac{\varepsilon^2 y\Phi_y\rho_x\tilde{\rho}}{(1+\varepsilon\rho)^2}\right)\right.\\
&\qquad\qquad+\varepsilon^3\left(\Phi_z-\frac{\varepsilon y\Phi_y\rho_z}{1+\varepsilon\rho}\right)\left(\frac{-\varepsilon y\Phi_y\tilde{\rho}_z}{1+\varepsilon\rho}+\frac{\varepsilon^2 y\Phi_y\rho_z\tilde{\rho}}{(1+\varepsilon\rho)^2}\right)\\
&\qquad\qquad+\varepsilon^2\left(y\Phi_y\left(\frac{-\varepsilon y\Phi_y\tilde{\rho}_x}{1+\varepsilon\rho}+\frac{\varepsilon^2 y\Phi_y\rho_x\tilde{\rho}}{(1+\varepsilon\rho)^2}\right)\right)_x+\varepsilon^3\left(y\Phi_y\left(\frac{-\varepsilon y\Phi_y\tilde{\rho}_z}{1+\varepsilon\rho}+\frac{\varepsilon^2 y\Phi_y\rho_z\tilde{\rho}}{(1+\varepsilon\rho)^2}\right)\right)_z
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon^{\frac{3}{2}}\left(\frac{-\varepsilon y\Phi_y\tilde{\rho}_x}{1+\varepsilon\rho}+\frac{\varepsilon^2 y\Phi_y\rho_x\tilde{\rho}}{(1+\varepsilon\rho)^2}\right)\frac{y\Phi_y}{1+\varepsilon\rho}-\varepsilon^{\frac{5}{2}}\left(\Phi_x-\frac{\varepsilon y\Phi_y}{1+\varepsilon\rho}\right)\frac{y\Phi_y\tilde{\rho}}{(1+\varepsilon\rho)^2}\\
&+\varepsilon^{2}\left(\frac{-\varepsilon y\Phi_y\tilde{\rho}_z}{1+\varepsilon\rho}+\frac{\varepsilon^2 y\Phi_y\rho_z\tilde{\rho}}{(1+\varepsilon\rho)^2}\right)\frac{y\Phi_y}{1+\varepsilon\rho}-\varepsilon^{3}\left(\Phi_z-\frac{\varepsilon y\Phi_y}{1+\varepsilon\rho}\right)\frac{y\Phi_y\tilde{\rho}}{(1+\varepsilon\rho)^2}\\
&\left.-\frac{\varepsilon^2\Phi_y^2\tilde{\rho}}{(1+\varepsilon\rho)^3}\right\}\mathrm{d}y. \qquad (44)
\end{aligned}
$$

We proceed by estimating the above quantity under the assumptions (40) and

$$|\rho|_{\delta,p,\varepsilon}\leq c\varepsilon^{-\frac{1}{4}-\Delta},$$

which follows from (40) and (41), together with the rules

$$\|\rho\|_{\delta,p,\varepsilon}\leq c\varepsilon^{-\frac{\delta}{2}}|\rho|_{\delta,p,\varepsilon}\qquad \|\rho_x\|_{\delta,p,\varepsilon}\leq c\varepsilon^{-\frac{1}{2}}|\rho|_{\delta,p,\varepsilon}\qquad \|\rho_z\|_{\delta,p,\varepsilon}\leq c\varepsilon^{-1}|\rho|_{\delta,p,\varepsilon} \qquad (45)$$

and

$$
\begin{aligned}
\left\|\frac{u}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon} &= \left\|u-\frac{\varepsilon u\rho}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon}\\
&\leq \|u\|_{\delta,p,\varepsilon}+c\varepsilon^{-\Delta}(\varepsilon\|\rho\|_{\delta,p,\varepsilon}+\varepsilon^2\|\rho\|^2_{\delta,p,\varepsilon}+\ldots)\\
&\leq c\|u\|_{\delta,p,\varepsilon}
\end{aligned}
$$

(with similar rules for the other denominators). We find for example that

$$
\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\frac{\varepsilon^2 y^2\Phi_y^2\tilde{\rho}_x}{1+\varepsilon\rho}\right]\right]\right|_{\delta,p,\varepsilon}\\
&\qquad\leq c\varepsilon^{-\frac{3}{2}}\left\|\frac{y^2\Phi_y^2\tilde{\rho}_x}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon}\\
&\qquad\leq c\varepsilon^{-\frac{3}{2}}\|y^2\Phi_y^2\tilde{\rho}_x\|_{\delta,p,\varepsilon}\\
&\qquad\leq c\varepsilon^{-\frac{3}{2}-\Delta}\|\Phi_y\|^2_{\delta,p,\varepsilon}\|\tilde{\rho}_x\|_{\delta,p,\varepsilon}\\
&\qquad\leq c\varepsilon^{\frac{3}{2}-\Delta}\|\tilde{\rho}_x\|_{\delta,p,\varepsilon}\\
&\qquad\leq c\varepsilon^{1-\Delta}|\tilde{\rho}|_{\delta,p,\varepsilon};
\end{aligned}
$$

estimating each term in this fashion we conclude that

$$\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{\delta,p,\varepsilon}\leq c\varepsilon^{\frac{1}{2}-\Delta}|\tilde{\rho}|_{\delta,p,\varepsilon},$$

from which (43) follows immediately.

Our fixed-point theorem states that

$$|\rho_\Psi\tilde{\Psi}|_{\delta,p,\varepsilon}\leq 2\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\int_0^1 y\hat{\tilde{\Psi}}\,\mathrm{d}y\right]\right|_{\delta,p,\varepsilon}\leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}$$

and

$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leq 2\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\mathrm{i}\mu\int_0^1 y\hat{\tilde{\Phi}}_2\,\mathrm{d}y + \varepsilon^{-1}\partial_2\hat{N}_1(\rho,\Phi)\tilde{\Phi}_2\right)\right]\right|_{\delta,p,\varepsilon};$$

using the calculation

$$\begin{aligned}
&\partial_1 N_2(\rho,\Phi)\tilde{\Phi} = \\
&\quad -\int_0^1 \left\{\varepsilon^2\left(\Phi_x - \frac{\varepsilon y\Phi_y\rho_x}{1+\varepsilon\rho}\right)\left(\tilde{\Phi}_x - \frac{\varepsilon y\tilde{\Phi}_y\rho_x}{1+\varepsilon\rho}\right) + \varepsilon^3\left(\Phi_z - \frac{\varepsilon y\Phi_y\rho_z}{1+\varepsilon\rho}\right)\left(\tilde{\Phi}_z - \frac{\varepsilon y\tilde{\Phi}_y\rho_z}{1+\varepsilon\rho}\right)\right. \\
&\qquad\qquad + \varepsilon^2\left(\left(\Phi_x - \frac{\varepsilon y\Phi_y\rho_x}{1+\varepsilon\rho}\right)y\tilde{\Phi}_y\right)_x + \varepsilon^2\left(\left(\tilde{\Phi}_x - \frac{\varepsilon y\tilde{\Phi}_y\rho_x}{1+\varepsilon\rho}\right)y\Phi_y\right)_x \\
&\qquad\qquad + \varepsilon^3\left(\left(\Phi_z - \frac{\varepsilon y\Phi_y\rho_z}{1+\varepsilon\rho}\right)y\tilde{\Phi}_y\right)_z + \varepsilon^3\left(\left(\tilde{\Phi}_z - \frac{\varepsilon y\tilde{\Phi}_y\rho_z}{1+\varepsilon\rho}\right)y\Phi_y\right)_z \\
&\qquad\qquad + \varepsilon^{\frac{3}{2}}\left(\Phi_x - \frac{\varepsilon y\Phi_y\rho_x}{1+\varepsilon\rho}\right)\frac{y\tilde{\Phi}_y}{1+\varepsilon\rho} + \varepsilon^{\frac{3}{2}}\left(\tilde{\Phi}_x - \frac{\varepsilon y\tilde{\Phi}_y\rho_x}{1+\varepsilon\rho}\right)\frac{y\Phi_y}{1+\varepsilon\rho} \\
&\qquad\qquad + \varepsilon^2\left(\Phi_z - \frac{\varepsilon y\Phi_y\rho_z}{1+\varepsilon\rho}\right)\frac{y\tilde{\Phi}_y}{1+\varepsilon\rho} + \varepsilon^2\left(\tilde{\Phi}_z - \frac{\varepsilon y\tilde{\Phi}_y\rho_z}{1+\varepsilon\rho}\right)\frac{y\Phi_y}{1+\varepsilon\rho} \\
&\qquad\qquad \left. + \frac{\varepsilon\Phi_y\tilde{\Phi}_y^2}{(1+\varepsilon\rho)^2}\right\}\mathrm{d}y
\end{aligned}$$

and arguing as above, we find that

$$|\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}. \qquad \square$$

We also need some information about the behaviour of ρ as a function of Ψ, Φ_1 and Φ_2 in L^2-based function spaces.

Corollary 2.12 *The solution $\rho = \rho(\Psi, \Phi_1, \Phi_2)$ to (39) identified in the previous theorem satisfies the estimate*

$$|\rho|_{0,2,\varepsilon} \leq c(\|\Phi_x\|_2 + \varepsilon^{-\frac{1}{2}}\|\Psi\|_2 + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Phi_y\|_2)(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})). \qquad (46)$$

Moreover ρ is a smooth function of (Ψ, Φ_1, Φ_2) with respect to the $V_\varepsilon^{0,2}(\mathbb{R}^2)$ and $L^2(\Sigma) \times U_\varepsilon^{0,2}(\mathbb{R}^2) \times W_\varepsilon^{1,2}(\Sigma)$ topologies and in particular its derivatives with respect to Ψ and Φ_2 satisfy the estimates

$$|\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_2, \qquad |\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}.$$

Proof. We begin by observing that

$$\begin{aligned}
X &= V_\varepsilon^{0,2}(\mathbb{R})^2 \cap \{\rho \in V_\varepsilon^{\delta,p}(\mathbb{R}^2) : |\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}\}, \\
Y_1 &= L^2(\Sigma) \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}\}, \\
Y_2 &= W_\varepsilon^{0,2}(\mathbb{R}^2) \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}\}, \\
Y_3 &= W_\varepsilon^{1,2}(\Sigma) \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\}
\end{aligned}$$

are closed subsets of respectively $\mathcal{X} = V_\varepsilon^{0,2}(\mathbb{R}^2)$, $\mathcal{Y}_1 = L^2(\Sigma)$, $\mathcal{Y}_2 = U_\varepsilon^{0,2}(\mathbb{R}^2)$ and $\mathcal{Y}_3 = W_\varepsilon^{1,2}(\Sigma)$. We may therefore apply our fixed-point to equation (39) with these definitions of $\mathcal{X}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and X, Y_1, Y_2, Y_3; the fixed point thus located clearly coincides with that identified in Theorem 2.11. Our task is to verify that

$$\begin{aligned}&|\mathcal{F}_1(0,\Psi,\Phi_1,\Phi_2)|_{0,2,\varepsilon}\\ &\qquad \leq c\big(\|\Phi_x\|_2 + \varepsilon^{-\frac{1}{2}}\|\Psi\|_2 + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Phi_y\|_2)(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})\big)\end{aligned}$$

and that

$$|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2]|_{V_\varepsilon^{0,2}(\mathbb{R}^2)\to V_\varepsilon^{0,2}(\mathbb{R}^2)} \leq \frac{1}{2}$$

whenever

$$|\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}. \tag{47}$$

Observe that

$$\begin{aligned}
&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\hat{N}_1(0,\Phi)\right]\right|_{0,2,\varepsilon}\\
&\leq \left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1 \left\{\frac{\varepsilon}{2}\Phi_x^2 + \frac{\varepsilon^2}{2}\Phi_z^2 + \varepsilon^{\frac{1}{2}}\Phi_x y\Phi_y + \varepsilon\Phi_z y\Phi_y + \frac{1}{2}\Phi_y^2\right\}\mathrm{d}y\right]\right]\right|_{0,2,\varepsilon}\\
&\qquad + \left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1 \varepsilon\Phi_x y\Phi_y\,\mathrm{d}y\right]\right]\right|_{0,2,\varepsilon}\\
&\qquad + \left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1 \varepsilon^{\frac{3}{2}}\Phi_z y\Phi_z\,\mathrm{d}y\right]\right]\right|_{0,2,\varepsilon}\\
&\leq c(\varepsilon\|\Phi_x^2\|_2 + \varepsilon^2\|\Phi_z^2\|_2 + \varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_y\|_2 + \varepsilon\|\Phi_z y\Phi_y\|_2 + \|\Phi_y^2\|_2\\
&\qquad + \varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_y\|_2 + \varepsilon\|\Phi_z y\Phi_y\|_2)\\
&\leq c(\varepsilon\|\Phi_x\|_\infty\|\Phi_x\|_2 + \varepsilon^2\|\Phi_z\|_\infty\|\Phi_z\|_2 + \|\Phi_y\|_\infty\|\Phi_y\|_2\\
&\qquad + \varepsilon^{\frac{1}{2}}\|\Phi_x\|_\infty\|\Phi_y\|_2 + \varepsilon\|\Phi_z\|_\infty\|\Phi_y\|_2)\\
&\leq c(\varepsilon^{1-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_x\|_2 + \varepsilon^{2-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}\|\Phi_z\|_2 + \varepsilon^{-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}\|\Phi_y\|_2\\
&\qquad + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_y\|_2 + \varepsilon^{1-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}\|\Phi_y\|_2)\\
&\leq c\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}} + \|\Phi_y\|_2),
\end{aligned}$$

and we similarly find that

$$\begin{aligned}&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x} + \mathrm{i}\mu\int_0^1 y\hat{\Psi}\,\mathrm{d}y + \int_0^1 \hat{\Phi}_{2x}\,\mathrm{d}y\right)\right]\right|_{0,2,\varepsilon}\\ &\qquad \leq c(\|\Phi_{1x}\|_2 + \|\Phi_{2x}\|_2 + \varepsilon^{-1/2}\|\Psi\|_2);\end{aligned}$$

the estimate for $|\mathcal{F}_1(0,\Psi,\Phi_1,\Phi_2)|_{0,2,\varepsilon}$ follows directly from the above calculations. The bound for $|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2]|_{V_\varepsilon^{0,2}(\mathbb{R}^2)\to V_\varepsilon^{0,2}(\mathbb{R}^2)}$ is obtained by estimating

$$\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{0,2,\varepsilon}$$

using the assumptions (47) together with the rules (45) and

$$\begin{aligned}\left\|\frac{1}{1+\varepsilon\rho}\right\|_\infty &= \left\|1-\frac{\varepsilon\rho}{1+\varepsilon\rho}\right\|_\infty\\ &\leq 1+\left\|\frac{\varepsilon\rho}{1+\varepsilon\rho}\right\|_\infty\\ &\leq 1+c\varepsilon^{-\Delta}\left\|\frac{\varepsilon\rho}{1+\varepsilon\rho}\right\|_{\delta,p,\varepsilon}\\ &\leq 1+c\varepsilon^{-\Delta}\|\varepsilon\rho\|_{\delta,p,\varepsilon}\\ &\leq c\end{aligned}$$

(with similar rules for the other denominators). We find for example that

$$\begin{aligned}&\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\frac{\varepsilon^2y^2\Phi_y^2\tilde{\rho}_x}{1+\varepsilon\rho}\right]\right]\right|_{1,2,\varepsilon}\\ &\qquad\leq c\varepsilon^{-\frac{3}{2}}\left\|\frac{y^2\Phi_y^2\tilde{\rho}_x}{1+\varepsilon\rho}\right\|_2\\ &\qquad\leq c\varepsilon^{-\frac{3}{2}}\left\|\frac{y^2\Phi_y^2}{1+\varepsilon\rho}\right\|_\infty\|\tilde{\rho}_x\|_2\\ &\qquad\leq c\varepsilon^{-\frac{3}{2}-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}^2\|\tilde{\rho}_x\|_2\\ &\qquad\leq c\varepsilon^{\frac{3}{2}-\Delta}\|\tilde{\rho}_x\|_2\\ &\qquad\leq c\varepsilon^{1-\Delta}|\tilde{\rho}|_{0,2,\varepsilon},\end{aligned}$$

and estimating each term in this fashion we conclude that

$$\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{0,2,\varepsilon}\leq c\varepsilon^{\frac{1}{2}-\Delta}|\tilde{\rho}|_{0,2,\varepsilon}. \tag{48}$$

According to our fixed-point theorem, the estimates for $\rho_\Psi\tilde{\Psi}$ and $\rho_{\Phi_2}\tilde{\Phi}_2$ are given by the formulae

$$\begin{aligned}|\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} &\leq 2\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\int_0^1 y\hat{\tilde{\Psi}}\,\mathrm{d}y\right]\right|_{0,2,\varepsilon}\\ &\leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_2,\end{aligned}$$

$$\begin{aligned}|\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} &\leq 2\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\mathrm{i}\mu\int_0^1 y\hat{\tilde{\Phi}}_2\,\mathrm{d}y+\varepsilon^{-1}\partial_2\hat{N}_1(\rho,\Phi)\tilde{\Phi}_2\right)\right]\right|_{0,2,\varepsilon}\\ &\leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon},\end{aligned}$$

where the second inequality is in each case obtained in the same fashion as (48). □

Finally, we record some further estimates for ρ which are used later; they are proved using the estimation techniques developed in Theorem 2.11 and Corollary 2.12.

Lemma 2.13 *Define*

$$\rho_{\mathrm{NL}}(\rho,\Phi_1,\Phi_2)=\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\hat{N}_1(\rho,\Phi_1,\Phi_2)\right],$$

so that

$$\rho=\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x}+\mathrm{i}\mu\int_0^1 y\hat{\Psi}\,\mathrm{d}y+\int_0^1\hat{\Phi}_{2x}\right)\right]+\rho_{\mathrm{NL}}(\rho,\Phi_1,\Phi_2).$$

The function ρ_{NL} *satisfies the estimates*

$$\begin{aligned}|\rho_{\mathrm{NL}}|_{\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}},\|\Psi\|_{\delta,p,\varepsilon},\|\Phi_y\|_{\delta,p,\varepsilon}),\\ |\rho_{\mathrm{NL}}|_{0,2,\varepsilon} &\leq c(\varepsilon\|\Phi_1\|^2_{U_\varepsilon^{0,4}}+\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}}+\|\Psi\|_2+\|\Phi_y\|_2)\\ &\qquad\times P_1(\varepsilon^{\frac{3}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}},\varepsilon^{\frac{1}{2}}\|\Phi_2\|_{U_\varepsilon^{\delta,p}},\|\Psi\|_{\delta,p,\varepsilon},\|\Phi_y\|_{\delta,p,\varepsilon})).\end{aligned}$$

2.3 Elimination of the variable Φ_2

Substituting $\rho=\rho(\Psi,\Phi_1,\Phi_2)$ into the integral form of the equation for Φ_2 and identifying Ψ with Φ_{2y}, one finds that

$$\begin{aligned}\hat{\Phi}_2=&-\int_0^1\frac{G_1}{\varepsilon^{5/2}}\hat{N}_4(\rho(\Phi_{2y},\Phi_1,\Phi_2),\Phi_1+\Phi_2)\,\mathrm{d}\xi-\int_0^1\frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_5(\rho(\Phi_{2y},\Phi_1,\Phi_2),\Phi_1+\Phi_2)\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_1(\rho(\Phi_{2y},\Phi_1,\Phi_2),\Phi_1+\Phi_2).\end{aligned}\tag{49}$$

In this section we show that the above equation can be solved for Φ_2 as a function of Φ_1. We proceed by replacing it with a pair of equivalent integral equations which have more favourable mapping properties (see below), namely

$$\begin{aligned}\hat{\Phi}_2=&-\int_0^1\frac{G_1}{\varepsilon^{5/2}}\hat{N}_6(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi-\int_0^1\frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi),\end{aligned}\tag{50}$$

$$\begin{aligned}\hat{\Psi}=&-\int_0^1\frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi-\int_0^1\frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi)\,\mathrm{d}\xi\\ &+\frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(\Psi,\Phi_1,\Phi_2),\Phi_1,\Phi_2,\Psi).\end{aligned}\tag{51}$$

The first equation is obtained by replacing the nonlinearities N_4, N_5 and N_1 with new nonlinear functions N_6, N_7 and N_8, while the second is obtained by differentiating the first with respect to y and replacing Φ_2 with Ψ on the left-hand side; the functions N_6, N_7 and N_8 are given by the formulae defining N_4, N_5 and N_1 with all occurrences of Φ_{2y} replaced by Ψ.

Proposition 2.14 *Any solution $\Phi_2^\star$ of (49) defines a solution $(\Phi_2^\star, \Phi_{2y}^\star)$ of (50), (51). Conversely, any solution $(\Phi_2^\star, \Psi^\star)$ of (50), (51) satisfies $\Psi^\star = \Phi_{2y}^\star$ and hence defines a solution of (49).*

The following lemma gives estimates on the norms of the Fourier-multiplier operators that appear in the above equations; its proof is given in Section 4.

Lemma 2.15 *The following statements hold for each $\delta \in [0,1]$ and $p \in (1,\infty)$.*

(i) For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the function

$$\mathcal{G}_4(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\Sigma)$ and satisfies the estimate

$$\|\mathcal{G}_4(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon}.$$

(ii) For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the function

$$\mathcal{G}_5(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\varepsilon^{\frac{1}{2}} k G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\Sigma)$ and satisfies the estimate

$$\|\mathcal{G}_5(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon}.$$

(iii) For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the function

$$\mathcal{G}_6(u) = \mathcal{F}^{-1}\left[\int_0^1 G_{1\xi} \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\Sigma)$ and satisfies the estimate

$$\|\mathcal{G}_6(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon}.$$

(iv) For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the function

$$\mathcal{G}_7(u) = \mathcal{F}^{-1}\left[\int_0^1 G_{1y\xi} \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W_\varepsilon^{\delta,p}(\Sigma)$ and satisfies the estimate

$$\|\mathcal{G}_7(u)\|_{\delta,p,\varepsilon} \leq c\varepsilon^2 \|u\|_{\delta,p,\varepsilon}.$$

(v) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_8(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_8(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon\|u\|_{\delta,p,\varepsilon}.$$

(vi) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_9(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}kG_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_9(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon\|u\|_{\delta,p,\varepsilon}.$$

(vii) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_{10}(u) = \mathcal{F}^{-1}\left[\frac{-\mu^2 G_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{10}(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}}\|u\|_{\delta,p,\varepsilon}.$$

(viii) For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function

$$\mathcal{G}_{11}(u) = \mathcal{F}^{-1}\left[\frac{-\varepsilon^{\frac{1}{2}}\mu kG_1|_{\xi=1}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{11}(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}}\|u\|_{\delta,p,\varepsilon}.$$

Our strategy in dealing with the coupled integral equations (50), (51) is to solve (51) for Ψ as a function of Φ_1, Φ_2, substitute $\Psi = \Psi(\Phi_1, \Phi_2)$ into (50) and solve this equation for Φ_2 as a function of Φ_1; the two equations are solved by the method used for the equation for ρ in Section 2.2 above. (Attempting to solve equation (49) directly using this method, one finds that the estimates for certain terms have insufficient powers of ε. This difficulty is overcome by the use of the equivalent equations (50), (51). Part (iv) of Lemma 2.15 ensures that an additional power of ε appears in the estimate of the problematic term in equation (51), and this additional power is inherited by equation (50) in the form of a good estimate for Ψ.) We carry out the first step by writing equation (51) as

$$\Psi = \mathcal{F}_2(\Psi, \Phi_1, \Phi_2) \tag{52}$$

and applying our fixed-point theorem.

Theorem 2.16 *Suppose that*

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}. \tag{53}$$

Equation (52) has a unique solution $\Psi = \Psi(\Phi_1, \Phi_2)$ *which satisfies the estimate*

$$\|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}). \tag{54}$$

Moreover Ψ *is a smooth function of* (Φ_1, Φ_2) *with respect to the* $W_\varepsilon^{\delta,p}(\Sigma)$ *and* $U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma)$ *topologies and in particular its first derivative with respect to* Φ_2 *satisfies the estimate*

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}.$$

Proof. We obtain this result by applying Theorem 1.5 with $\mathcal{X} = W_\varepsilon^{\delta,p}(\Sigma)$, $\mathcal{Y}_1 = U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{Y}_2 = W_\varepsilon^{1+\delta,p}(\Sigma)$ and X, Y_1, Y_2 closed origin-centred balls of radius $\mathcal{O}(\varepsilon^{\frac{1}{2}-\Delta})$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$, $\mathcal{O}(\varepsilon^{-\Delta})$; one has to verify that

$$\|\mathcal{F}_2(0, \Phi_1, \Phi_2)\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}) \tag{55}$$

and that

$$\|\mathrm{d}_1\mathcal{F}_2[\Psi, \Phi_1, \Phi_2]\|_{W_\varepsilon^{\delta,p}(\Sigma)\to W_\varepsilon^{\delta,p}(\Sigma)} \leq \frac{1}{2}$$

whenever (53) and (54) hold.

We therefore begin by examining

$$\mathcal{F}^{-1}\Bigg[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho(0,\Phi), \Phi, 0)\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(0,\Phi), \Phi, 0)\,\mathrm{d}\xi + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(0,\Phi), \Phi, 0)\Bigg],$$

where we use the expressions

$$\begin{aligned}
N_6(\rho, \Phi, 0) &= \varepsilon^{\frac{5}{2}}(\rho\Phi_x)_x + \varepsilon^{\frac{7}{2}}(\rho\Phi_z)_z, \\
N_7(\rho, \Phi, 0) &= \varepsilon^{\frac{5}{2}}y\rho_x\Phi_x + \varepsilon^{\frac{7}{2}}y\rho_z\Phi_z, \\
N_8(\rho, \Phi, 0) &= \beta\varepsilon^2\left[\frac{-(\varepsilon^3\rho_x^2 + \varepsilon^4\rho_z^2)\rho_x}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}(1+\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2})}\right]_x \\
&\quad + \beta\varepsilon^3\left[\frac{-(\varepsilon^3\rho_x^2 + \varepsilon^4\rho_z^2)\rho_z}{\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}(1+\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2})}\right]_z - \int_0^1 (\varepsilon^2\Phi_x^2 + \varepsilon^3\Phi_z^2)\,\mathrm{d}y
\end{aligned}$$

and the estimate

$$|\rho(0,\Phi)|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}(\|\Phi_x\|_{\delta,p,\varepsilon} + (\|\Phi_y\|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}})^2). \tag{56}$$

The calculations

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho_0,\Phi,0)\,\mathrm{d}\xi\right]\right\|_{\delta,p,\varepsilon}\\
&\qquad\leq \left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\hat{N}_6(\rho_0,\Phi,0)\,\mathrm{d}\xi\right]\right\|_{1+\delta,p,\varepsilon}\\
&\qquad\leq c(\varepsilon\|\rho_0\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^{\frac{3}{2}}\|\rho_0\Phi_z\|_{\delta,p,\varepsilon})\\
&\qquad\leq c\varepsilon^{1-\Delta}\|\rho_0\|_{\delta,p,\varepsilon}(\|\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^{\frac{1}{2}}\|\Phi_z\|_{\delta,p,\varepsilon})\\
&\qquad\leq c\varepsilon^{1-\Delta}|\rho_0|_{\delta,p,\varepsilon}(\|\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^{\frac{1}{2}}\|\Phi_z\|_{\delta,p,\varepsilon})\\
&\qquad\leq c\varepsilon^{\frac{1}{2}-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}},\|\Phi_y\|_{\delta,p,\varepsilon}),
\end{aligned}$$

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho_0,\Phi,0)\,\mathrm{d}\xi\right]\right\|_{\delta,p,\varepsilon}\\
&\qquad\leq c(\varepsilon^2\|\rho_{0x}\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^3\|\rho_{0z}\Phi_z\|_{\delta,p,\varepsilon})\\
&\qquad\leq c(\varepsilon^{2-\Delta}\|\rho_{0x}\|_{\delta,p,\varepsilon}\|\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^{3-\Delta}\|\rho_{0z}\|_{\delta,p,\varepsilon}\|\Phi_z\|_{\delta,p,\varepsilon})\\
&\qquad\leq c\varepsilon^{\frac{3}{2}-\Delta}|\rho_0|_{\delta,p,\varepsilon}(\|\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^{\frac{1}{2}}\|\Phi_z\|_{\delta,p,\varepsilon})\\
&\qquad\leq c\varepsilon^{\frac{1}{2}-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}},\|\Phi_y\|_{\delta,p,\varepsilon}),
\end{aligned}$$

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho_0,\Phi,0)\right]\right\|_{\delta,p,\varepsilon}\\
&\qquad\leq \left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho_0,\Phi,0)\right]\right\|_{1+\delta,p,\varepsilon}\\
&\qquad\leq c(\varepsilon^{\frac{7}{2}-\Delta}\|\rho_{0x}\|^3_{\delta,p,\varepsilon}+\varepsilon^{\frac{9}{2}-\Delta}\|\rho_{0z}\|^2_{\delta,p,\varepsilon}\|\rho_{0x}\|_{\delta,p,\varepsilon}+\varepsilon^{4-\Delta}\|\rho_{0x}\|^2_{\delta,p,\varepsilon}\|\rho_{0z}\|_{\delta,p,\varepsilon}\\
&\qquad\qquad+\varepsilon^{5-\Delta}\|\rho_{0z}\|^3_{\delta,p,\varepsilon}+\varepsilon^{1-\Delta}\|\Phi_x\|^2_{\delta,p,\varepsilon}+\varepsilon^{2-\Delta}\|\Phi_z\|^2_{\delta,p,\varepsilon})\\
&\qquad\leq c(\varepsilon^{2-\Delta}|\rho_0|^3_{\delta,p,\varepsilon}+\varepsilon^{1-\Delta}\|\Phi\|^2_{U_\varepsilon^{\delta,p}})\\
&\qquad\leq c\varepsilon^{\frac{1}{2}-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}},\|\Phi_y\|_{\delta,p,\varepsilon}),
\end{aligned}$$

in which ρ_0 is an abbreviation for $\rho(0,\Phi)$, are obtained using Lemma 2.15 together with the properties of our function spaces and yield inequality (55).

The next step is to estimate

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)\bar{\rho}\,\mathrm{d}\xi-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_3\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi\right.\right.\\
&\qquad-\int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)\bar{\rho}\,\mathrm{d}\xi-\int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_3\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi\\
&\qquad\left.\left.+\frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)\bar{\rho}+\frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_3\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Psi}\right]\right\|_{\delta,p,\varepsilon}, \qquad (57)
\end{aligned}$$

where $\bar{\rho}=\rho_\Psi\tilde{\Psi}$, using the calculations

$$\partial_3 N_6(\rho,\Phi,\Psi)\tilde{\Psi}=-\varepsilon^{\frac{5}{2}}(y\tilde{\Psi}\rho_x)_x-\varepsilon^{\frac{7}{2}}(y\tilde{\Psi}\rho_z)_z,$$

$$\partial_3 N_7(\rho,\Phi,\Psi)\tilde{\Psi} = -\frac{2\varepsilon^{\frac{7}{2}}y^2\tilde{\Psi}\rho_x}{1+\varepsilon\rho} - \frac{2\varepsilon^{\frac{9}{2}}y^2\tilde{\Psi}\rho_z}{1+\varepsilon\rho} - \frac{\varepsilon^{\frac{3}{2}}\rho\tilde{\Psi}}{1+\varepsilon\rho},$$

$$\begin{aligned}
&\partial_3 N_8(\rho,\Phi,\Psi)\tilde{\Psi} = \\
&\quad -\int_0^1 \Bigg\{ -\varepsilon^2\left(\Phi_x - \frac{\varepsilon y\Psi\rho_x}{1+\varepsilon\rho}\right)\frac{\varepsilon y\tilde{\Psi}\rho_x}{1+\varepsilon\rho} - \varepsilon^3\left(\Phi_z - \frac{\varepsilon y\Psi\rho_z}{1+\varepsilon\rho}\right)\frac{\varepsilon y\tilde{\Psi}\rho_z}{1+\varepsilon\rho} \\
&\qquad + \varepsilon^2\left(\frac{y\Phi_z\tilde{\Psi}}{1+\varepsilon\rho} - \frac{2\varepsilon y^2\Psi\tilde{\Psi}\rho_x}{(1+\varepsilon\rho)^2}\right)_x + \varepsilon^3\left(\frac{y\Phi_z\tilde{\Psi}}{1+\varepsilon\rho} - \frac{2\varepsilon y^2\Psi\tilde{\Psi}\rho_z}{(1+\varepsilon\rho)^2}\right)_z \\
&\qquad + \frac{\varepsilon^{\frac{3}{2}}y\Phi_x\tilde{\Psi}}{1+\varepsilon\rho} - \frac{2\varepsilon^{\frac{5}{2}}y^2\Psi\tilde{\Psi}\rho_x}{(1+\varepsilon\rho)^2} + \frac{\varepsilon^2 y\Phi_z\tilde{\Psi}}{1+\varepsilon\rho} - \frac{2\varepsilon^3y^2\Psi\tilde{\Psi}\rho_z}{(1+\varepsilon\rho)^2} + \frac{\varepsilon\Psi\tilde{\Psi}}{(1+\varepsilon)^2}\Bigg\}\,\mathrm{d}y,
\end{aligned}$$

$$\begin{aligned}
&\partial_1 N_6(\rho,\Phi,\Psi)\bar{\rho} = \varepsilon^{\frac{5}{2}}(\tilde{\rho}\Phi_x)_x + \varepsilon^{\frac{7}{2}}(\tilde{\rho}\Phi_z)_z - \varepsilon^{\frac{5}{2}}(y\Psi\tilde{\rho}_x)_x - \varepsilon^{\frac{7}{2}}(y\Psi\tilde{\rho}_z)_z, \\
&\partial_1 N_7(\rho,\Phi,\Psi)\bar{\rho} = \\
&\quad \varepsilon^{\frac{5}{2}}\left(\Phi_x - \frac{2\varepsilon y\Psi\rho_x}{1+\varepsilon\rho}\right)y\tilde{\rho}_x + \frac{\varepsilon^{\frac{9}{2}}y^2\Psi\rho_x^2\tilde{\rho}}{(1+\varepsilon\rho)^2} + \varepsilon^{\frac{7}{2}}\left(\Phi_z - \frac{2\varepsilon y\Psi\rho_z}{1+\varepsilon\rho}\right)y\tilde{\rho}_z + \frac{\varepsilon^{\frac{11}{2}}y^2\Psi\rho_z^2\tilde{\rho}}{(1+\varepsilon\rho)^2} \\
&\quad - \frac{\varepsilon^{\frac{3}{2}}\tilde{\rho}\Psi}{1+\varepsilon\rho} + \frac{\varepsilon^{\frac{5}{2}}\rho\tilde{\rho}\Psi}{(1+\varepsilon\rho)^2}
\end{aligned}$$

(an expression for $\partial_1 N_8(\rho,\Phi,\Psi)\bar{\rho}$ is easily deduced from the formula (44) for $\partial_1 N_1(\rho,\Phi,\Psi)\tilde{\rho}$). Estimating the quantity (57) using the method explained in Theorem 2.11 together with the estimates (53) and

$$|\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \qquad \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}$$

(which follow from (53), (54), (56)), one finds that it is bounded by

$$c(\varepsilon^{\frac{3}{4}-\Delta}|\bar{\rho}|_{\delta,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}) \leq c\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon},$$

in which the further inequality $|\rho_\Psi\tilde{\Psi}|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}|\Psi\|_{\delta,p,\varepsilon}$ has been used (see Theorem 2.11).

Our fixed-point theorem states that

$$\begin{aligned}
&\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \\
&\leq\; 2\Bigg\|\mathcal{F}^{-1}\Bigg[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_2\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi \\
&\qquad - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_2\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi \\
&\qquad + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)\tilde{\rho} + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_2\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Phi}_2\Bigg]\Bigg\|_{1+\delta,p,\varepsilon},
\end{aligned}$$

where $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$. Observe that

$$\begin{aligned}
&\partial_2 N_6(\rho,\Phi,\Psi)\tilde{\Phi} = \varepsilon^{\frac{5}{2}}(\rho\tilde{\Phi}_x)_x + \varepsilon^{\frac{7}{2}}(\rho\tilde{\Phi}_z)_z, \\
&\partial_2 N_7(\rho,\Phi,\Psi)\tilde{\Phi} = \varepsilon^{\frac{5}{2}}y\rho_x\tilde{\Phi}_x + \varepsilon^{\frac{7}{2}}y\rho_z\tilde{\Phi}_z, \\
&\partial_2 N_8(\rho,\Phi,\Psi)\tilde{\Phi} =
\end{aligned}$$

$$- \int_0^2 \left\{ \varepsilon^2 \left(\Phi_x - \frac{\varepsilon y \Psi \rho_x}{1+\varepsilon\rho} \right) \tilde{\Phi}_x + \varepsilon^3 \left(\Phi_z - \frac{\varepsilon y \Psi \rho_z}{1+\varepsilon\rho} \right) \tilde{\Phi}_z \right.$$
$$\left. + \varepsilon^2 (y\Psi\tilde{\Phi}_x)_x + \varepsilon^3 (y\Psi\tilde{\Phi}_z)_z + \frac{\varepsilon^{\frac{3}{2}} y \Psi \tilde{\Phi}_x}{1+\varepsilon\rho} + \frac{\varepsilon^2 y \Psi \tilde{\Phi}_z}{1+\varepsilon\rho} \right\} \mathrm{d}y$$

(expressions for $\partial_1 N_6(\rho,\Phi,\Psi)\tilde{\rho}$, $\partial_1 N_7(\rho,\Phi,\Psi)\tilde{\rho}$, $\partial_1 N_8(\rho,\Phi,\Psi)\tilde{\rho}$ have already been computed); arguing as above, we find that

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \leq c(\varepsilon^{\frac{3}{4}-\Delta}|\tilde{\rho}|_{\delta,p,\varepsilon} + \varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}) \leq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon},$$

where we have used the estimate $|\rho_{\Phi_2}\tilde{\Phi}_2| \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}$ (see Theorem 2.11). □

Corollary 2.17 *The solution $\Psi = \Psi(\Phi_1, \Phi_2)$ to (52) identified in the previous theorem satisfies the estimate*

$$\|\Psi\|_2 \leq c(\varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}(\|\Phi_2\|_{1,2,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,2}})P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}, \|\Phi_2\|_{1+\delta,p,\varepsilon})). \quad (58)$$

Moreover Ψ is a smooth function of (Φ_1, Φ_2) with respect to the $L^2(\Sigma)$ and $[U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)] \times W_\varepsilon^{1,2}(\Sigma)$ topologies and in particular its derivative with respect to Φ_2 satisfies the estimate

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_2 \leq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}.$$

Proof. We apply our fixed-point theorem to (52), working in the closed subsets

$$\begin{aligned} X &= L^2(\Sigma) \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}\}, \\ Y_1 &= [U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)] \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}\}, \\ Y_2 &= W_\varepsilon^{1,2}(\Sigma) \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\} \end{aligned}$$

of respectively $\mathcal{X} = L^2(\Sigma)$, $\mathcal{Y}_1 = U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$, $\mathcal{Y}_2 = W_\varepsilon^{1,2}(\Sigma)$. We therefore verify that

$$\begin{aligned} &\|\mathcal{F}_2(0, \Phi_1, \Phi_2)\|_2 \\ &\quad \leq c(\varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}(\|\Phi_2\|_{1,2,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,2}})P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}, \|\Phi_2\|_{1+\delta,p,\varepsilon})) \quad (59) \end{aligned}$$

and that

$$\|\mathrm{d}_1\mathcal{F}_2[\Psi, \Phi_1, \Phi_2]\|_{L^2(\Sigma)\to L^2(\Sigma)} \leq \frac{1}{2}$$

whenever

$$\|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}$$

and hence $|\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}$.

In order to estimate

$$\left\| \mathcal{F}^{-1} \left[- \int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}} \hat{N}_6(\rho(0,\Phi), \Phi, 0)\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}} \hat{N}_7(\rho(0,\Phi), \Phi, 0)\,\mathrm{d}\xi \right.\right.$$
$$\left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \hat{N}_8(\rho(0,\Phi), \Phi, 0) \right] \right\|_2$$

we recall the equation

$$\rho(0,\Phi)=\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]+\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}z\right]+\rho_{\mathrm{NL}}(0,\Phi)$$

and the inequalities

$$\begin{aligned}|\rho(0,\Phi)|_{\delta,p,\varepsilon} &\leq c(\|\Phi_x\|_{\delta,p,\varepsilon}+\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}+\|\Phi_y\|_{\delta,p,\varepsilon})^2),\\ |\rho(0,\Phi)|_{0,2,\varepsilon} &\leq c(\|\Phi_x\|_2+\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{0,2}}+\|\Phi_y\|_2)(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}+\|\Phi_y\|_{\delta,p,\varepsilon})),\\ |\rho_{\mathrm{NL}}(0,\Phi)|_{\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}},\|\Phi_y\|_{\delta,p,\varepsilon}).\end{aligned}$$

One finds that

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[\int_0^1\frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho_0,\Phi,0)\,\mathrm{d}\xi\right]\right\|_2\\
&\leq \left\|\mathcal{F}^{-1}\left[\int_0^1\frac{G_1}{\varepsilon^{5/2}}\hat{N}_6(\rho_0,\Phi,0)\,\mathrm{d}\xi\right]\right\|_{1,2,\varepsilon}\\
&\leq c(\varepsilon\|\rho_0\Phi_x\|_2+\varepsilon^{\frac{3}{2}}\|\rho_0\Phi_z\|_2)\\
&\leq c\Bigg(\varepsilon\left\|\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}\right\|_2+\varepsilon\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right]\Phi_{1x}\right\|_2\\
&\qquad+\varepsilon^{\frac{3}{2}}\left\|\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}\right\|_2+\varepsilon^{\frac{3}{2}}\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right]\Phi_{1z}\right\|_2\\
&\qquad+\varepsilon\|\rho_{\mathrm{NL}\,0}\Phi_{1x}\|_2+\varepsilon\|\rho_0\Phi_{2x}\|_2+\varepsilon^{\frac{3}{2}}\|\rho_{\mathrm{NL}\,0}\Phi_{1z}\|_2+\varepsilon^{\frac{3}{2}}\|\rho_0\Phi_{2z}\|_2\Bigg)\\
&\leq c\Bigg(\varepsilon\|\Phi_{1x}^2\|_2+\varepsilon^{\frac{3}{2}}\|\Phi_{1x}\Phi_{1z}\|_2+\varepsilon\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right]\right\|_\infty\|\Phi_1\|_{U_\varepsilon^{0,2}}\\
&\qquad+\varepsilon\|\rho_{\mathrm{NL}\,0}\|_\infty\|\Phi_1\|_{U_\varepsilon^{0,2}}+\varepsilon\|\rho_0\|_\infty\|\Phi_2\|_{U_\varepsilon^{0,2}}\Bigg)\\
&\leq c(\varepsilon\|\Phi_1\|_{U_\varepsilon^{0,4}}^2+\varepsilon^{1-\Delta}(\|\Phi_{2x}\|_{\delta,p,\varepsilon}+|\rho_{\mathrm{NL}\,0}|_{\delta,p,\varepsilon})\|\Phi_1\|_{U_\varepsilon^{0,2}}+\varepsilon^{1-\Delta}|\rho_0|_{\delta,p,\varepsilon}\|\Phi_2\|_{U_\varepsilon^{0,2}})\\
&\leq c(\varepsilon\|\Phi_1\|_{U_\varepsilon^{0,4}}^2+\varepsilon^{\frac{1}{2}-\Delta}(\|\Phi_2\|_{1,2,\varepsilon}+\varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,2}})P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}},\|\Phi_2\|_{1+\delta,p,\varepsilon})),\end{aligned}$$

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[\int_0^1\frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho_0,\Phi,0)\,\mathrm{d}\xi\right]\right\|_2\\
&\leq c(\varepsilon^2\|\rho_{0x}\Phi_x\|_2+\varepsilon^3\|\rho_{0z}\Phi_z\|_2)\\
&\leq c\Bigg(\varepsilon^2\left\|\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]_x\Phi_{1x}\right\|_2+\varepsilon^2\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right]_x\Phi_{1x}\right\|_2\\
&\qquad+\varepsilon^3\left\|\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]_z\Phi_{1z}\right\|_2+\varepsilon^3\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right]_z\Phi_{1z}\right\|_2\\
&\qquad+\varepsilon^2\|\rho_{\mathrm{NL}\,0x}\Phi_{1x}\|_2+\varepsilon^2\|\rho_{0x}\Phi_{2x}\|_2+\varepsilon^3\|\rho_{\mathrm{NL}\,0z}\Phi_{1z}\|_2+\varepsilon^3\|\rho_{0z}\Phi_{2z}\|_2\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\leq c(\varepsilon^{\frac{3}{2}}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{\frac{3}{2}-\Delta}(\|\Phi_{2x}\|_{\delta,p,\varepsilon} + |\rho_{\mathrm{NL}\,0}|_{\delta,p,\varepsilon})\|\Phi_1\|_{U^{0,2}_\varepsilon} + \varepsilon^{\frac{3}{2}-\Delta}|\rho_0|_{\delta,p,\varepsilon}\|\Phi_2\|_{U^{0,2}_\varepsilon})\\
&\leq c(\varepsilon^{\frac{3}{2}}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{1-\Delta}(\|\Phi_2\|_{1,2,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U^{0,2}_\varepsilon})P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon}, \|\Phi_2\|_{1+\delta,p,\varepsilon})),
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho_0,\Phi,0)\right]\right\|_2\\
&\quad\leq \left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_{1}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho_0,\Phi,0)\right]\right\|_{1,2,\varepsilon}\\
&\quad\leq c(\varepsilon^{\frac{7}{2}-\Delta}\|\rho_{0x}^3\|_2 + \varepsilon^{\frac{9}{2}-\Delta}\|\rho_{0z}^2\rho_{0x}\|_2 + \varepsilon^{4-\Delta}\|\rho_{0x}^2\rho_{0z}\|_2 + \varepsilon^{5-\Delta}\|\rho_{0z}^3\|_2\\
&\qquad\quad + \varepsilon^{1-\Delta}\|\Phi_x^2\|_2 + \varepsilon^{2-\Delta}\|\Phi_z^2\|_2\\
&\quad\leq c(\varepsilon^{2-\Delta}|\rho|^2_{\delta,p,\varepsilon}|\rho|_{0,2,\varepsilon}\\
&\qquad\quad + \varepsilon^{1-\Delta}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon}\|\Phi_2\|_{U^{\delta,p}_\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon}\|\Phi\|_{U^{\delta,p}_\varepsilon})\\
&\quad\leq c(\varepsilon^{1-\Delta}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{\frac{1}{2}-\Delta}(\|\Phi_2\|_{1,2,\varepsilon} + \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U^{0,2}_\varepsilon})P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon}, \|\Phi_2\|_{1+\delta,p,\varepsilon})),
\end{aligned}
$$

where $\rho_{\mathrm{NL}\,0}$ is an abbreviation for $\rho_{\mathrm{NL}}(0,\Phi)$, and (59) follows from these inequalities. The estimate for $\|\mathrm{d}_1\mathcal{F}_2[\Psi,\Phi_1,\Phi_2]\|_{L^2(\Sigma)\to L^2(\Sigma)}$ is obtained using the method developed to estimate $|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2]|_{V^{0,2}_\varepsilon(\mathbb{R}^2)\to V^{0,2}_\varepsilon(\mathbb{R}^2)}$ in Corollary 2.12; one finds that

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[-\int_0^1\frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)\bar{\rho}\,\mathrm{d}\xi - \int_0^1\frac{G_{1y}}{\varepsilon^{5/2}}\partial_3\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi\right.\right.\\
&\qquad -\int_0^1\frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)\bar{\rho}\,\mathrm{d}\xi - \int_0^1\frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_3\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi\\
&\qquad\left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)\bar{\rho} + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_3\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Psi}\right]\right\|_2\\
&\leq c(\varepsilon^{\frac{3}{4}-\Delta}|\bar{\rho}|_{0,2,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_2)\\
&\leq c\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_2,
\end{aligned}
$$

where $\bar{\rho} = \rho_\Psi\tilde{\Psi}$ and the estimate $|\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_2$ has also been used (see Corollary 2.12).

Finally we note that

$$
\begin{aligned}
&\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_2\\
&\leq 2\left\|\mathcal{F}^{-1}\left[-\int_0^1\frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1\frac{G_{1y}}{\varepsilon^{5/2}}\partial_2\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi\right.\right.\\
&\qquad -\int_0^1\frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1\frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_2\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi\\
&\qquad\left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)\tilde{\rho} + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_2\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Phi}_2\right]\right\|_{1,2,\varepsilon}\\
&\leq c(\varepsilon^{\frac{3}{4}-\Delta}|\tilde{\rho}|_{0,2,\varepsilon} + \varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon})\\
&\leq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon},
\end{aligned}
$$

in which $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$ and the last line follows from the estimate $|\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}$ (see Corollary 2.12). □

We now substitute $\Psi = \Psi(\Phi_1, \Phi_2)$ into (50), write the resulting equation as

$$\Phi_2 = \mathcal{F}_3(\Phi_1, \Phi_2) \tag{60}$$

and solve this equation for Φ_2 as a function of Φ_1 using our fixed-point theorem.

Theorem 2.18 *Suppose that*

$$\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}. \tag{61}$$

Equation (60) has a unique solution $\Phi_2 = \Phi_2(\Phi_1)$ which satisfies the estimate

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}). \tag{62}$$

Moreover Φ_2 depends smoothly upon Φ_1 with respect to the $W_\varepsilon^{1+\delta,p}(\Sigma)$ and $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ topologies.

Proof. This result is established by applying Theorem 1.5 with $\mathcal{X} = W_\varepsilon^{1+\delta,p}(\Sigma)$, $\mathcal{Y} = U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and X, Y closed origin-centred balls of radius $\mathcal{O}(\varepsilon^{-\Delta})$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$; we show that

$$\|\mathcal{F}_3(\Phi_1, 0)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}) \tag{63}$$

and that

$$\|\mathrm{d}_2\mathcal{F}_3[\Phi_1, \Phi_2]\|_{W_\varepsilon^{1+\delta,p}(\Sigma)\to W_\varepsilon^{1+\delta,p}(\Sigma)} \leq \frac{1}{2}$$

whenever (61) and (62) hold.

Let us first examine

$$\mathcal{F}^{-1}\biggl[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\hat{N}_6(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi$$
$$+ \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\biggr],$$

using the estimates

$$\|\Psi(\Phi_1)\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}), \tag{64}$$

$$|\rho(\Psi(\Phi_1), \Phi_1)|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}(\|\Phi_{1x}\|_{\delta,p,\varepsilon} + P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})). \tag{65}$$

The estimation methods used in Theorem 2.16 yield

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\hat{N}_6(\rho_1, \Phi_1, \Psi_1)\,\mathrm{d}\xi\right]\right\|_{1+\delta,p,\varepsilon} \\
&\quad\leq c(\varepsilon^{1-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Phi_1\|_{U_\varepsilon^{\delta,p}} + \varepsilon^{\frac{1}{2}-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|_{\delta,p,\varepsilon}) \\
&\quad\leq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}),
\end{aligned}$$

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho_1,\Phi_1,\Psi_1)\,\mathrm{d}\xi\right]\right\|_{1+\delta,p,\varepsilon} \\
&\qquad \leq \quad c(\varepsilon^{\frac{1}{2}-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Phi_1\|_{U_\varepsilon^{\delta,p}} + \varepsilon^{1-\Delta}|\rho_1|^2_{\delta,p,\varepsilon}\|\Psi_1\|_{\delta,p,\varepsilon} + \varepsilon^{-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|_{\delta,p,\varepsilon}) \\
&\qquad \leq \quad c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}),
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho_1,\Phi_1,\Psi_1)\right]\right\|_{1+\delta,p,\varepsilon} \\
&\qquad \leq \quad c(\varepsilon^{2-\Delta}|\rho_1|^3_{\delta,p,\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_1\|^2_{U_\varepsilon^{\delta,p}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \\
&\qquad\qquad + \varepsilon^{-\Delta}\|\Psi_1\|^2_{\delta,p,\varepsilon} + \varepsilon^{1-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|^2_{\delta,p,\varepsilon} + \varepsilon^{2-\Delta}|\rho_1|^2_{\delta,p,\varepsilon}\|\Psi_1\|^2_{\delta,p,\varepsilon} \\
&\qquad\qquad + \varepsilon^{\frac{3}{2}-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}\|\Psi_1\|_{\delta,p,\varepsilon}) \\
&\qquad \leq \quad c\varepsilon^{-\Delta}P_2(\varepsilon^{\frac{1}{2}}|\rho_1|_{\delta,p,\varepsilon}, \varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}, \|\Psi_1\|_{\delta,p,\varepsilon}) \\
&\qquad \leq \quad c\varepsilon^{\frac{1}{2}-\Delta}P_2(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}),
\end{aligned}
$$

where ρ_1 and Ψ_1 are abbreviations for respectively $\rho(\Psi(\Phi_1),\Phi_1)$ and $\Psi(\Phi_1)$, and inequality (63) is an immediate consequence of these estimates.

Writing $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, $\bar{\rho} = \rho_\Psi\tilde{\Psi}$, $\tilde{\Psi} = \Psi_{\Phi_2}\tilde{\Phi}_2$ and using the estimates (61) and

$$
|\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}, \qquad \|\Phi_2\|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}
$$

(which follow from (61), (62), (64), (65)), we find that

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)(\tilde{\rho}+\bar{\rho})\,\mathrm{d}\xi - \int_0^1 \frac{G_1}{\varepsilon^{5/2}}\partial_2\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi\right.\right. \\
&\qquad - \int_0^1 \frac{G_1}{\varepsilon^{5/2}}\partial_3\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)(\tilde{\rho}+\bar{\rho})\,\mathrm{d}\xi \\
&\qquad - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\partial_2\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\partial_3\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi \\
&\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)(\tilde{\rho}+\bar{\rho}) + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_2\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Phi}_2 \\
&\qquad \left.\left. + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_3\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Psi}\right]\right\|_{1+\delta,p,\varepsilon} \\
&\leq c(\varepsilon^{\frac{1}{4}-\Delta}(|\tilde{\rho}|_{\delta,p,\varepsilon} + \bar{\rho}|_{\delta,p,\varepsilon}) + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}) \\
&\leq c(\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon} + \varepsilon^{-\frac{1}{4}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}) \\
&\leq c\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon},
\end{aligned}
$$

in which the further inequalities

$$
|\rho_{\Phi_2}\tilde{\Phi}_2|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}, \qquad |\rho_\Psi\tilde{\Psi}|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon},
$$
$$
\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{\delta,p,\varepsilon}
$$

have been used (see Theorems 2.11 and 2.16). □

Corollary 2.19 *The solution $\Phi_2 = \Phi_2(\Phi_1)$ to (60) identified in the previous theorem satisfies the estimate*

$$\|\Phi_2\|_{1,2,\varepsilon} \leq c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})). \tag{66}$$

Moreover Φ_2 depends smoothly upon Φ_1 with respect to the $W_\varepsilon^{1,2}(\Sigma)$ and $U_\varepsilon^{0,2}(\mathbb{R}^2) \times U_\varepsilon^{0,4}(\mathbb{R}^2)$ topologies.

Proof. We again note that

$$\begin{aligned} X &= W_\varepsilon^{1,2}(\Sigma) \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\}, \\ Y &= [U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)] \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}\} \end{aligned}$$

are closed subsets of respectively $\mathcal{X} = W^{1,2}(\Sigma)$, $\mathcal{Y} = U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$ and apply our fixed-point equation to (52) with these definitions of $\mathcal{X}$, $\mathcal{Y}$ and X, Y, verifying that

$$\|\mathcal{F}_3(\Phi_1, 0)\|_2 \leq c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}))$$

and that

$$\|\mathrm{d}_2\mathcal{F}_3[\Phi_1, \Phi_2]\|_{W_\varepsilon^{1,2}(\Sigma)\to W_\varepsilon^{1,2}(\Sigma)} \leq \frac{1}{2}$$

whenever

$$\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}$$

and hence

$$|\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}.$$

We begin by estimating

$$\begin{aligned} \mathcal{F}^{-1}\bigg[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \hat{N}_6(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \hat{N}_7(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi \\ + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \hat{N}_8(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\bigg], \end{aligned}$$

where we use the inequalities

$$\begin{aligned} \|\Psi(\Phi_1)\|_{\delta,p,\varepsilon} &\leq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}), \\ \|\Psi(\Phi_1)\|_2 &\leq c(\varepsilon\|\Phi_1\|^2_{U_\varepsilon^{0,4}} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U_\varepsilon^{0,2}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})), \\ |\rho(\Psi_1(\Phi_1), \Phi_1)|_{\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}(\|\Phi_{1x}\|_{\delta,p,\varepsilon} + P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}), \\ |\rho_{\mathrm{NL}}(\Psi_1(\Phi_1), \Phi_1)|_{\delta,p,\varepsilon} &\leq c\varepsilon^{\frac{1}{2}-\Delta} P_2(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}). \end{aligned}$$

The estimation techniques used in Corollary 2.17 yield

$$\begin{aligned} &\left\|\mathcal{F}^{-1}\bigg[\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \hat{N}_6(\rho_1, \Phi_1, \Psi_1)\,\mathrm{d}\xi\bigg]\right\|_{1,2,\varepsilon} \\ &\quad\leq c(\varepsilon\|\Phi_1\|^2_{U_\varepsilon^{0,4}} + \varepsilon^{\frac{1}{2}-\Delta}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Phi_1\|_{U_\varepsilon^{0,2}} \\ &\qquad + \varepsilon^{1-\Delta}|\rho_{\mathrm{NL}\,1}|_{\delta,p,\varepsilon}\|\Phi_1\|_{U_\varepsilon^{0,2}} + \varepsilon^{\frac{1}{2}-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|_2) \end{aligned}$$

$$
\leq \quad c(\varepsilon\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})),
$$

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho_1,\Phi_1,\Psi_1)\,\mathrm{d}\xi\right]\right\|_{1,2,\varepsilon}\\
&\qquad\leq \quad c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{-\Delta}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Phi_1\|_{U^{0,2}_\varepsilon}\\
&\qquad\qquad + \varepsilon^{\frac{1}{2}-\Delta}|\rho_{\mathrm{NL}\,1}|_{\delta,p,\varepsilon}\|\Phi_1\|_{U^{0,2}_\varepsilon} + \varepsilon^{-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|_2)\\
&\qquad\leq \quad c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})),
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho_1,\Phi_1,\Psi_1)\right]\right\|_{1,2,\varepsilon}\\
&\qquad\leq \quad c(\varepsilon^{2-\Delta}|\rho_1|^2_{\delta,p,\varepsilon}|\rho_1|_{0,2,\varepsilon} + \varepsilon^{1-\Delta}\|\Phi_1\|_{U^{\delta,p}_\varepsilon}\|\Phi_1\|_{U^{0,2}_\varepsilon} + \varepsilon^{\frac{1}{2}-\Delta}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Phi_1\|_{U^{0,2}_\varepsilon}\\
&\qquad\qquad + \varepsilon^{-\Delta}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Psi_1\|_2 + \varepsilon^{1-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Psi_1\|_2\\
&\qquad\qquad + \varepsilon^{2-\Delta}|\rho_1|^2_{\delta,p,\varepsilon}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Psi_1\|_2 + \varepsilon^{\frac{3}{2}-\Delta}|\rho_1|_{\delta,p,\varepsilon}\|\Psi_1\|_{\delta,p,\varepsilon}\|\Phi_1\|_{U^{0,2}_\varepsilon})\\
&\qquad\leq \quad c(\varepsilon^{\frac{1}{2}-\Delta}\|\Psi_1\|_2 + \varepsilon\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})\\
&\qquad\leq \quad c(\varepsilon^{\frac{1}{2}}\|\Phi_1\|^2_{U^{0,4}_\varepsilon} + \varepsilon^{\frac{1}{2}-\Delta}\|\Phi_1\|_{U^{0,2}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})),
\end{aligned}
$$

where $\rho_{\mathrm{NL}\,1}$ is an abbreviation for $\rho_{\mathrm{NL}}(\Psi(\Phi_1),\Phi_1)$, and (59) follows from these inequalities.

The estimate for $\|\mathrm{d}_2\mathcal{F}_3[\Phi_1,\Phi_2]\|_{W^{1,2}_\varepsilon(\Sigma)\to W^{1,2}_\varepsilon(\Sigma)}$ is obtained using the method developed to estimate $\|\mathrm{d}_1\mathcal{F}_2[\Psi,\Phi_1,\Phi_2]\|_{L^2(\Sigma)\to L^2(\Sigma)}$ in Corollary 2.17; one finds that

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)(\tilde{\rho}+\bar{\rho})\,\mathrm{d}\xi - \int_0^1 \frac{G_1}{\varepsilon^{5/2}}\partial_2\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi\right.\right.\\
&\qquad - \int_0^1 \frac{G_1}{\varepsilon^{5/2}}\partial_3\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)(\tilde{\rho}+\bar{\rho})\,\mathrm{d}\xi\\
&\qquad - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\partial_2\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}}\partial_3\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi\\
&\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)(\tilde{\rho}+\bar{\rho}) + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_2\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Phi}_2\\
&\qquad \left.\left. + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_3\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Psi}\right]\right\|_{1,2,\varepsilon}\\
&\leq \quad c(\varepsilon^{\frac{1}{4}-\Delta}(|\tilde{\rho}|_{0,2,\varepsilon} + |\bar{\rho}|_{0,2,\varepsilon}) + \varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon} + \varepsilon^{-\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_2)\\
&\leq \quad c(\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon} + \varepsilon^{-\frac{1}{4}-\Delta}\|\tilde{\Psi}\|_{\delta,p,\varepsilon})\\
&\leq \quad c\varepsilon^{\frac{1}{4}-\Delta}\|\tilde{\Phi}_2\|_2,
\end{aligned}
$$

where $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, $\bar{\rho} = \rho_\Psi\tilde{\Psi}$, $\tilde{\Psi} = \Psi_{\Phi_2}\tilde{\Phi}_2$ and the further inequalities

$$
|\rho_{\Phi_2}\tilde{\Phi}_2|_{0,2,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}, \quad |\rho_\Psi\tilde{\Psi}|_{0,2,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}-\Delta}\|\tilde{\Psi}\|_2, \quad \|\Psi_{\Phi_2}\tilde{\Phi}_2\|_2 \leq c\varepsilon^{\frac{3}{4}-\Delta}\|\tilde{\Phi}_2\|_{1,2,\varepsilon}
$$

have been used (see Corollaries 2.12 and 2.17). □

2.4 Regularity theory

The (integral form of the) reduced equation for Φ_1 is obtained by substituting $\rho = \rho(\Phi_1)$ and $\Phi_2 = \Phi_2(\Phi_1)$ (where $\rho(\Phi_1)$ is an abbreviation for $\rho(\Phi_{2y}(\Phi_1), \Phi_2(\Phi_1), \Phi_1)$ and Ψ has been identified with Φ_{2y}) into the integral form of the equation for Φ_1. One finds that

$$\hat{\Phi}_1 = \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \varepsilon^{-\frac{1}{2}}\hat{N}_4(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))\,\mathrm{d}\xi - \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))\right); \tag{67}$$

according to the material presented in Sections 2.2 and 2.3 above, the quantity in brackets on the right-hand side of this equation is well defined provided that

$$\|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \tag{68}$$

whence

$$|\rho|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}, \qquad \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}. \tag{69}$$

The corresponding weak formulation of the reduced equation for Φ_1 (see Definition 2.7(i)) requires that $\Phi_1 \in X$; in view of the embedding (24) we therefore study the integral and weak formulations of this equation in the closed origin-centred ball $\{\Phi_1 \in X : |\!|\!|\Phi_1|\!|\!| \leq c\}$ of X.

Any solution of the integral form of the reduced equation for Φ_1 defines a weak solution $(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))$ of the scaled water-wave problem (26)–(29), and in Section 3 this aspect of the existence theory is completed with the confirmation that (67) indeed has a nonzero solution. In this section we complete the analysis of the reduction procedure by presenting regularity theory which asserts that Φ_1, Φ_2 and ρ actually belong to the smaller function spaces $U_\varepsilon^{5,p}(\mathbb{R}^2)$, $W_\varepsilon^{2,p}(\Sigma)$ and $V_\varepsilon^{1,p}(\mathbb{R}^2)$ and solve the strong forms of their equations; it follows that $(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))$ is a strong solution of the equations (26)–(29).

Our first regularity result (Proposition 2.21 below) shows that Φ_1 belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$. In order to establish this result we need the following lemma, which deals with Fourier-multiplier operators appearing in the integral form of the equation for Φ_1; its proof is given in Section 4.

Lemma 2.20

(i) For each $u \in L^p(\mathbb{R}^2)$ the function

$$\mathcal{G}_{12}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right]$$

belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{12}(u)\|_{U_\varepsilon^{2,p}} \leq c\|u\|_p.$$

(ii) For each $u \in L^p(\mathbb{R}^2)$ the function

$$\mathcal{G}_{13}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{Q}\mathcal{F}[u]\right]$$

belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{13}(u)\|_{U_\varepsilon^{2,p}} \leq c\|u\|_p.$$

(iii) For each $u \in L^p(\mathbb{R}^2)$ the function

$$\mathcal{G}_{14}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{(1+\varepsilon+\beta q^2)Q}\mathcal{F}[u]\right]$$

belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{14}(u)\|_{U_\varepsilon^{2,p}} \leq c\|u\|_p.$$

(iv) For each $u \in L^p(\mathbb{R}^2)$ the function

$$\mathcal{G}_{15}(u) = \mathcal{F}^{-1}\left[\frac{-\mu^2}{(1+\varepsilon+\beta q^2)Q}\mathcal{F}[u]\right]$$

belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{15}(u)\|_{U_\varepsilon^{2,p}} \leq c\varepsilon^{-\frac{1}{2}}\|u\|_p.$$

(v) For each $u \in L^p(\mathbb{R}^2)$ the function

$$\mathcal{G}_{16}(u) = \mathcal{F}^{-1}\left[\frac{-\varepsilon^{\frac{1}{2}}\mu k}{(1+\varepsilon+\beta q^2)Q}\mathcal{F}[u]\right]$$

belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimate

$$\|\mathcal{G}_{16}(u)\|_{U_\varepsilon^{2,p}} \leq c\varepsilon^{-\frac{1}{2}}\|u\|_p.$$

Proposition 2.21 *A solution of the integral form of the equation for Φ_1 which satisfies $|||\Phi_1||| \leq c$ belongs to $U_\varepsilon^{2,p}(\mathbb{R}^2)$ and satisfies the estimates*

$$\|\Phi_1\|_{U_\varepsilon^{1,p}} \leq c\varepsilon^{-\frac{3}{8}-\Delta}, \qquad \|\Phi_1\|_{U_\varepsilon^{2,p}} \leq c\varepsilon^{-\frac{1}{2}-\Delta}.$$

Proof. Using Lemma 2.20(i)-(ii) and the estimates (69) we find that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\frac{1}{Q}\int_0^1 \varepsilon^{-\frac{5}{2}}N_4(\rho,\Phi)\,\mathrm{d}y\right]\right\|_{U_\varepsilon^{2,p}} \\
&\quad\leq\ c(\|\rho\Phi_x\|_p + \varepsilon^{\frac{1}{2}}\|\rho\Phi_z\|_p + \|\Phi_y\rho_x\|_p + \varepsilon^{\frac{1}{2}}\|\Phi_y\rho_z\|_p) \\
&\quad\leq\ c\varepsilon^{-\Delta}(|\rho|_{\delta,p,\varepsilon}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \varepsilon^{-\frac{1}{2}}\|\Phi_y\|_{\delta,p,\varepsilon}|\rho|_{\delta,p,\varepsilon}) \\
&\quad\leq\ c\varepsilon^{-\frac{1}{2}-\Delta},
\end{aligned}$$

and a similar calculation using Lemma 2.20(iii)-(v) and (69) shows that

$$\left\|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q(1+\varepsilon+\beta q^2)}\varepsilon^{-2}\hat{N}_1(\rho,\Phi)\right]\right\|_{U_\varepsilon^{2,p}} \leq c\varepsilon^{-\frac{1}{2}-\Delta}.$$

An inspection of the reduced equation (67) shows that

$$\|\Phi_1\|_{U^{2,p}_\varepsilon} \le c\varepsilon^{-\frac{1}{2}-\Delta}, \tag{70}$$

and the remaining estimate

$$\|\Phi_1\|_{U^{1,p}_\varepsilon} \le c\varepsilon^{-\frac{3}{8}-\Delta}$$

follows by interpolation between (70) and

$$\|\Phi_1\|_{U^{0,p}_\varepsilon} \le \|\Phi_1\|_{U^{\delta,p}_\varepsilon} \le c\varepsilon^{-\frac{1}{4}-\Delta}$$

(see equation (68)). □

The next step is to reappraise the integral equations for ρ, Ψ and Φ_2 in the light of the improved regularity of Φ_1. We proceed in the spirit of Corollaries 2.12, 2.17 and 2.19, which show how these integral equations, which were originally solved in $V^{\delta,p}_\varepsilon(\mathbb{R}^2)$, $W^{\delta,p}_\varepsilon(\Sigma)$ and $W^{1+\delta,p}_\varepsilon(\Sigma)$, are also solvable in $V^{0,2}_\varepsilon(\mathbb{R}^2)$, $L^2(\Sigma)$ and $W^{0,2}_\varepsilon(\Sigma)$; here we give three Lemmata which show that they are solvable in $V^{1,p}_\varepsilon(\mathbb{R}^2)$, $W^{1,p}_\varepsilon(\Sigma)$ and $W^{2,p}_\varepsilon(\Sigma)$.

Lemma 2.22 *Suppose that*

$$\|\Phi_1\|_{1,p,\varepsilon} \le c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\Phi_2\|_{2,p,\varepsilon} \le c\varepsilon^{-\frac{1}{8}-\Delta}, \quad \|\Psi\|_{1,p,\varepsilon} \le c\varepsilon^{\frac{3}{8}-\Delta}. \tag{71}$$

The solution $\rho = \rho(\Psi, \Phi_1, \Phi_2)$ to (39) identified in Theorem 2.11 satisfies the estimate

$$|\rho|_{1,p,\varepsilon} \le c(\|\Phi_x\|_{1,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{1,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U^{1,p}_\varepsilon} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U^{\delta,p}_\varepsilon} + \|\Phi_y\|_{\delta,p,\varepsilon})). \tag{72}$$

Moreover ρ depends smoothly upon (Ψ_1, Φ_1, Φ_2) with respect to the $V^{1,p}_\varepsilon(\mathbb{R}^2)$ and $W^{1,p}_\varepsilon(\Sigma) \times U^{1,p}_\varepsilon(\mathbb{R}^2) \times W^{2,p}_\varepsilon(\Sigma)$ toplogies and in particular its derivatives with respect to Ψ and Φ_2 satisfy the estimates

$$|\rho_\Psi \tilde{\Psi}|_{1,p,\varepsilon} \le c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon}, \qquad |\rho_{\Phi_2}\tilde{\Phi}_2|_{1,p,\varepsilon} \le c\varepsilon^{-\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon}.$$

Proof. We apply our fixed-point theorem to (39), working in the closed subsets

$$\begin{aligned}
X &= \{\rho \in V^{1,p}_\varepsilon(\mathbb{R}^2) : |\rho|_{1,p,\varepsilon} \le c\varepsilon^{-\frac{3}{8}-\Delta}\} \cap \{\rho \in V^{\delta,p}_\varepsilon(\mathbb{R}^2) : |\rho|_{\delta,p,\varepsilon} \le c\varepsilon^{-\frac{1}{4}-\Delta}\},\\
Y_1 &= \{\Psi_2 \in W^{1,p}_\varepsilon(\Sigma) : \|\Psi\|_{1,p,\varepsilon} \le c\varepsilon^{\frac{3}{8}-\Delta}\} \cap \{\Psi \in W^{\delta,p}_\varepsilon(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \le c\varepsilon^{\frac{1}{2}-\Delta}\},\\
Y_2 &= \{\Phi_1 \in U^{1,p}_\varepsilon(\mathbb{R}^2) : \|\Phi_1\|_{U^{1,p}_\varepsilon} \le c\varepsilon^{-\frac{3}{8}-\Delta}\} \cap \{\Phi_1 \in U^{\delta,p}_\varepsilon(\mathbb{R}^2) : \|\Phi_1\|_{U^{\delta,p}_\varepsilon} \le c\varepsilon^{-\frac{1}{4}-\Delta}\},\\
Y_3 &= \{\Phi_2 \in W^{2,p}_\varepsilon(\Sigma) : \|\Phi_2\|_{2,p,\varepsilon} \le c\varepsilon^{-\frac{1}{8}-\Delta}\} \cap \{\Phi_2 \in W^{1+\delta,p}_\varepsilon(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \le c\varepsilon^{-\Delta}\}
\end{aligned}$$

of respectively $\mathcal{X} = V^{1,p}_\varepsilon(\mathbb{R}^2)$, $\mathcal{Y}_1 = W^{1,p}_\varepsilon(\Sigma)$. $\mathcal{Y}_2 = U^{1,p}_\varepsilon(\mathbb{R}^2)$, $\mathcal{Y}_3 = W^{2,p}_\varepsilon(\Sigma)$. Our task is to verify that

$$\begin{aligned}
&|\mathcal{F}_1(0, \Psi, \Phi_1, \Phi_2)|_{1,p,\varepsilon}\\
&\qquad \le c(\|\Phi_x\|_{1,p,\varepsilon} + \varepsilon^{-\frac{1}{2}}\|\Psi\|_{1,p,\varepsilon} + \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U^{1,p}_\varepsilon} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U^{\delta,p}_\varepsilon} + \|\Phi_y\|_{\delta,p,\varepsilon}))
\end{aligned}$$

and that

$$|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2]|_{V_\varepsilon^{1,p}(\mathbb{R}^2)\to V_\varepsilon^{1,p}(\mathbb{R}^2)} \le \frac{1}{2}$$

whenever

$$|\rho|_{\delta,p,\varepsilon} \le c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \le c\varepsilon^{-\frac{1}{4}-\Delta}, \quad \|\Phi_2\|_{1+\delta,p,\varepsilon} \le c\varepsilon^{-\Delta}, \quad \|\Psi\|_{\delta,p,\varepsilon} \le c\varepsilon^{\frac{1}{2}-\Delta} \tag{73}$$

and (71), (72) hold.

Observe that

$$\begin{aligned}&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x}+\mathrm{i}\mu\int_0^1 y\hat{\Psi}\,\mathrm{d}y+\int_0^1\hat{\Phi}_{2x}\,\mathrm{d}y\right)\right]\right|_{1,p,\varepsilon}\\ &\qquad\le c(\|\Phi_{1x}\|_{1,p,\varepsilon}+\|\Phi_{2x}\|_{1,p,\varepsilon}+\varepsilon^{-1/2}\|\Psi\|_{1,p,\varepsilon})\end{aligned}$$

and

$$\begin{aligned}&\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\hat{N}_1(0,\Phi)\right]\right|_{1,p,\varepsilon}\\
\le\;& \left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\left\{\frac{\varepsilon}{2}\Phi_x^2+\frac{\varepsilon^2}{2}\Phi_z^2+\varepsilon^{\frac{1}{2}}\Phi_x y\Phi_y+\varepsilon\Phi_z y\Phi_y+\frac{1}{2}\Phi_y^2\right\}\mathrm{d}y\right]\right]\right|_{1,p,\varepsilon}\\
&\quad+\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\varepsilon\Phi_x y\Phi_y\,\mathrm{d}y\right]\right]\right|_{1,p,\varepsilon}\\
&\quad+\left|\mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}\left[\int_0^1\varepsilon^{\frac{3}{2}}\Phi_z y\Phi_z\,\mathrm{d}y\right]\right]\right|_{1,p,\varepsilon}\\
\le\;& c(\varepsilon\|\Phi_x^2\|_{1,p,\varepsilon}+\varepsilon^2\|\Phi_z^2\|_{1,p,\varepsilon}+\varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_y\|_{1,p,\varepsilon}+\varepsilon\|\Phi_z y\Phi_y\|_{1,p,\varepsilon}+\|\Phi_y^2\|_{1,p,\varepsilon}\\
&\qquad+\varepsilon^{\frac{1}{2}}\|\Phi_x y\Phi_y\|_{1,p,\varepsilon}+\varepsilon\|\Phi_z y\Phi_y\|_{1,p,\varepsilon})\\
\le\;& c(\varepsilon^{1-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_x\|_{1,p,\varepsilon}+\varepsilon^{2-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}\|\Phi_z\|_{1,p,\varepsilon}+\varepsilon^{-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}\|\Phi_y\|_{1,p,\varepsilon}\\
&\qquad+\varepsilon^{\frac{1}{2}-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_y\|_{1,p,\varepsilon}+\varepsilon^{\frac{1}{2}-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}\|\Phi_x\|_{1,p,\varepsilon}\\
&\qquad+\varepsilon^{1-\Delta}\|\Phi_z\|_{\delta,p,\varepsilon}\|\Phi_y\|_{1,p,\varepsilon}+\varepsilon^{1-\Delta}\|\Phi_y\|_{\delta,p,\varepsilon}\|\Phi_z\|_{1,p,\varepsilon})\\
\le\;& \varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}}+\|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}+\|\Phi_y\|_{\delta,p,\varepsilon}),\end{aligned}$$

where we have estimated for example

$$\begin{aligned}\|\Phi_x^2\|_{1,p,\varepsilon} &= \|\Phi_x^2\|_p+2\|\Phi_x\Phi_{xx}\|_p+2\varepsilon^{\frac{1}{2}}\|\Phi_x\Phi_{xz}\|_p\\ &\le \|\Phi_x\|_\infty(\|\Phi_x\|_p+\|\Phi_{xx}\|_p+\varepsilon^{\frac{1}{2}}\|\Phi_{xz}\|_p)\\ &\le \varepsilon^{-\Delta}\|\Phi_x\|_{\delta,p,\varepsilon}\|\Phi_x\|_{1,p,\varepsilon}.\end{aligned}$$

The estimate for $|\mathcal{F}_1(0,\Psi,\Phi_1,\Phi_2)|_{1,p,\varepsilon}$ follows directly from the above calculations.

The bound for $|\mathrm{d}_1\mathcal{F}_1[\rho,\Psi,\Phi_1,\Phi_2]|_{V_\varepsilon^{1,p}(\mathbb{R}^2)\to V_\varepsilon^{1,p}(\mathbb{R}^2)}$ is obtained by estimating

$$\left|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\varepsilon^{-1}\partial_1\hat{N}_1(\rho,\Phi)\tilde{\rho}\right]\right|_{1,p,\varepsilon}$$

using the assumptions (71), (73) and

$$|\rho|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}$$

(which follows from (71) and (72)), together with the rule

$$\begin{aligned}
&\left\| \frac{u}{1+\varepsilon\rho} \right\|_{1,p,\varepsilon} \\
&= \left\| \frac{u}{1+\varepsilon\rho} \right\|_p + \left\| \frac{u_x}{1+\varepsilon\rho} \right\|_p + \varepsilon^{\frac{1}{2}} \left\| \frac{u_z}{1+\varepsilon\rho} \right\|_p + \varepsilon \left\| \frac{\rho_x u}{(1+\varepsilon\rho)^2} \right\|_p + \varepsilon^{\frac{3}{2}} \left\| \frac{\rho_z u}{(1+\varepsilon\rho)^2} \right\|_p \\
&\leq c\left(\left\| \frac{1}{1+\varepsilon\rho} \right\|_\infty \|u\|_{1,p,\varepsilon} + \varepsilon(\|\rho_x\|_\infty + \varepsilon^{\frac{1}{2}}\|\rho_z\|_\infty) \left\| \frac{1}{(1+\varepsilon\rho)^2} \right\|_\infty \|u\|_p \right) \\
&\leq c\|u\|_{1,p,\varepsilon}
\end{aligned}$$

(with similar rules for the other denominators). We find for example that

$$\begin{aligned}
&\left| \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2} \mathcal{F}\left[\int_0^1 \frac{\varepsilon^2 y^2 \Phi_y^2 \tilde{\rho}_x}{1+\varepsilon\rho} \right] \right] \right|_{1,p,\varepsilon} \\
&\qquad \leq c\varepsilon^{\frac{3}{2}} \left\| \frac{y^2 \Phi_y^2 \tilde{\rho}_x}{1+\varepsilon\rho} \right\|_{1,p,\varepsilon} \\
&\qquad \leq c\varepsilon^{\frac{3}{2}} \|y^2 \Phi_y^2 \tilde{\rho}_x\|_{1,p,\varepsilon} \\
&\qquad \leq c(\varepsilon^{\frac{3}{2}-\Delta} \|\Phi_y\|_{\delta,p,\varepsilon}^2 \|\tilde{\rho}_x\|_{1,p,\varepsilon} + \|\Phi_y\|_{1,p,\varepsilon} \|\Phi_y\|_{\delta,p,\varepsilon} \|\tilde{\rho}_x\|_{\delta,p,\varepsilon}) \\
&\qquad \leq c\varepsilon^{\frac{11}{8}-\Delta} \|\tilde{\rho}_x\|_{1,p,\varepsilon} \\
&\qquad \leq c\varepsilon^{\frac{7}{8}-\Delta} |\tilde{\rho}|_{1,p,\varepsilon},
\end{aligned}$$

and estimating each term in this fashion one concludes that

$$\left| \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2} \varepsilon^{-1} \partial_1 \hat{N}_1(\rho, \Phi)\tilde{\rho} \right] \right|_{1,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta} |\tilde{\rho}|_{1,p,\varepsilon}. \tag{74}$$

According to our fixed-point theorem, the estimates for $\rho_\Psi \tilde{\Psi}$ and $\rho_{\Phi_2} \tilde{\Phi}_2$ are given by the formulae

$$\begin{aligned}
|\rho_\Psi \tilde{\Psi}|_{1,p,\varepsilon} &\leq 2\left| \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2} \int_0^1 y\hat{\tilde{\Psi}}\, \mathrm{d}y \right] \right|_{1,p,\varepsilon} \\
&\leq c\varepsilon^{-\frac{1}{2}} \|\tilde{\Psi}\|_{1,p,\varepsilon}, \\
|\rho_{\Phi_2} \tilde{\Phi}_2|_{1,p,\varepsilon} &\leq 2\left| \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2} \left(\mathrm{i}\mu \int_0^1 y\hat{\tilde{\Phi}}_2\, \mathrm{d}y + \varepsilon^{-1}\partial_2 \hat{N}_1(\rho, \Phi)\tilde{\Phi}_2 \right) \right] \right|_{1,p,\varepsilon} \\
&\leq c\varepsilon^{-\frac{1}{8}-\Delta} \|\tilde{\Phi}_2\|_{2,p,\varepsilon},
\end{aligned}$$

where the final inequality is obtained in the same fashion as (74). □

Before proceeding to the equations for Ψ and Φ_2, let us record some further estimates which are useful in the analysis of these equations; they are proved using the estimation techniques developed above.

Proposition 2.23 *The function $\rho = \rho(\Psi, \Phi_1\Phi_2)$ discussed in the previous lemma satisfies the further inequalities*

$$\|\rho\|_{U_\varepsilon^{0,p}} \leq c(\|\Phi_x\|_{1,p,\varepsilon}+\varepsilon^{-\frac{1}{2}}\|\Psi\|_{1,p,\varepsilon}+\varepsilon^{-\Delta}(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}}+\|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}}+\|\Phi_y\|_{\delta,p,\varepsilon})) \quad (75)$$

and

$$\|\rho_{\Phi_2}\tilde{\Phi}_2\|_{U_\varepsilon^{0,p}} \leq c\varepsilon^{-\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon}, \qquad \|\rho_\Psi\tilde{\Psi}\|_{U_\varepsilon^{0,p}} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Phi}_2\|_{1,p,\varepsilon}.$$

Lemma 2.24 *Suppose that*

$$\|\Phi_1\|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}, \qquad \|\Phi_2\|_{2,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{8}-\Delta}. \quad (76)$$

The solution $\Psi = \Psi(\Phi_1, \Phi_2)$ to (52) identified in Theorem 2.16 satisfies the estimate

$$\|\Psi\|_{1,p,\varepsilon} \leq c(\varepsilon^{\frac{3}{4}-\Delta}\|\Phi\|_{U_\varepsilon^{1,p}} + \varepsilon^{\frac{3}{4}-\Delta}\|\Phi_y\|_{1,p,\varepsilon})P_1(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}). \quad (77)$$

Moreover Ψ depends smoothly upon (Φ_1, Φ_2) with respect to the $W_\varepsilon^{1,p}(\Sigma)$ and $U_\varepsilon^{1,p}(\mathbb{R}^2) \times W_\varepsilon^{2,p}(\Sigma)$ topologies and in particular its derivative with respect to Φ_2 satisfies the estimate

$$\|\Psi_{\Phi_2}\tilde{\Phi}_2\|_{1,p,\varepsilon} \leq c\varepsilon^{\frac{5}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon}.$$

Proof. We obtain this result by applying our fixed-point theorem to (52) with $\mathcal{X} = W_\varepsilon^{1,p}(\Sigma)$, $\mathcal{Y}_1 = U_\varepsilon^{1,p}(\mathbb{R}^2)$, $\mathcal{Y}_2 = W_\varepsilon^{2,p}(\Sigma)$ and

$$\begin{aligned}
X &= \{\Psi \in W_\varepsilon^{1,p}(\Sigma) : \|\Psi\|_{1,p,\varepsilon} \leq c\varepsilon^{\frac{3}{8}-\Delta}\} \cap \{\Psi \in W_\varepsilon^{\delta,p}(\Sigma) : \|\Psi\|_{\delta,p,\varepsilon} \leq c\varepsilon^{\frac{1}{2}-\Delta}\},\\
Y_1 &= \{\Phi_1 \in U_\varepsilon^{1,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{1,p}} \leq c\varepsilon^{-\frac{3}{8}-\Delta}\} \cap \{\Phi_1 \in U_\varepsilon^{\delta,p}(\mathbb{R}^2) : \|\Phi_1\|_{U_\varepsilon^{\delta,p}} \leq c\varepsilon^{-\frac{1}{4}-\Delta}\},\\
Y_2 &= \{\Phi_2 \in W_\varepsilon^{2,p}(\Sigma) : \|\Phi_2\|_{2,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{8}-\Delta}\} \cap \{\Phi_2 \in W_\varepsilon^{1+\delta,p}(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\}.
\end{aligned}$$

Employing the methods developed in the proofs of Theorem 2.16 and Lemma 2.22 together with the estimates

$$\begin{aligned}
|\rho_0|_{\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}(\|\Phi_x\|_{\delta,p,\varepsilon} + (\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})^2),\\
|\rho_0|_{1,p,\varepsilon} &\leq c\varepsilon^{-\Delta}(\|\Phi_x\|_{1,p,\varepsilon} + (\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})),\\
\|\rho_0\|_{U_\varepsilon^{0,p}} &\leq c\varepsilon^{-\Delta}(\|\Phi_x\|_{1,p,\varepsilon} + (\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{1,p}} + \|\Phi_y\|_{1,p,\varepsilon})(\varepsilon^{\frac{1}{2}}\|\Phi\|_{U_\varepsilon^{\delta,p}} + \|\Phi_y\|_{\delta,p,\varepsilon})),
\end{aligned}$$

one finds that

$$\begin{aligned}
&\|\mathcal{F}_2(0, \Phi_1, \Phi_2)\|_{1,p,\varepsilon}\\
&\quad = \left\|\mathcal{F}^{-1}\Bigg[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\hat{N}_6(\rho(\Phi), \Phi, 0)\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\hat{N}_7(\rho(\Phi), \Phi, 0)\,\mathrm{d}\xi\right.\\
&\qquad\qquad \left.+ \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\hat{N}_8(\rho(\Phi), \Phi, 0)\Bigg]\right\|_{1,p,\varepsilon}\\
&\quad \leq c(\varepsilon^{\frac{3}{4}-\Delta}\|\Phi\|_{U_\varepsilon^{1,p}} + \varepsilon^{\frac{3}{4}-\Delta}\|\Phi_y\|_{1,p,\varepsilon})P_1(\varepsilon^{\frac{1}{4}}\|\Phi\|_{U_\varepsilon^{\delta,p}}, \|\Phi_y\|_{\delta,p,\varepsilon}),
\end{aligned}$$

in which $\rho_0 = \rho(0, \Phi)$. Similarly, using inequalities (73), (76) and

$$|\rho|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\rho\|_{U_\varepsilon^{0,p}} \leq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\Psi\|_{1,p,\varepsilon} \leq c\varepsilon^{\frac{3}{8}-\Delta}$$

(which follow from (72), (75) and (77)), we find that

$$\begin{aligned}
&\|\mathrm{d}_2\mathcal{F}_2[\rho,\Psi,\Phi_1,\Phi_2]\tilde{\Psi}\|_{1,p,\varepsilon} \\
&= \left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)\bar{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_3\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi \right.\right. \\
&\qquad - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)\bar{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_3\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Psi}\,\mathrm{d}\xi \\
&\qquad \left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)\bar{\rho} + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_3\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Psi}\right]\right\|_{\delta,p,\varepsilon} \\
&\leq c\varepsilon^{\frac{1}{8}-\Delta}\|\tilde{\Psi}\|_{1,p,\varepsilon} \\
&\leq \frac{1}{2}\|\tilde{\Psi}\|_{1,p,\varepsilon}
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathrm{d}_3\mathcal{F}_2[\rho,\Psi,\Phi_1,\Phi_2]\tilde{\Psi}\|_{1,p,\varepsilon} \\
&\leq 2\left\|\mathcal{F}^{-1}\left[-\int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_1\hat{N}_6(\rho,\Phi,\Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y}}{\varepsilon^{5/2}}\partial_2\hat{N}_6(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi \right.\right. \\
&\qquad - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_1\hat{N}_7(\rho,\Phi,\Psi)\tilde{\rho}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1y\xi}}{\varepsilon^{5/2}}\partial_2\hat{N}_7(\rho,\Phi,\Psi)\tilde{\Phi}_2\,\mathrm{d}\xi \\
&\qquad \left.\left. + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_1\hat{N}_8(\rho,\Phi,\Psi)\tilde{\rho} + \frac{\mathrm{i}\mu G_{1y}|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)}\partial_2\hat{N}_8(\rho,\Phi,\Psi)\tilde{\Phi}_2\right]\right\|_{1+\delta,p,\varepsilon} \\
&\leq \varepsilon^{\frac{5}{8}-\Delta}\|\tilde{\Phi}_2\|_{2,p,\varepsilon};
\end{aligned}$$

here $\bar{\rho} = \rho_\Psi\tilde{\Psi}$ and $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$ are estimated by

$$|\bar{\rho}|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{\delta,p,\varepsilon}, \quad |\bar{\rho}|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon}, \quad \|\bar{\rho}\|_{U_\varepsilon^{0,p}} \leq c\varepsilon^{-\frac{1}{2}}\|\tilde{\Psi}\|_{1,p,\varepsilon},$$

$$|\tilde{\rho}|_{\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\|\tilde{\Phi}_2\|_{\delta,p,\varepsilon}, \quad |\tilde{\rho}|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{8}}\|\tilde{\Phi}_2\|_{1,p,\varepsilon}, \quad \|\tilde{\rho}\|_{U_\varepsilon^{0,p}} \leq c\varepsilon^{-\frac{1}{8}}\|\tilde{\Phi}_2\|_{1,p,\varepsilon}. \qquad \square$$

Lemma 2.25 *Suppose that*

$$\|\Phi_1\|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}. \tag{78}$$

The solution $\Phi_2 = \Phi_2(\Phi_1)$ to (60) identified in Theorem 2.18 satisfies the estimate

$$\|\Phi_2\|_{2,p,\varepsilon} \leq c\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}}P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}). \tag{79}$$

Moreover Φ_2 depends smoothly upon Φ_1 with respect to the $W_\varepsilon^{2,p}(\Sigma)$ and $U_\varepsilon^{1,p}(\mathbb{R}^2)$ topologies.

Proof. This result is established by applying our fixed-point theorem to (60) with $\mathcal{X} = W^{2,p}_\varepsilon(\Sigma)$, $\mathcal{Y} = U^{1,p}_\varepsilon(\mathbb{R}^2)$ and

$$\begin{aligned} X &= \{\Phi_2 \in W^{2,p}_\varepsilon(\Sigma) : \|\Phi_2\|_{2,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{8}-\Delta}\} \cap \{\Phi_2 \in W^{1+\delta,p}_\varepsilon(\Sigma) : \|\Phi_2\|_{1+\delta,p,\varepsilon} \leq c\varepsilon^{-\Delta}\}, \\ Y &= \{\Phi_1 \in U^{1,p}_\varepsilon(\mathbb{R}^2) : \|\Phi_1\|_{U^{1,p}_\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}\} \cap \{\Phi_1 \in U^{\delta,p}_\varepsilon(\mathbb{R}^2) : \|\Phi_1\|_{U^{\delta,p}_\varepsilon} \leq c\varepsilon^{-\frac{1}{4}-\Delta}\}. \end{aligned}$$

The methods developed in the proofs of Theorem 2.18 and Corollary 2.22 together with the estimates (64), (65) and

$$\begin{aligned} |\rho(\Psi(\Phi_1), \Phi_1)|_{1,p,\varepsilon} &\leq c(\|\Phi_{1x}\|_{1,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U^{1,p}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})), \\ \|\rho(\Psi(\Phi_1), \Phi_1)\|_{U^{0,p}_\varepsilon} &\leq c(\|\Phi_{1x}\|_{1,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U^{1,p}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon})) \end{aligned}$$

yield

$$\begin{aligned} &\|\mathcal{F}_3(\Phi_1, 0)\|_{2,p,\varepsilon} \\ &\quad = \left\| \mathcal{F}^{-1}\left[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \hat{N}_6(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi \right.\right. \\ &\qquad\qquad - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \hat{N}_7(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1))\,\mathrm{d}\xi \\ &\qquad\qquad \left.\left. + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \hat{N}_8(\rho(\Psi(\Phi_1), \Phi_1), \Phi_1, \Psi(\Phi_1)) \right] \right\|_{2,p,\varepsilon} \\ &\quad \leq c\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U^{1,p}_\varepsilon} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U^{\delta,p}_\varepsilon}). \end{aligned}$$

Furthermore, writing $\tilde{\rho} = \rho_{\Phi_2}\tilde{\Phi}_2$, $\bar{\rho} = \rho_\Psi \tilde{\Psi}$, $\tilde{\Psi} = \Psi_{\Phi_2}\tilde{\Phi}_2$ and using the estimates (73), (78) and

$$|\rho|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\rho\|_{U^{0,p}_\varepsilon} \leq c\varepsilon^{-\frac{3}{8}-\Delta}, \quad \|\Phi_2\|_{1,p,\varepsilon} \leq c\varepsilon^{-\frac{1}{2}-\Delta}, \quad \|\Psi\|_{1,p,\varepsilon} \leq c\varepsilon^{\frac{3}{8}-\Delta}$$

(which follow from (72), (75), (77) and (79)) we find that

$$\begin{aligned} &\left\| \mathcal{F}^{-1}\left[-\int_0^1 \frac{G_1}{\varepsilon^{5/2}} \partial_1 \hat{N}_6(\rho, \Phi, \Psi)(\tilde{\rho} + \bar{\rho})\,\mathrm{d}\xi - \int_0^1 \frac{G_1}{\varepsilon^{5/2}} \partial_2 \hat{N}_6(\rho, \Phi, \Psi)\tilde{\Phi}_2\,\mathrm{d}\xi \right.\right. \\ &\qquad - \int_0^1 \frac{G_1}{\varepsilon^{5/2}} \partial_3 \hat{N}_6(\rho, \Phi, \Psi)\tilde{\Psi}\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \partial_1 \hat{N}_7(\rho, \Phi, \Psi)(\tilde{\rho} + \bar{\rho})\,\mathrm{d}\xi \\ &\qquad - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \partial_2 \hat{N}_7(\rho, \Phi, \Psi)\tilde{\Phi}_2\,\mathrm{d}\xi - \int_0^1 \frac{G_{1\xi}}{\varepsilon^{5/2}} \partial_3 \hat{N}_7(\rho, \Phi, \Psi)\tilde{\Psi}\,\mathrm{d}\xi \\ &\qquad + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \partial_1 \hat{N}_8(\rho, \Phi, \Psi)(\tilde{\rho} + \bar{\rho}) + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \partial_2 \hat{N}_8(\rho, \Phi, \Psi)\tilde{\Phi}_2 \\ &\qquad \left.\left. + \frac{\mathrm{i}\mu G_1|_{\xi=1}}{\varepsilon^2(1+\varepsilon+\beta q^2)} \partial_3 \hat{N}_8(\rho, \Phi, \Psi)\tilde{\Psi} \right] \right\|_{1+\delta,p,\varepsilon} \\ &\leq c\varepsilon^{\frac{1}{8}-\Delta}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon} \\ &\leq \frac{1}{2}\|\tilde{\Phi}_2\|_{1+\delta,p,\varepsilon}, \end{aligned}$$

in which the estimates for $\rho_{\Phi_2}\tilde{\Phi}_2$, $\rho_\Psi\tilde{\Psi}$ and $\Psi_{\Phi_2}\tilde{\Phi}_2$ stated in Theorems 2.11 and 2.16 and Lemmata 2.22 and 2.24 have also been used. □

Altogether, the above results show that

$$\|\Phi_2\|_{2,p,\varepsilon} \leq c\varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}}), \qquad \|\Phi_{2y}\|_{2,p,\varepsilon} \leq c\varepsilon^{\frac{3}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})$$

and

$$\begin{aligned} |\rho(\Phi_1)|_{1,p,\varepsilon} &\leq c(\|\Phi_{1x}\|_{1,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})), \\ \|\rho(\Phi_1)\|_{U_\varepsilon^{0,p}} &\leq c(\|\Phi_{1x}\|_{1,p,\varepsilon} + \varepsilon^{\frac{1}{4}-\Delta}\|\Phi_1\|_{U_\varepsilon^{1,p}} + P_1(\varepsilon^{\frac{1}{4}}\|\Phi_1\|_{U_\varepsilon^{\delta,p}})), \end{aligned}$$

where $\rho(\Phi_1)$ is an abbreviation for $\rho(\Phi_{2y}(\Phi_1), \Phi_2(\Phi_1), \Phi_1)$ and Ψ has been identified with Φ_{2y}. Observe that $\rho(\Phi_1)$ is a weak solution of the equation for ρ (with $\Phi_2 = \Phi_2(\Phi_1)$) which meets the additional regularity requirements of a strong solution; a familiar argument asserts that it is a strong solution. One similarly finds that $\Phi_2(\Phi_1)$ is a strong solution of the equation for Φ_2 (with $\rho = \rho(\Phi_1)$), and that $(\rho(\Phi_1), \Phi_1 + \Phi_2(\Phi_1))$ is a strong solution of the original equations (26)–(29). Finally, it is possible to repeat the proof of Proposition 2.21 in a 'bootstrap' fashion to conclude that Φ_1 belongs to $U_\varepsilon^{5,p}(\mathbb{R}^2)$ and is therefore a strong solution of equation (67); this step is however only of academic interest since it does not play a role in the regularity theory for (26)–(29).

3 Solution of the reduced equation

3.1 Variational structure

The key to finding solutions of the integral form of the reduced equation for Φ_1 (equation (67)) lies in its variational structure. This variational structure arises from the fact that the original hydrodynamic problem (1)–(4) in the parameter regime (6) itself follows from a formal variational principle, namely

$$\begin{aligned} \delta\Bigg\{ \int_{\mathbb{R}^2} \Bigg(\int_0^{1+\rho} (-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2))\,\mathrm{d}y \\ + \frac{1}{2}(1+\varepsilon)\rho^2 + \beta(\sqrt{1+\rho_x^2+\rho_z^2} - 1)\Bigg)\,\mathrm{d}x\,\mathrm{d}z \Bigg\} = 0, \end{aligned}$$

where the variation is taken in (ρ, ϕ) (see Luke [28]). In this section we identify the variational structure of the reduced equation for Φ_1 by reviewing the steps in its derivation and showing that the variational structure is preserved at each step of the reduction procedure. In Section 3.2 below we apply the direct methods of the calculus of variations to the relevant variational functional to confirm the existence of a nonzero weak solution of the reduced equation for Φ_1, which according to Proposition 2.8(i) is also a solution of equation (67).

The first step is to introduce the change of variable

$$y = \tilde{y}(1 + \rho(x,z)), \qquad \phi(x,y,z) = \Phi(x,\tilde{y},z)$$

and the scaled coordinates

$$(\tilde{\rho}(\tilde{x},\tilde{z}),\tilde{\Phi}(\tilde{x},y,\tilde{z}))=(\varepsilon^{-1}\rho(x,z),\varepsilon^{-\frac{1}{2}}\Phi(x,y,z)),\qquad(\tilde{x},\tilde{z})=(\varepsilon^{\frac{1}{2}}x,\varepsilon z),$$

which transform the hydrodynamic equations into equations (26)–(29) and the functional in the above variational principle into

$$\begin{aligned}\mathcal{V}(\rho,\Phi)&=\\&\int_{\mathbb{R}^2}\left\{\int_0^1\left(\frac{\varepsilon}{2}\left[\Phi_x-\frac{\varepsilon y\rho_x\Phi_y}{1+\varepsilon\rho}\right]^2+\frac{\Phi_y^2}{2(1+\varepsilon\rho)^2}+\frac{\varepsilon^2}{2}\left[\Phi_z-\frac{\varepsilon y\rho_z\Phi_y}{1+\varepsilon\rho}\right]^2\right)(1+\varepsilon\rho)\,\mathrm{d}y\right.\\&\qquad\left.+\frac{1}{2}\varepsilon(1+\varepsilon)\rho^2+\beta\varepsilon^{-1}[\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}-1]+\varepsilon\int_0^1(\rho_x y\Phi_y-\rho\Phi_x)\,\mathrm{d}y\right\}\mathrm{d}x\,\mathrm{d}z,\end{aligned}$$

where the tildes have been dropped for notational simplicity.

Proposition 3.1 *The weak solutions of equations (26)–(29) are precisely the critical points of the smooth functional* $\mathcal{V}:[V_\varepsilon^{0,2}(\mathbb{R}^2)\times U_\varepsilon^{0,2}(\Sigma)]\cap[V_\varepsilon^{\delta,p}(\mathbb{R}^2)\times U_\varepsilon^{\delta,p}(\Sigma)]\to\mathbb{R}$.

The Euler-Lagrange equations for $\mathcal{V}$, namely

$$\mathrm{d}_1\mathcal{V}[\rho,\Phi]=0, \tag{80}$$

$$\mathrm{d}_2\mathcal{V}[\rho,\Phi]=0, \tag{81}$$

correspond to the weak forms of the equations for ρ and Φ and are given explicitly by equations (30) and (31). Proposition 2.3 asserts that (80) is equivalent to the integral form of the equation for ρ, and the second step in the reduction procedure is to solve this equation for ρ as a function of Φ and insert $\rho=\rho(\Phi)$ into the equation for Φ, whose weak form is therefore

$$\mathrm{d}_2\mathcal{V}[\rho(\Phi),\Phi]=0. \tag{82}$$

The following proposition shows that this step in the reduction procedure preserves the variational structure in a natural way.

Proposition 3.2 *Define a smooth functional* $\mathcal{W}:U_\varepsilon^{0,2}(\Sigma)\cap U_\varepsilon^{\delta,p}(\Sigma)\to\mathbb{R}$ *by the formula* $\mathcal{W}(\Phi)=\mathcal{V}(\rho(\Phi),\Phi)$. *The critical points of* $\mathcal{W}$ *are precisely the solutions of equation (82).*

Proof. Observe that

$$\begin{aligned}\mathrm{d}\mathcal{W}[\Phi]&=\mathrm{d}_1\mathcal{V}[\rho(\Phi),\Phi](\mathrm{d}\rho[\Phi])+\mathrm{d}_2\mathcal{V}[\rho(\Phi),\Phi]\\&=\mathrm{d}_2\mathcal{V}[\rho(\Phi),\Phi]\end{aligned}$$

since the defining property of $\rho(\Phi)$ is that it solves equation (80). □

The final step is the decomposition

$$\Phi(x,y,z)=\Phi_1(x,z)+\Phi_2(x,y,z)$$

defined by equations (34), (35); the integral form of the equation for Φ_2 (with $\rho=\rho(\Phi)$) is solved for Φ_2 as a function of Φ_1, and inserting $\Phi_2=\Phi_2(\Phi_1)$ into the equation for Φ_1 we obtain the reduced equation for Φ_1. This approach (writing a variable as the sum of two components X and Y, solving one of the equations to yield the functional relationship $Y = Y(X)$, and inserting this function into the other equation to obtain a 'reduced equation' for X) is reminiscent of the classical Lyapunov-Schmidt reduction. There is a variational version of the Lyapunov-Schmidt reduction which asserts that the variational structure of the original equation is inherited by that of the reduced equation in a natural fashion (that is, the reduced variational functional is obtained by substituting $Y = Y(X)$ into the original variational functional), provided that the quadratic part of the original variational functional contains no mixed terms in X and Y. The following argument shows how this strategy can be used to detect the variational structure of our reduced equation for Φ_1; in effect we show how the quadratic part of the functional $\mathcal{W}$ can be replaced by the sum of a quadratic form for Φ_1 and a quadratic form for Φ_2.

Let us briefly proceed formally. Suppose that $\Phi_2(\Phi_1)$ solves the strong form (36)–(38) of the problem for Φ_2. A straightforward calculation shows that this problem is equivalent to the boundary-value problem

$$-\hat{\Phi}_{2yy} + q^2\hat{\Phi}_2 + \frac{q^2(1+\varepsilon)}{\varepsilon^2 QS}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \hat{H}, \qquad 0<y<1,$$

$$\hat{\Phi}_{2y} - \frac{\varepsilon\mu^2\hat{\Phi}_2}{1+\varepsilon+\beta q^2} + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 QS(1+\varepsilon+\beta q^2)}\left(q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = \hat{h}, \quad y=1,$$

$$\hat{\Phi}_{2y} = 0, \qquad y=0,$$

in which

$$S = 1 - \frac{q^2(1+\varepsilon)}{\varepsilon^2 Q} + \frac{(1+\varepsilon)\varepsilon\mu^2}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)}$$

and

$$H = \varepsilon^{-\frac{1}{2}}N_2(\rho(\Phi),\Phi), \qquad h = \varepsilon^{-\frac{1}{2}}N_3(\rho(\Phi),\Phi) - \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\hat{N}_1(\rho(\Phi),\Phi)\right];$$

the left-hand sides of these equations constitute a formally self-adjoint operator associated with the quadratic form

$$Q_2(\Phi_2) = \frac{1}{2}\int_{\mathbb{R}^2}\left\{\int_0^1 (|\hat{\Phi}_{2y}|^2 + q^2|\hat{\Phi}_2|^2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}|\hat{\Phi}_2|_{y=1}|^2 \right.$$
$$\left. + \frac{1+\varepsilon}{\varepsilon^2 QS}\left|q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right|^2\right\}\mathrm{d}\mu\,\mathrm{d}k.$$

(Notice that the quantity S vanishes for certain values of μ and k; we return to this issue below.) The weak formulation of the above boundary-value problem is obtained by multiplying it by a test function $\Psi_2 \in W^{1,2}_\varepsilon(\Sigma)$, integrating over Σ and using integration by parts to transfer 'additional' derivatives to Ψ_2.

Similarly, the left-hand side of the strong form of the equation for Φ_1, namely

$$\frac{\varepsilon^2}{1+\varepsilon}[-c_0\varepsilon(\partial_x^2+\varepsilon\partial_z^2)^3 + (\beta-\tfrac{1}{3})(\partial_x^2+\varepsilon\partial_z^2)^2 - (1+\varepsilon)\partial_z^2 - \partial_x^2]\Phi_1 = \int_0^1 H\,\mathrm{d}y + h,$$

constitutes a formally self-adjoint operator associated with the quadratic form $\varepsilon^2 Q_1$, where

$$Q_1(\Phi_1) = \frac{1}{2(1+\varepsilon)} |||\Phi_1|||^2.$$

The weak formulation of this equation is obtained in the usual fashion (see Definition 2.7(i)).

Let us now write $\mathcal{W}(\Phi) = \mathcal{W}_2(\Phi) + \mathcal{W}_{\mathrm{NL}}(\Phi)$, where $\mathcal{W}_2$ denotes the quadratic part of $\mathcal{W}$, and note that

$$\mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi](\Psi) = -\int_{\mathbb{R}^2} \left\{ \int_0^1 (\hat{H}_0 \bar{\hat{\Psi}}_y + \hat{H}_1 \bar{\hat{\Psi}})\,\mathrm{d}y + \hat{h}_1 \bar{\hat{\Psi}}|_{y=1} \right\} \mathrm{d}\mu\,\mathrm{d}k, \tag{83}$$

where

$$H_0 = \varepsilon^{-\frac{1}{2}} N_5(\rho(\Phi), \Phi), \quad H_1 = \varepsilon^{-\frac{1}{2}} N_4(\rho(\Phi), \Phi), \quad h_1 = \mathcal{F}\left[\frac{-\mathrm{i}\mu}{1+\varepsilon+\beta q^2} \hat{N}_1(\rho(\Phi), \Phi)\right].$$

An inspection of the weak form of the equation for Φ_1 and the weak form of the reformulated problem for Φ_2 shows that they formally correspond to respectively

$$\varepsilon^2 \mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2](\Psi_1) = 0, \qquad \mathrm{d}Q_2[\Phi_2](\Psi_2) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2](\Psi_2) = 0,$$

so that the weak form of the reduced equation for Φ_1 formally corresponds to

$$\varepsilon^2 \mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1) = 0. \tag{84}$$

Repeating the arguments used in Proposition 3.2, one finds that the solutions of (84) are precisely the critical points of the functional

$$I(\Phi_1) = \varepsilon^2 Q_1(\Phi_1) + Q_2(\Phi_2(\Phi_1)) + \mathcal{W}_{\mathrm{NL}}(\Phi_1 + \Phi_2(\Phi_1)),$$

since

$$\begin{aligned} \mathrm{d}I[\Phi_1](\Psi_1) &= \varepsilon^2 \mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1) \\ &\qquad + (\mathrm{d}Q_2[\Phi_2(\Phi_1)] + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2])(\mathrm{d}\Phi_2[\Phi_1](\Psi_1)) \\ &= \varepsilon^2 \mathrm{d}Q_1[\Phi_1](\Psi_1) + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1), \end{aligned}$$

where the second line follows from the defining property of $\Phi_2(\Phi_1)$ as a solution of the integral and hence of the weak form of the equation for Φ_2.

It remains to treat the difficulty posed by the vanishing denominator in the formula for Q_2. To this end we use the identity

$$\left(q^2 \int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2 \hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\right) = S\left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1\right), \tag{85}$$

which is satisfied by $\Phi_2(\Phi_1)$; it is obtained by integrating (36) with respect to y over $(0,1)$, substituting for $\hat{\Phi}_{2y}|_{y=0}$, $\hat{\Phi}_{2y}|_{y=1}$ according to (37), (38) and noting that

$$\int_0^1 \hat{H}\,\mathrm{d}y + \hat{h} = \int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1.$$

Using (85) to eliminate S we obtain the alternative formula

$$Q_2(\Phi_2)=\frac{1}{2}\int_{\mathbb{R}^2}\Bigg\{\int_0^1(|\hat{\Phi}_{2y}|^2+q^2|\hat{\Phi}_2|^2)\,\mathrm{d}y-\frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}|\hat{\Phi}_2|_{y=1}|^2 \\ +\frac{1+\varepsilon}{\varepsilon^2 Q}\bigg(\int_0^1\hat{H}_1\,\mathrm{d}y+\hat{h}_1\bigg)\bigg(q^2\int_0^1\bar{\hat{\Phi}}_2\,\mathrm{d}y-\frac{\varepsilon\mu^2\bar{\hat{\Phi}}_2|_{y=1}}{1+\varepsilon+\beta q^2}\bigg)\Bigg\}\,\mathrm{d}\mu\,\mathrm{d}k$$

for $Q_2(\Phi_2(\Phi_1))$.

The above argument, which is formal in nature, delivers a candidate for the variational functional corresponding to the reduced equation for Φ_1. Rather than making the argument rigorous, we proceed by confirming directly that critical points of I (which, with the new definition of $Q_2(\Phi_2(\Phi_1)))$, is a smooth functional on X) are weak solutions of the reduced equation for Φ_1. This result is stated in Lemma 3.4 below; the following proposition, which asserts that a suitable version of (85) holds for solutions of the integral form of the problem for Φ_2, is required for its proof.

Proposition 3.3 *The solution $\Phi_2(\Phi_1)$ of the integral form of the problem for Φ_2 satisfies the identity*

$$\frac{1}{Q^{1/2}}\bigg(q^2\int_0^1\hat{\Phi}_2\,\mathrm{d}y-\frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\bigg)=\frac{S}{Q^{1/2}}\bigg(\int_0^1\hat{H}_1\,\mathrm{d}y+\hat{h}_1\bigg). \tag{86}$$

Proof. With slightly more generality, we establish the result for the boundary-value problem for Φ_2 obtained by replacing N_5 by an arbitrary function in $L^2(\Sigma)$, N_4 by an arbitrary function of the form

$$\hat{N}_4=\mathrm{i}\mu\hat{N}_4^1+\mathrm{i}\varepsilon^{\frac{1}{2}}k\hat{N}_4^2,\qquad N_4^1,N_4^2\in L^2(\Sigma)$$

and N_1 by an arbitrary function of the form

$$\hat{N}_1=\hat{N}_1^1+\mathrm{i}\mu\hat{N}_1^2+\mathrm{i}\varepsilon^{\frac{1}{2}}k\hat{N}_1^3,\qquad N_1^1,N_1^2,N_1^3\in L^2(\mathbb{R}^2).$$

It is a straightforward exercise to show that

$$F(N_1^1,N_1^2,N_1^3,N_4^1,N_4^2,N_5)=\frac{1}{Q^{1/2}}\bigg(q^2\int_0^1\hat{\Phi}_2\,\mathrm{d}y-\frac{\varepsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\varepsilon+\beta q^2}\bigg)-\frac{S}{Q^{1/2}}\bigg(\int_0^1\hat{H}_1\,\mathrm{d}y+\hat{h}_1\bigg),$$

where Φ_2 is the solution of the integral form of the problem, is a continuous function $(L^2(\mathbb{R}^2))^3\times(L^2(\Sigma))^3\rightarrow L^2(\Sigma)$ (the Fourier-multiplier operators appearing in this equation are handled using Parseval's formula). Now suppose that N_1^1, N_1^2, N_1^3 belong to the dense subset $W_0^{1,2}(\mathbb{R}^2)$ of $L^2(\mathbb{R}^2)$ and that N_4^1, N_4^2, N_5 belong to the dense subset $W_0^{1,2}(\Sigma)$ of $L^2(\Sigma)$. Using Lemma 2.15 in a 'bootstrap' fashion, we find that Φ_2 belongs to $W^{2,2}(\Sigma)$; because it is a weak solution of the problem for Φ_2 with the required additional regularity it solves the strong form of the problem in $L^2(\Sigma)$ and hence the identity (85) in $L^2(\Sigma)$. It follows that F vanishes for $(N_1^1,N_1^2,N_1^3,N_4^1,N_4^2,N_5)\in(W_0^{1,2}(\mathbb{R}^2))^3\times(W_0^{1,2}(\Sigma))^3$; a standard density argument asserts that it also vanishes for each $(N_1^1,N_1^2,N_1^3,N_4^1,N_4^2,N_5)\in(L^2(\mathbb{R}^2))^3\times(L^2(\Sigma))^3\rightarrow L^2(\Sigma)$. □

Lemma 3.4 *Each critical point of $I : X \to \mathbb{R}$ is a weak solution of the reduced equation for Φ_1.*

Proof. Observe that

$$\begin{aligned}
\mathrm{d}I[\Phi_1](\Psi_1) &= \frac{\varepsilon^2}{1+\varepsilon}\langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle + \mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1 + \Psi_2) \\
&\quad + \int_{\mathbb{R}^2} \left\{ \int_0^1 (\hat{\Phi}_{2y}\bar{\hat{\Psi}}_{2y} + q^2\hat{\Phi}_2\bar{\hat{\Psi}}_2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}\hat{\Phi}_2\bar{\hat{\Psi}}_2|_{y=1} \right\} \mathrm{d}\mu\,\mathrm{d}k \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} \left\{ \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \mathcal{F}[\partial H_1 \Psi]\,\mathrm{d}y + \mathcal{F}[\partial h_1 \Psi] \right)\left(q^2\int_0^1 \bar{\hat{\Phi}}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\bar{\hat{\Phi}}_2|_{y=1}}{1+\varepsilon+\beta q^2} \right) \right. \\
&\qquad\qquad \left. + \frac{1+\varepsilon}{\varepsilon^2 Q}\left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1 \right)\left(q^2\int_0^1 \bar{\hat{\Psi}}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\bar{\hat{\Psi}}_2|_{y=1}}{1+\varepsilon+\beta q^2} \right) \right\} \mathrm{d}\mu\,\mathrm{d}k, \qquad (87)
\end{aligned}$$

where $\Psi_2 = \mathrm{d}\Phi_2[\Phi_1](\Psi_1)$ and $\Psi = \Psi_1 + \Psi_2$. Differentiation of equation (86) with respect to Φ_1 yields

$$\frac{1}{Q^{1/2}}\left(\int_0^1 q^2\hat{\Psi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Psi}_2|_{y=1}}{1+\varepsilon+\beta q^2} \right) = \frac{S}{Q^{1/2}}\left(\int_0^1 \mathcal{F}[\partial H_1 \Psi]\,\mathrm{d}y + \mathcal{F}[\partial \hat{h}_1 \Psi] \right)$$

and eliminating S between this equation and (86), we find that

$$\begin{aligned}
&\frac{1}{Q^{1/2}}\left(\int_0^1 \bar{\hat{H}}_1\,\mathrm{d}y + \bar{\hat{h}}_1 \right)\left(\int_0^1 q^2\hat{\Psi}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\hat{\Psi}_2|_{y=1}}{1+\varepsilon+\beta q^2} \right) \\
&\qquad = \frac{1}{Q^{1/2}}\left(\int_0^1 \mathcal{F}[\partial H_1 \Psi]\,\mathrm{d}y + \mathcal{F}[\partial \hat{h}_1 \Psi] \right)\left(\int_0^1 q^2\bar{\hat{\Phi}}_2\,\mathrm{d}y - \frac{\varepsilon\mu^2\bar{\hat{\Phi}}_2|_{y=1}}{1+\varepsilon+\beta q^2} \right). \qquad (88)
\end{aligned}$$

It follows from (83) that

$$\begin{aligned}
&\mathrm{d}\mathcal{W}_{\mathrm{NL}}[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1) \\
&\qquad = -\int_{\mathbb{R}^2} \left\{ \left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1 \right)\bar{\hat{\Psi}}_1 + \int_0^1 (\hat{H}_0\bar{\hat{\Psi}}_{2y} + \hat{H}_1\bar{\hat{\Psi}}_2)\,\mathrm{d}y + \hat{h}_1\bar{\hat{\Psi}}_2|_{y=1} \right\} \mathrm{d}\mu\,\mathrm{d}k, \quad (89)
\end{aligned}$$

and combining equations (87)–(89), one finds that

$$\begin{aligned}
\mathrm{d}I[\Phi_1](\Psi_1) &= \frac{\varepsilon^2}{1+\varepsilon}\langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle \\
&\quad + \int_{\mathbb{R}^2} \left\{ \int_0^1 (\hat{\Phi}_{2y}\bar{\hat{\Psi}}_{2y} + q^2\hat{\Phi}_2\bar{\hat{\Psi}}_2)\,\mathrm{d}y - \frac{\varepsilon\mu^2}{1+\varepsilon+\beta q^2}\hat{\Phi}_2\bar{\hat{\Psi}}_2|_{y=1} \right. \\
&\qquad \left. + \operatorname{Re}\left(\left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1 \right)\left(\frac{(1+\varepsilon)q^2}{\varepsilon^2 Q}\int_0^1 \bar{\hat{\Psi}}_2\,\mathrm{d}y - \frac{(1+\varepsilon)\varepsilon\mu^2\bar{\hat{\Psi}}_2|_{y=1}}{\varepsilon^2 Q(1+\varepsilon+\beta q^2)} \right) \right) \right\} \mathrm{d}\mu\,\mathrm{d}k \\
&\quad - \int_{\mathbb{R}^2} \left\{ \left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1 \right)\bar{\hat{\Psi}}_1 + \int_0^1 (\hat{H}_0\bar{\hat{\Psi}}_{2y} + \hat{H}_1\bar{\hat{\Psi}}_2)\,\mathrm{d}y + \hat{h}_1\bar{\hat{\Psi}}_2|_{y=1} \right\} \mathrm{d}\mu\,\mathrm{d}k \\
&= \frac{\varepsilon^2}{1+\varepsilon}\langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle - \int_{\mathbb{R}^2} \left(\int_0^1 \hat{H}_1\,\mathrm{d}y + \hat{h}_1 \right)\bar{\hat{\Psi}}_1\,\mathrm{d}\mu\,\mathrm{d}k,
\end{aligned}$$

in which the facts that the quantity whose real part is being taken is already real and that $\Phi_2(\Phi_1)$ is a weak solution of the problem for Φ_2 have been used. $\square$

In our subsequent analysis we replace I by the equivalent functional

$$J(\Phi_1) = \varepsilon^{-2} I(\Phi_1) = Q_1(\Phi_1) + \varepsilon^{-2} Q_2(\Phi_2(\Phi_1)) + \varepsilon^{-2}\mathcal{W}_{\mathrm{NL}}(\Phi_1 + \Phi_2(\Phi_1)),$$

and we conclude this section by computing a convenient formula for J. Using the fact that $\Phi_2(\Phi_1)$ solves the weak formulation of the problem for Φ_2, we find from Definition 2.7(ii) with $\Psi_2 = \Phi_2$ that

$$Q_2(\Phi_2) = \frac{1}{2}\int_{\mathbb{R}^2}\left\{\int_0^1 (\hat{H}_0\bar{\hat{\Phi}}_{2y} + \hat{H}_1\bar{\hat{\Phi}}_2)\,\mathrm{d}y + \hat{h}_1\bar{\hat{\Phi}}_2|_{y=1}\right\}\mathrm{d}\mu\,\mathrm{d}k. \tag{90}$$

Bearing this equation in mind, note that

$$\begin{aligned}
&\varepsilon^{-2}\int_{\mathbb{R}^2}\int_0^1 (\hat{H}_0\bar{\hat{\Phi}}_{2y} + \hat{H}_1\bar{\hat{\Phi}}_2)\,\mathrm{d}y\,\mathrm{d}\mu\,\mathrm{d}k \\
&\quad = \int_{\mathbb{R}^2}\int_0^1 \Bigg\{ y(\Phi_x + \Phi_{2x})\rho_x\Phi_{2y} + \varepsilon y(\Phi_z + \Phi_{2z})\rho_z\Phi_{2y} \\
&\qquad\qquad - \rho\Phi_x\Phi_{2x} - \varepsilon\rho\Phi_z\Phi_{2z} - \frac{\varepsilon y^2\rho_x^2\Phi_{2y}^2}{1+\varepsilon\rho} - \frac{\varepsilon^2 y^2\rho_z^2\Phi_{2y}^2}{1+\varepsilon\rho} - \frac{\rho\Phi_{2y}^2}{\varepsilon(1+\varepsilon\rho)}\Bigg\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z
\end{aligned} \tag{91}$$

and

$$\int_{\mathbb{R}^2}\hat{h}_1\bar{\hat{\Phi}}_2|_{y=1}\,\mathrm{d}\mu\,\mathrm{d}k = \varepsilon^{-1}\int_{\mathbb{R}^2}\rho_{\mathrm{NL}\,x}\Phi_2|_{y=1}\,\mathrm{d}x\,\mathrm{d}z. \tag{92}$$

A suitable formula for $\mathcal{W}_{\mathrm{NL}}$ is obtained by using the expression

$$\rho = \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\left(\hat{\Phi}_{1x} + \mathrm{i}\mu\int_0^1 y\hat{\Phi}_{2y}\,\mathrm{d}y + \int_0^1 \hat{\Phi}_{2x}\right)\right] + \rho_{\mathrm{NL}}(\rho, \Phi_1, \Phi_2)$$

in the formula

$$\begin{aligned}
&\mathcal{W}(\Phi) = \\
&\int_{\mathbb{R}^2}\Bigg\{\int_0^1\left(\frac{\varepsilon}{2}\Phi_x^2 + \frac{\varepsilon^2}{2}\Phi_z^2 + \frac{1}{2}\Phi_y^2 + \varepsilon(\rho_x y\Phi_y - \rho\Phi_x)\right)\mathrm{d}y + \frac{1}{2}\varepsilon(1+\varepsilon)\rho^2 + \frac{\beta}{2}\varepsilon^2\rho_x^2 + \frac{\beta}{2}\varepsilon^3\rho_z^2 \\
&\quad + \int_0^1\Bigg(\frac{\varepsilon^2}{2}\rho\Phi_x^2 + \frac{1}{2}\varepsilon^3\rho\Phi_z^2 - \frac{\varepsilon\rho\Phi_y^2}{2(1+\varepsilon\rho)} + \frac{\varepsilon^3 y^2\Phi_y^2\rho_x^2}{2(1+\varepsilon\rho)} + \frac{\varepsilon^4 y^2\Phi_y^2\rho_z^2}{2(1+\varepsilon\rho)} \\
&\qquad\qquad - \varepsilon^2 y\Phi_y\Phi_x\rho_x - \varepsilon^3 y\Phi_y\Phi_z\rho_z\Bigg)\mathrm{d}y \\
&\quad - \frac{\beta\varepsilon^{-1}(\varepsilon^3\rho_x^2 + \varepsilon^4\rho_z^2)^2}{2(\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}+1)^2}\Bigg\}\,\mathrm{d}x\,\mathrm{d}z;
\end{aligned}$$

one finds that

$$\varepsilon^{-2}\mathcal{W}_{\mathrm{NL}}(\Phi) =$$

$$
\begin{aligned}
\int_{\mathbb{R}^2} \Bigg\{ & \frac{1}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2 + \frac{\varepsilon}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}^2 \\
& + \int_0^1 \left(\frac{1}{2}\rho\Phi_{2x}^2 + \frac{\varepsilon}{2}\rho\Phi_{2z}^2 - \frac{\rho\Phi_{2y}^2}{2\varepsilon(1+\varepsilon\rho)} + \frac{\varepsilon y^2\Phi_{2y}^2\rho_x^2}{2(1+\varepsilon\rho)} + \frac{\varepsilon^2 y^2\Phi_{2y}^2\rho_z^2}{2(1+\varepsilon\rho)}\right. \\
& \qquad\qquad \left. - y\Phi_{2y}\Phi_x\rho_x - \varepsilon y\Phi_{2y}\Phi_z\rho_z\right)\mathrm{d}y \\
& + \frac{1}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2 + \frac{\varepsilon}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}^2 + \frac{1}{2}\rho_{\mathrm{NL}}\Phi_{1x}^2 + \frac{\varepsilon}{2}\rho_{\mathrm{NL}}\Phi_{1z}^2 \\
& - \frac{\beta\varepsilon^{-3}(\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)^2}{2(\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}+1)^2}\Bigg\}\,\mathrm{d}x\,\mathrm{d}z + \frac{\varepsilon^{-1}}{2}\|\rho_{\mathrm{NL}}\|_2^2 + \frac{1}{2}\|\rho_{\mathrm{NL}}\|_{1,2,\varepsilon}^2. \qquad (93)
\end{aligned}
$$

Combining (90)–(92), (93), we arrive at our final formula for $J : X \to \mathbb{R}$, namely

$$
J(\Phi_1) = J_2(\Phi_1) + J_3(\Phi_1) + J_4(\Phi_1),
$$

where

$$
\begin{aligned}
J_2(\Phi_1) &= Q_1(\Phi_1), \qquad (94)\\
J_3(\Phi_1) &= \int_{\mathbb{R}^2} \left\{\frac{1}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2 + \frac{\varepsilon}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{1x}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}^2\right\}\mathrm{d}x\,\mathrm{d}z, \qquad (95)\\
J_4(\Phi_1) &= \int_{\mathbb{R}^2} \Bigg\{ \int_0^1 \left(\frac{1}{2}\rho\Phi_{2x}^2 + \frac{\varepsilon}{2}\rho\Phi_{2z}^2 - \frac{\rho\Phi_{2y}^2}{2\varepsilon(1+\varepsilon\rho)} - y\Phi_{1x}\rho_x\Phi_{2y} - \varepsilon y\Phi_{1z}\rho_z\Phi_{2y}\right. \\
& \qquad\qquad \left. - \rho\Phi_x\Phi_{2z} - \varepsilon\rho\Phi_z\Phi_{2z}\right)\mathrm{d}y \\
& \qquad + \frac{1}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2 + \frac{\varepsilon}{2}\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1z}^2 \\
& \qquad + \frac{1}{2}\rho_{\mathrm{NL}}\Phi_{1x}^2 + \frac{\varepsilon}{2}\rho_{\mathrm{NL}}\Phi_{1z}^2 - \frac{\varepsilon^{-1}}{2}\rho_{\mathrm{NL}\,x}\Phi_2|_{y=1} \\
& \qquad - \frac{\beta\varepsilon^{-3}(\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2)^2}{2(\sqrt{1+\varepsilon^3\rho_x^2+\varepsilon^4\rho_z^2}+1)^2}\Bigg\}\,\mathrm{d}x\,\mathrm{d}z + \frac{\varepsilon^{-1}}{2}\|\rho_{\mathrm{NL}}\|_2^2 + \frac{1}{2}\|\rho_{\mathrm{NL}}\|_{1,2,\varepsilon}^2 \qquad (96)
\end{aligned}
$$

are respectively its quadratic, cubic and higher-order parts (recall that Φ_2 and ρ_{NL} are quadratic functions of Φ_1). This formula shows that J_3 and J_4 define smooth functionals on $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, and since X is continuously embedded in $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ one concludes that J has a semilinear structure.

3.2 Critical-point theory

In this section we complete our existence theory by showing that the functional $J : X \to \mathbb{R}^2$ has at least one non-trivial critical point. We employ a well-established strategy from the calculus of variations, namely an application of the mountain-pass lemma (to find a Palais-Smale sequence) and the concentration-compactness principle (to deduce the existence of a nonzero critical point). This strategy has been used to obtain solitary-wave solutions to several model equations for water waves, in particular by Kichenassamy [22], Groves [14] and Pego & Quintero [32], and here we follow the theory presented by Groves. The present situation is however complicated by the presence of non-local terms in J and the fact that it is defined only upon a neighbourhood of the origin in its function space. We henceforth denote the radius of this neighbourhood by the distinguished symbol M and write $J : \bar{B}_M(0) \subset X \to \mathbb{R}$; note that although M may be taken arbitrarily large, the greatest permissible magnitude of ε decreases as M is increased.

We begin by collecting together several auxiliary results necessary for the subsequent application of the calculus of variations. Let us first note two topological facts concerning J. Examining the formulae (94), (95), we find that J_2 and J_3 admit natural extensions from $\bar{B}_M(0)$ to the whole of X, and we henceforth consider them as functions $X \to \mathbb{R}$. Recall also that the cubic and higher-order parts J_3 and J_4 of J define smooth functionals on (a neighbourhood of the origin in) $U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2) \cap U_\varepsilon^{\delta,p}(\mathbb{R}^2)$. Turning to an algebraic property of J, we may eliminate $J_3(\Phi_1)$ between

$$J(\Phi_1) = J_2(\Phi_1) + J_3(\Phi_1) + J_4(\Phi_1)$$

and

$$\begin{aligned} \langle\!\langle\!\langle J'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle &= \langle\!\langle\!\langle J_2'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_3'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle \\ &= 2J_2(\Phi_1) + 3J_3(\Phi_1) + \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle, \end{aligned}$$

to obtain the identities

$$\begin{aligned} J(\Phi_1) &= \frac{1}{3}J_2(\Phi_1) + J_4(\Phi_1) - \frac{1}{3}\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \frac{1}{3}\langle\!\langle\!\langle J'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle, \\ J_2(\Phi_1) &= 3J(\Phi_1) - 3J_4(\Phi_1) + \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle \end{aligned}$$

which are exploited repeatedly below.

Observe that

$$|J_2(\Phi_1)| = \frac{1}{2(1+\varepsilon)} |\!|\!| \Phi_1 |\!|\!|^2, \tag{97}$$

$$|J_3(\Phi_1)| \le c\|\Phi_1\|^3_{U_\varepsilon^{0,3}} \le c|\!|\!| \Phi_1 |\!|\!|^3; \tag{98}$$

the following proposition presents corresponding estimates for the higher-order terms in J.

Proposition 3.5 *The inequalities*

$$|J_4(\Phi_1)| \le c\varepsilon^{\frac{1}{4}-\Delta} P_4(|\!|\!| \Phi_1 |\!|\!|), \tag{99}$$

$$|\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle| \le c\varepsilon^{\frac{1}{4}-\Delta} P_4(|\!|\!| \Phi_1 |\!|\!|) \tag{100}$$

hold for each $\Phi_1 \in \bar{B}_M(0) \subset X$.

Proof. We proceed by estimating each term in the explicit formula (96) for J_4 using the inequalities

$$\|\Phi_2\|_{1+\delta,p,\varepsilon} \le c\varepsilon^{-\Delta} P_2(|\!|\!| \Phi_1 |\!|\!|),$$

$$\begin{aligned}
\|\Phi_{2y}\|_{\delta,p,\varepsilon} &\leq c\varepsilon^{\frac{1}{2}-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|),\\
|\rho|_{\delta,p,\varepsilon} &\leq c(\varepsilon^{-\frac{1}{4}-\Delta}|\!|\!|\Phi_1|\!|\!| + \varepsilon^{-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|)),\\
|\rho_{\mathrm{NL}}|_{\delta,p,\varepsilon} &\leq c\varepsilon^{-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|),
\end{aligned}$$

$$\begin{aligned}
\|\Phi_2\|_{1,2,\varepsilon} &\leq c\varepsilon^{\frac{1}{2}-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|),\\
\|\Phi_{2y}\|_2 &\leq c\varepsilon^{1-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|),\\
|\rho|_{0,2,\varepsilon} &\leq c(|\!|\!|\Phi_1|\!|\!| + \varepsilon^{\frac{1}{2}-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|)),\\
|\rho_{\mathrm{NL}}|_{0,2,\varepsilon} &\leq c\varepsilon^{1-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|),
\end{aligned}$$

which are obtained by combining the estimates presented in Theorem 2.9 and Proposition 2.13 with the embeddings (24), (25). We find for example that

$$\begin{aligned}
\left\| \int_0^1 \frac{\rho\Phi_{2y}^2}{2\varepsilon(1+\varepsilon\rho)}\,\mathrm{d}y \right\|_1 &\leq c\varepsilon^{-1}\left\|\frac{\rho}{2(1+\varepsilon\rho)}\right\|_\infty \|\Phi_{2y}\|_2^2\\
&\leq c\varepsilon^{-1}\|\rho\|_\infty\|\Phi_{2y}\|_2^2\\
&\leq c\varepsilon^{-1-\Delta}|\rho|_{\delta,p,\varepsilon}\|\Phi_{2y}\|_2^2\\
&\leq c\varepsilon^{-1-\Delta}(\varepsilon^{-\frac{1}{4}}|\!|\!|\Phi_1|\!|\!| + \varepsilon^{-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|))(\varepsilon^{1-\Delta}P_2(|\!|\!|\Phi_1|\!|\!|))^2\\
&\leq c\varepsilon^{\frac{3}{4}-\Delta}P_4(|\!|\!|\Phi_1|\!|\!|),
\end{aligned}$$

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\Phi_{1x}^2\right\|_1 &\leq \left\|\mathcal{F}^{-1}\left[\frac{\hat{\Phi}_{2x}|_{y=1}}{1+\varepsilon+\beta q^2}\right]\right\|_2 \|\Phi_{1x}^2\|_2\\
&\leq c\varepsilon^{-\frac{1}{4}}\|\mu^{\frac{1}{2}}\hat{\Phi}_2|_{y=1}\|_2\|\Phi_{1x}^2\|_2\\
&\leq c\varepsilon^{-\frac{1}{4}}\|\Phi_2\|_{1,2,\varepsilon}\|\Phi_{1x}\|_4^2\\
&\leq c\varepsilon^{-\frac{1}{4}-\Delta}P_4(|\!|\!|\Phi_1|\!|\!|),
\end{aligned}$$

and the remaining terms are estimated in a similar fashion; altogether we have that

$$|J_4(\Phi_1)| \leq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|\!|\!|\Phi_1|\!|\!|).$$

The second estimate is obtained by noting that

$$\begin{aligned}
\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle &= \langle\!\langle\!\langle J'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle - 2J_2(\Phi_1) - 3J_3(\Phi_1)\\
&= -\int_{\mathbb{R}^2}\left(\int_0^1 H_1\,\mathrm{d}y + h_1\right)\Phi_1\,\mathrm{d}x\,\mathrm{d}z - 3J_3(\Phi_1), \qquad (101)
\end{aligned}$$

where we have used the fact that

$$\langle\!\langle\!\langle J'(\Phi_1), \Psi_1\rangle\!\rangle\!\rangle = \frac{1}{1+\varepsilon}\langle\!\langle\!\langle \Phi_1, \Psi_1\rangle\!\rangle\!\rangle - \int_{\mathbb{R}^2}\left(\int_0^1 H_1\,\mathrm{d}y + h_1\right)\Psi_1\,\mathrm{d}x\,\mathrm{d}z.$$

An expression for $\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle$ is therefore obtained by substituting the explicit formulae for J_3, H_1 and h_1 into the right-hand side of (101). Estimating each term in this expression using the rules explained above, we arrive at the requisite inequality

$$|\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1\rangle\!\rangle\!\rangle| \leq c\varepsilon^{\frac{1}{4}-\Delta}P_4(|\!|\!|\Phi_1|\!|\!|). \qquad \square$$

Let us now recall the mountain-pass lemma as stated by Brezis & Nirenberg [5, p. 943].

Lemma 3.6 *Consider a Banach space $\mathcal{X}$ and a functional $\mathcal{J} \in C^1(\mathcal{X}, \mathbb{R})$ with the properties that $\mathcal{J}(0) = 0$, that 0 is a strict local minimum of $\mathcal{J}$ and that there is an element $x \in \mathcal{X}$ with $\mathcal{J}(x) < 0$. There exists a Palais-Smale sequence $\{x_m\} \subset \mathcal{X}$ such that $\mathcal{J}(x_m) \to a$, $\mathcal{J}'(x_m) \to 0$ as $m \to \infty$, where*

$$a = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \mathcal{J}(\gamma(s)), \qquad \Gamma = \{\gamma \in C([0,1], \mathcal{X}) : \gamma(0) = 0, \mathcal{J}(\gamma(1)) < 0\}.$$

A functional that satisfies the hypotheses of Lemma 3.6 is said to have a *mountain-pass structure.*

It is not possible to apply Lemma 3.6 directly to $J : \bar{B}_M(0) \subset X \to \mathbb{R}$ since it is not defined upon the whole of X. Notice however that it does meet the geometric requirements of a mountain-pass functional: it follows from (97)–(99) that 0 is a strict local minimum of J, and choosing $\Phi_1^\star$ such that $J_3(\Phi_1^\star) \neq 0$, we find that there exists a real number $\lambda^\star$ which has the property that $J(\lambda^\star \Phi_1^\star) < 0$. We proceed by extending J to a smooth functional $\tilde{J} : X \to \mathbb{R}$ in such a way that J and $\tilde{J}$ coincide on a sufficiently large neighbourhood of the origin; the new functional therefore inherits the geometric structure of J and can be treated using Lemma 3.6.

Define

$$M_1 = \sup\{J(\Phi_1) : |||\Phi_1||| \leq 2|||\lambda^\star \Phi_1^\star|||\},$$

choose $M_2 \geq \max(2|||\lambda^\star \Phi_1^\star|||, (24(1+\varepsilon)\tilde{M})^{\frac{1}{2}})$ and let $\psi : X \to \mathbb{R}$ be a smooth ‘cut-off’ function with the properties that

$$\begin{aligned} \psi(x) = 1, &\qquad |||x||| \leq M_2, \\ \psi(x) = 0, &\qquad |||x||| \geq M_2 + 1. \end{aligned}$$

The new functional $\tilde{J} : X \to \mathbb{R}$ is defined by the formula

$$\tilde{J}(\Phi_1) = \tilde{J}_2(\Phi_1) + \tilde{J}_3(\Phi_1) + \tilde{J}_4(\Phi_1),$$

where

$$\tilde{J}_2(\Phi_1) = J_2(\Phi_1), \quad \tilde{J}_3(\Phi_1) = J_3(\Phi_1), \quad \tilde{J}_4(\Phi_1) = \psi(\Phi_1) J_4(\Phi_1).$$

Because $\tilde{J}$ coincides with J on $\bar{B}_N(0) \subset X$, one concludes that 0 is a strict local minimum of $\tilde{J}$ and that $\tilde{J}(\lambda^\star \Phi_1^\star) < 0$. The functional $\tilde{J}$ therefore has a mountain-pass structure, and Lemma 3.6 implies the existence of a Palais-Smale sequence $\{\Phi_{1m}\} \subset X$ such that $\tilde{J}(\Phi_{1m}) \to a_\varepsilon$, $\tilde{J}'(\Phi_{1m}) \to 0$ as $m \to \infty$, where

$$a_\varepsilon = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \tilde{J}(\gamma(s)), \qquad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \tilde{J}(\gamma(1)) < 0\}.$$

(Here, and in the remainder of this section, we attach the subscript ε to certain quantities as a reminder of their ε-dependence.)

The functional $\tilde{J}$ clearly satisfies the same identities as J, namely

$$\tilde{J}(\Phi_1) = \frac{1}{3}\tilde{J}_2(\Phi_1) + \tilde{J}_4(\Phi_1) - \frac{1}{3}\langle\!\langle\!\langle \tilde{J}_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \frac{1}{3}\langle\!\langle\!\langle \tilde{J}'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle, \tag{102}$$

$$\tilde{J}_2(\Phi_1) = 3\tilde{J}(\Phi_1) - 3\tilde{J}_4(\Phi_1) + \langle\!\langle\!\langle \tilde{J}_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle \tilde{J}'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle; \tag{103}$$

we now use these identities to establish some bounds for a and the Palais-Smale sequence $\{\Phi_{1m}\}$ which are needed later.

Proposition 3.7

(i) The constant a_ε *satisfies* $0 < a_\varepsilon \leq M_1$.

(ii) There exists a positive constant C_ε *such that* $|||\Phi_{1m}||| \geq C_\varepsilon$ *for all* $m \in \mathbb{N}$.

(iii) The Palais-Smale sequence $\{\Phi_{1m}\}$ *satisfies* $|||\Phi_{1m}||| \leq M_2$ *for all sufficiently large values of* m.

Proof. (i) The positivity of a_ε follows from the fact that 0 is a strict local minimum of $\tilde{J}$, while the upper bound for a_ε follows from the calculation

$$\begin{aligned} a_\varepsilon &= \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \tilde{J}(\gamma(s)) \\ &\leq \max_{s\in[0,1]} \tilde{J}(s\lambda^\star\Phi_1^\star) \\ &\leq \sup\{\tilde{J}(\Phi_1) : |||\Phi_1||| \leq 2|||\lambda^\star\Phi_1^\star|||\} \\ &= M_1. \end{aligned}$$

(ii) Suppose that there were no positive lower bound for $|||\Phi_{1m}|||$. It would be possible to extract a subsequence (still denoted by $\{\Phi_{1m}\}$) such that $|||\Phi_{1m}||| \to 0$ and hence $J(\Phi_{1m}) \to 0$ as $m \to \infty$, which would imply that $a_\varepsilon = 0$ and contradict part (i).

(iii) The first step is to show that $|||\Phi_{1m}|||$ is bounded above (without loss of generality one may assume that any upper bound is independent of ε). Suppose that there were no upper bound for $|||\Phi_{1m}|||$. It would be possible to extract a subsequence (still denoted by $\{\Phi_{1m}\}$) such that $|||\Phi_{1m}||| \to \infty$ as $m \to \infty$; in particular $|||\Phi_{1m}||| \geq M_2 + 1$ for all sufficiently large values of m, so that $\tilde{J}_4(\Phi_{1m}) = 0$ and

$$\tilde{J}_2(\Phi_{1m}) = 3\tilde{J}(\Phi_{1m}) - \langle\langle\langle \tilde{J}'(\Phi_{1m}), \Phi_{1m} \rangle\rangle\rangle$$

(see equation (103)). It would follow that

$$\frac{1}{2(1+\varepsilon)} |||\Phi_{1m}|||^2 \leq 3|\tilde{J}(\Phi_{1m})| + |||\tilde{J}'(\Phi_{1m})||| \, |||\Phi_{1m}|||$$

and hence that

$$\frac{1}{2(1+\varepsilon)} \leq \frac{3|\tilde{J}(\Phi_{1m})|}{|||\Phi_{1m}|||^2} + \frac{|||\tilde{J}'(\Phi_{1m})|||}{|||\Phi_{1m}|||};$$

this inequality is a contradiction since its right-hand side tends to zero as $m \to \infty$.

The specific upper bound stated in the proposition is obtained using the fact that $|||\Phi_{1m}|||$ is bounded above. Observe that

$$|\tilde{J}(\Phi_{1m})| \geq \frac{1}{3}|\tilde{J}_2(\Phi_{1m})| - |\tilde{J}_4(\Phi_{1m})| - \frac{1}{3}|\langle\langle\langle \tilde{J}_4'(\Phi_{1m}), \Phi_{1m} \rangle\rangle\rangle| - \frac{1}{3}|\langle\langle\langle \tilde{J}'(\Phi_{1m}), \Phi_{1m} \rangle\rangle\rangle|$$

(see equation (102)) and

$$\begin{aligned} \langle\langle\langle \tilde{J}_4'(\Phi_{1m}), \Phi_{1m} \rangle\rangle\rangle &= \psi'(\Phi_{1m})\langle\langle\langle J_4(\Phi_{1m}), \Phi_{1m} \rangle\rangle\rangle + \psi(\Phi_{1m})\langle\langle\langle J_4'(\Phi_{1m}), \Phi_{1m} \rangle\rangle\rangle \\ &\leq c(\psi(\Phi_{1m}) + \psi'(\Phi_{1m}))\varepsilon^{\frac{1}{4}-\Delta} P_4(|||\Phi_{1m}|||) \\ &\leq c\varepsilon^{\frac{1}{4}-\Delta} P_4(|||\Phi_{1m}|||). \end{aligned}$$

Substituting the second inequality into the first, we find that

$$\begin{aligned}|\tilde{J}(\Phi_{1m})| &\geq \frac{1}{6(1+\varepsilon)}(1-c\varepsilon^{\frac{1}{4}-\Delta}P_4(|||\Phi_{1m}|||))|||\Phi_{1m}|||^2 - \frac{1}{3}|||\tilde{J}'(\Phi_{1m})||| \, |||\Phi_{1m}||| \\ &\geq \frac{1}{12(1+\varepsilon)}|||\Phi_{1m}|||^2 - \frac{1}{3}|||\tilde{J}'(\Phi_{1m})||| \, |||\Phi_{1m}|||\end{aligned}$$

(because $|||\Phi_{1m}|||$ is bounded). The left-hand side of this expression approaches a_ε as $m \to \infty$ while the second term on its right-hand side vanishes as $m \to \infty$ (because $\tilde{J}'(\Phi_{1m}) \to 0$ and $|||\Phi_{1m}|||$ is bounded); we conclude that

$$|||\Phi_{1m}|||^2 \leq 24(1+\varepsilon)a_\varepsilon \leq 24(1+\varepsilon)M_1 \leq M_2^2$$

for sufficiently large values of m. □

Proposition 3.7(iii) implies that $\tilde{J}(\Phi_{1m}) = J(\Phi_{1m})$ for sufficiently large values of m; hence, by extracting a subsequence if necessary, one may assume that $\{\Phi_{1m}\}$ is a Palais-Smale sequence for the original functional J, so that $J(\Phi_{1m}) \to a_\varepsilon$ and $J'(\Phi_{1m}) \to 0$ as $m \to \infty$. In the following discussion we therefore return to the original functional $J : \bar{B}_M(0) \subset X \to \mathbb{R}$.

Let us now turn to the second element of the variational theory, namely the concentration-compactness principle (Lions [26, 27]).

Theorem 3.8 *Any sequence $\{u_m\} \subset L^1(\mathbb{R}^2)$ of non-negative functions with the property that*

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} u_m(x,z)\,\mathrm{d}x\,\mathrm{d}z = \ell > 0$$

contains a subsequence for which one of the following phenomena occurs.

Vanishing: For each $R > 0$ one has that

$$\lim_{m\to\infty}\left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_R(\tilde{x},\tilde{z})} u_m(x,z)\,\mathrm{d}x\,\mathrm{d}z\right) = 0.$$

Concentration: There is a sequence $\{(x_m,z_m)\} \subset \mathbb{R}^2$ with the property that for each $\tilde{\epsilon} > 0$ there exists a positive real number R with

$$\int_{B_R(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \geq \ell - \tilde{\epsilon}$$

for each $m \in \mathbb{N}$.

Dichotomy: There are sequences $\{(x_m,z_m)\} \subset \mathbb{R}^2$, $\{R_m\},\{S_m\} \subset \mathbb{R}$ and a real number $\lambda \in (0,\ell)$ with the properties that R_m, $S_m \to \infty$, $R_m/S_m \to 0$,

$$\int_{B_{R_m}(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \to \lambda,$$

$$\int_{B_{S_m}(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \to \lambda,$$

as $m \to \infty$. Furthermore, for each $\tilde{\epsilon} > 0$ there is a positive, real number R such that

$$\int_{B_R(0,0)} u_m(x+x_m, z+z_m)\,\mathrm{d}x\,\mathrm{d}z \geq \lambda - \tilde{\epsilon}$$

for each $m \in \mathbb{N}$.

It follows from Proposition 3.7(ii), (iii) that a subsequence of our Palais-Smale sequence (still denoted by $\{\Phi_{1m}\}$) satisfies $|||\Phi_{1m}|||^2 \to \ell_\varepsilon$ as $m \to \infty$, where $\ell_\varepsilon \neq 0$. This observation suggests exploring the convergence properties of $\{\Phi_{1m}\}$ by applying Theorem 3.8 to the sequence $\{u_m\}$ defined by

$$\begin{aligned} u_m = {} & c_0(\varepsilon\Phi_{1mxxx}^2 + 3\varepsilon^2\Phi_{1mxxz}^2 + 3\varepsilon^3\Phi_{1mxzz}^2 + \varepsilon^4\Phi_{1mzzz}^2) \\ & + (\beta - \tfrac{1}{3})(\Phi_{1mxx}^2 + 2\varepsilon\Phi_{1mxz}^2 + \varepsilon^2\Phi_{1mzz}^2) + \Phi_{1mx}^2 + (1+\varepsilon)\Phi_{1mz}^2. \end{aligned}$$

The consequences of 'vanishing', 'concentration' and 'dichotomy' are investigated in turn below, where $\{u_m\}$ is replaced by the subsequence identified by the relevant clause in Theorem 3.8 and we use the notation given there, writing ℓ_ε, λ_ε as a reminder of the ε-dependence of these quantities. Lemma 3.9 states that 'vanishing' does not occur, while Lemma 3.10 asserts that 'concentration' leads to the weak convergence of $\{\Phi_{1m}\}$ to a nonzero critical point of J. The discussion of 'dichotomy' is more involved and requires several steps, the conclusion of which is again the existence of a nonzero critical point of J.

Lemma 3.9 *The sequence $\{u_m\}$ does not have the 'vanishing' property.*

Proof. Observe that

$$\left(\int_{B_1(\tilde{x},\tilde{z})} u_m\,\mathrm{d}x\,\mathrm{d}z\right)^2 \leq \left(\sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_1(\tilde{x},\tilde{z})} u_m\,\mathrm{d}x\,\mathrm{d}z\right) \int_{B_1(\tilde{x},\tilde{z})} u_m\,\mathrm{d}x\,\mathrm{d}z$$

for each $(\tilde{x},\tilde{z}) \in \mathbb{R}^2$. Cover $\mathbb{R}^2$ by unit balls in such a fashion that each point of $\mathbb{R}^2$ is contained in at most three balls. Summing over all the balls, we find that

$$\begin{aligned} |||\Phi_{1m}|||^4 &\leq c|||\Phi_{1m}|||^2 \sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_1(\tilde{x},\tilde{z})} u_m\,\mathrm{d}x\,\mathrm{d}z \\ &\leq c \sup_{(\tilde{x},\tilde{z})\in\mathbb{R}^2} \int_{B_1(\tilde{x},\tilde{z})} u_m\,\mathrm{d}x\,\mathrm{d}z \\ &\to 0 \end{aligned}$$

as $m \to \infty$, which contradicts the fact that $|||\Phi_{1m}||| \geq C_\varepsilon$ for all $m \in \mathbb{N}$. □

Lemma 3.10 *Suppose that $\{u_m\}$ has the 'concentration' property. The sequence $\{\Phi_{1m}(x_m + \cdot, z_m + \cdot)\}$ converges weakly to a nonzero critical point of J.*

Proof. With a slight abuse of notation, let us abbreviate $\{\Phi_{1m}(x_m+\cdot, z_m+\cdot)\}$ to $\{\Phi_{1m}\}$. Clearly $|\!|\!|\Phi_{1m}|\!|\!|^2 \to \ell_\varepsilon$ as $m \to \infty$, so that $\{\Phi_{1m}\}$ admits a subsequence (still denoted by $\{\Phi_{1m}\}$) which is weakly convergent in X; here we denote its weak limit by Φ_1 and confirm that $\Phi_1 \neq 0$, $J'(\Phi_1) = 0$.

The first step is to show that $\Phi_{1m} \to \Phi_1$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2) \cap U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$. Choose $\tilde{\varepsilon} > 0$. The 'concentration' property asserts the existence of $R > 0$ such that

$$|\!|\!|\Phi_{1m}|\!|\!|_{\{|(x,z)|\geq R\}} < \tilde{\varepsilon}$$

for each $m \in \mathbb{N}$. By replacing R with a larger number if necessary we also have that

$$|\!|\!|\Phi_1|\!|\!|_{\{|(x,z)|\geq R\}} < \tilde{\varepsilon}$$

because Φ_1 is an element of X. It follows from the continuity of the embedding $X_{\{|(x,z)|\geq R\}} \subset U_\varepsilon^{\delta,p}(\{|(x,z)| \geq R\})$ that

$$\begin{aligned} \|\Phi_{1m} - \Phi_1\|_{U_\varepsilon^{\delta,p}(\{|(x,z)|\geq R\})} &\leq c_\varepsilon |\!|\!|\Phi_{1m} - \Phi_1|\!|\!|_{\{|(x,z)|\geq R\}} \\ &\leq c_\varepsilon |\!|\!|\Phi_{1m}|\!|\!|_{\{|(x,z)|\geq R\}} + c_\varepsilon |\!|\!|\Phi_1|\!|\!|_{\{|(x,z)|\geq R\}} \\ &\leq c_\varepsilon \tilde{\varepsilon} \end{aligned}$$

for each $m \in \mathbb{N}$. (Here, and in the remainder of this paper, the symbol c_ε is used to denote a general positive constant which may depend upon ε.) Furthermore, since $X_{B_R(0,0)}$ is compactly embedded in $U_\varepsilon^{\delta,p}(B_R(0,0))$ and $\Phi_{1m} \rightharpoonup \Phi_1$ in $X_{B_R(0,0)}$, one has that $\Phi_{1m} \to \Phi_1$ in $U_\varepsilon^{\delta,p}(B_R(0,0))$; the inequality

$$\|\Phi_{1m} - \Phi_1\|_{U_\varepsilon^{\delta,p}(B_R(0,0))} \leq \tilde{\varepsilon}$$

therefore holds for all sufficiently large values of m. The previous two inequalities assert that

$$\|\Phi_{1m} - \Phi_1\|_{\delta,p,\varepsilon} \leq c_\varepsilon \tilde{\varepsilon}$$

for all sufficiently large values of m, so that $\Phi_{1m} \to \Phi_1$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$, and a similar argument yields the strong convergence in $U_\varepsilon^{0,2}(\mathbb{R}^2)$ and $U_\varepsilon^{0,4}(\mathbb{R}^2)$ (and in fact in any Sobolev space which is locally compactly embedded in X).

It follows from the strong convergence of $\{\Phi_{1m}\}$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2) \cap U_\varepsilon^{0,2}(\mathbb{R}^2) \cap U_\varepsilon^{0,4}(\mathbb{R}^2)$ and the fact that J_3, J_4 are continuous functionals on (a sufficiently large neighbourhood of the origin in) this space that

$$J_3(\Phi_{1m}) \to J_3(\Phi_1), \quad J_4(\Phi_{1m}) \to J_4(\Phi_1), \qquad J_3'(\Phi_{1m}) \to J_3'(\Phi_1), \quad J_4'(\Phi_{1m}) \to J_4'(\Phi_1)$$

as $m \to \infty$, and noting that

$$\langle\!\langle\!\langle \Phi_{1m}, \Psi_1 \rangle\!\rangle\!\rangle \to \langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle$$

as $m \to \infty$ for each fixed $\Psi_1 \in X$ (by the definition of weak convergence), we find that

$$\begin{aligned} \langle\!\langle\!\langle J'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle &= \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_{1m}, \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_3'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_4'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle \\ &\to \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_1, \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_3'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J_4'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle \\ &= \langle\!\langle\!\langle J'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle \end{aligned}$$

as $m \to \infty$. On the other hand one has that

$$\langle\!\langle\!\langle J'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle \to 0$$

as $m \to \infty$, and it follows from the uniqueness of limits that

$$\langle\!\langle\!\langle J'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle = 0.$$

We conclude that $J'(\Phi_1) = 0$ since this equation holds for every $\Psi_1 \in X$.

It remains to confirm that $\Phi_1 \neq 0$. Notice that

$$\langle\!\langle\!\langle J_2'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle = -\langle\!\langle\!\langle J_3'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle + \langle\!\langle\!\langle J'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle,$$

from which it follows that

$$\begin{aligned} \frac{1}{1+\varepsilon} |\!|\!| \Phi_1 |\!|\!|^2 &= -3J_3(\Phi_1) - \langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle \\ &\leq 3|J_3(\Phi_1)| + |\langle\!\langle\!\langle J_4'(\Phi_1), \Phi_1 \rangle\!\rangle\!\rangle| \end{aligned}$$

and hence that

$$\frac{1}{1+\varepsilon} \leq c\left(|\!|\!| \Phi_1 |\!|\!| + c\varepsilon^{\frac{1}{4}-\Delta} \frac{P_4(|\!|\!| \Phi_1 |\!|\!|)}{|\!|\!| \Phi_1 |\!|\!|^2} \right)$$

(see equations (98), (100)); the right-hand side of this equation would vanish for $\Phi_1 = 0$ and contradict the positivity of its left-hand side. □

We now examine the remaining case ('dichotomy'), again abbreviating $\{u_m(x_m + \cdot, z_m + \cdot)\}$ and $\{\Phi_{1m}(x_m + \cdot, z_m + \cdot)\}$ to respectively $\{u_m\}$ and $\{\Phi_{1m}\}$. We begin by recalling an argument due to Groves [14, §3.3] which shows that this scenario also leads to the existence of a nonzero critical point of J; it relies upon a convergence result (equation (104) below) whose proof in the current situation is complicated by the presence of non-local terms in J.

Let $\{\chi_m\} \subset C_0^\infty(\mathbb{R}^2, \mathbb{R})$ be a sequence of 'cut-off' functions with the properties that

$$\begin{aligned} &\chi_m(x,z) = 1, && |(x,z)| \leq R_m, \\ &0 < \chi_m(x,z) < 1, && R_m < |(x,z)| < S_m, \\ &\chi_m(x,z) = 0, && |(x,z)| \geq S_m \end{aligned}$$

and $|\chi_m'(x,z)|, |\chi_m''(x,z)| \leq c$ for each $m \in \mathbb{N}$ and each $(x,z) \in \mathbb{R}^2$. (The existence of a sequence $\{\chi_m\}$ with these properties is assured by the facts that R_m, S_m, $S_m - R_m \to \infty$ as $m \to \infty$.) Define sequences $\{\Phi_{1m}^{(1)}\}$, $\{\Phi_{1m}^{(2)}\}$ and $\{u_m^{(1)}\}$ by the formulae

$$\Phi_{1m}^{(1)} = \Phi_{1m}\chi_m, \qquad \Phi_{1m}^{(2)} = \Phi_{1m}(1 - \chi_m)$$

and

$$\begin{aligned} u_m^{(1)} = c_0 &\Big(\varepsilon(\Phi_{1mxxx}^{(1)})^2 + 3\varepsilon^2(\Phi_{1mxxz}^{(1)})^2 + 3\varepsilon^3(\Phi_{1mxzz}^{(1)})^2 + \varepsilon^4(\Phi_{1mzzz}^{(1)})^2 \Big) \\ &+ (\beta - \tfrac{1}{3})\Big((\Phi_{1mxx}^{(1)})^2 + 2\varepsilon(\Phi_{1mxz}^{(1)})^2 + \varepsilon^2(\Phi_{1mzz}^{(1)})^2 \Big) + (\Phi_{1mx}^{(1)})^2 + (1+\varepsilon)(\Phi_{1mz}^{(1)})^2. \end{aligned}$$

The method described by Groves [14, Proposition 12 and Lemma 14] shows that

$$|||\Phi_{1m}^{(1)}|||^2 \to \lambda_\varepsilon, \qquad |||\Phi_{1m}^{(2)}|||^2 \to \ell_\varepsilon - \lambda_\varepsilon$$

as $m \to \infty$, that there are positive constants $C_\varepsilon^{(1)}$, $C_\varepsilon^{(2)}$ such that

$$|||\Phi_{1m}^{(1)}||| \geq C_\varepsilon^{(1)}, \qquad |||\Phi_{1m}^{(2)}||| \geq C_\varepsilon^{(2)}$$

for all $m \in \mathbb{N}$, that $|||\Phi_{1m}^{(1)}|||$ and $|||\Phi_{1m}^{(2)}|||$ are bounded above (by replacing M_2 with a larger number if necessary we may assume that the upper bounds do not exceed M_2) and that $\{u_m^{(1)}\}$ has the 'concentration' property: for each $\tilde{\varepsilon} > 0$ there exists a positive number R such that

$$\int_{B_R(0,0)} u_m(x,z)\,\mathrm{d}x\,\mathrm{d}z \geq \lambda_\varepsilon - \tilde{\epsilon}$$

for each $m \in \mathbb{N}$. Suppose that

$$\langle\!\langle\!\langle J'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\!\rangle\!\rangle \to 0 \tag{104}$$

as $m \to \infty$ for each $\Psi_1 \in X$; repeating the argument used in the proof of Lemma 3.10, we find that the weak limit $\Phi_1^{(1)}$ of $\{\Phi_{1m}^{(1)}\}$ in X is a nonzero critical point of J.

It therefore remains to establish the limit (104). This task is accomplished by showing that

$$\langle\!\langle\!\langle J'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\!\rangle\!\rangle \to 0 \tag{105}$$

as $m \to \infty$ for each $\Psi_1 \in C_0^\infty(\mathbb{R}^2)$ (and hence, by a density argument, for each $\Psi_1 \in X$); the desired result follows from this limit together with the fact that

$$\langle\!\langle\!\langle J'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\!\rangle\!\rangle = \langle\!\langle\!\langle J'(\Phi_{1m}), \Psi_1 \rangle\!\rangle\!\rangle \to 0$$

as $m \to \infty$. It is a straightforward matter to show that $\langle\!\langle\!\langle J_2'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J_2'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\!\rangle\!\rangle$ vanishes as $m \to \infty$. Observe that

$$\begin{aligned}
&\langle\!\langle\!\langle J_2'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle J_2'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\!\rangle\!\rangle \\
&\quad = \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1 \rangle\!\rangle\!\rangle - \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_{1m}^{(1)}, \Psi_1 \rangle\!\rangle\!\rangle \\
&\quad = \frac{1}{1+\varepsilon} \langle\!\langle\!\langle \Phi_{1m}^{(2)}, \Psi_1 \rangle\!\rangle\!\rangle,
\end{aligned}$$

and since the integrand in the formula for $\langle\!\langle\!\langle \Phi_{1m}^{(2)}, \Psi_1 \rangle\!\rangle\!\rangle$ is calculated by pointwise multiplication of derivatives of Φ_{1m} by derivatives of Ψ_1, we find that $\langle\!\langle\!\langle \Phi_{1m}^{(2)}, \Psi_1 \rangle\!\rangle\!\rangle$ vanishes whenever R_m exceeds the radius of support of Ψ_1, so that in particular the above expression vanishes as $m \to \infty$. The same argument shows that $\langle\!\langle\!\langle (J_3 + J_4)'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle (J_3 + J_4)'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\!\rangle\!\rangle \to 0$ as $m \to \infty$ provided that the integrand defining $\langle\!\langle\!\langle J'(\Phi_1), \Psi_1 \rangle\!\rangle\!\rangle$ contains only *local* operations with respect to (x, z), that is differentiation, integration with respect to y, pointwise addition and pointwise multiplication. The presence of the functional relationships $\rho = \rho(\Phi_1)$, $\Phi_2 = \Phi_2(\Phi_1)$ however means that *nonlocal* effects also have to be taken into account and the simple argument given above no longer suffices.

The functional relationships $\rho = \rho(\Phi_1)$, $\Phi_2 = \Phi_2(\Phi_1)$ are constructed using the basic Fourier-multiplier operators $\mathcal{G}_i$ described in Lemmata 2.10 and 2.15. The next result asserts that these operators, although nonlocal, enjoy a particular property of local operators, namely that $\|\Psi_1 \mathcal{G}_i(\Phi_{1m}^{(2)})\|_{1+\delta,p,\varepsilon} \to 0$ as $m \to \infty$ for each $\Psi_1 \in C_0^\infty(\mathbb{R}^2)$; its proof is deferred to Section 4.

Lemma 3.11 *Choose $N > 0$, suppose that $\{R_m\}$ is a sequence of positive, real numbers such that $R_m \to \infty$ as $m \to \infty$ and let $\chi_N : \mathbb{R}^2 \to \mathbb{R}$, $\chi_{R_m} : \mathbb{R}^2 \to \mathbb{R}$ be smooth 'cut-off' functions whose support is contained in respectively $\bar{B}_N(0)$ and $\bar{B}_{R_m}(0)$. The functions*

$$\mathcal{G}_i^{N,m}(u) = \chi_N \mathcal{G}_i((1 - \chi_{R_m})u), \qquad i = 1, \ldots, 6, 8, \ldots, 11$$

satisfy

$$\|\mathcal{G}_i^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N,m} \|u\|_{\delta,p,\varepsilon}$$

for each $\delta \in [0, 1]$ and each sufficiently large value of p, in which the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \to \infty$.

Our final result shows that the 'local' property of the basic Fourier-multiplier operators described in Lemma 3.11 is sufficient to guarantee the asymptotic behaviour (105) required of J.

Theorem 3.12 *One has that*

$$\langle\!\langle\!\langle (J_3 + J_4)'(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}), \Psi_1 \rangle\!\rangle\!\rangle - \langle\!\langle\!\langle (J_3 + J_4)'(\Phi_{1m}^{(1)}), \Psi_1 \rangle\!\rangle\!\rangle \to 0$$

as $m \to \infty$ for each $\Psi_1 \in C_0^\infty(\mathbb{R}^2)$.

Proof. Recall that $\rho(\Phi_1)$ and $\Phi_2(\Phi_1)$ are constructed by solving fixed-point problems using the contraction-mapping principle, in other words using an iteration scheme. The key to proving this theorem is to approximate $\rho(\Phi_1)$ and $\Phi_2(\Phi_1)$ by the result of a finite number of iterations in the scheme. Let us therefore begin by reviewing the four main steps in the construction of $\rho(\Phi_1)$ and $\Phi_2(\Phi_1)$. In the entirety of the discussion ρ, Ψ, Φ_1 and Φ_2 are supposed to lie in origin-centred balls of respective radii $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$ in $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{O}(\varepsilon^{\frac{1}{2}-\Delta})$ in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$, $\mathcal{O}(\varepsilon^{-\frac{1}{4}-\Delta})$ in $U_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and $\mathcal{O}(\varepsilon^{-\Delta})$ in $W_\varepsilon^{1+\delta,p}(\Sigma)$; all estimates hold uniformly in and suprema are taken over these sets.

(i) One solves a fixed-point problem of the form

$$\rho = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, \rho)$$

in $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$, in which $\mathcal{G}_i : W_\varepsilon^{\delta,p}(\mathbb{R}^2) \to V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is a Fourier-multiplier operator and $\mathcal{N}_i : U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma) \times W_\varepsilon^{\delta,p}(\Sigma) \times V_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ is a 'local' nonlinear function (that is, a function of its arguments that involves only differentiation, integration with respect to y, pointwise addition and pointwise multiplication). This fixed-point problem is solved using the iteration scheme

$$\begin{aligned} \rho_0 &= \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, 0), \\ \rho_{n+1} &= \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, \rho_n), \qquad n = 1, 2, \ldots, \end{aligned}$$

which converges uniformly in Φ_1, Φ_2, Ψ to the unique solution $\rho_\infty(\Phi_1, \Phi_2, \Psi)$. There are estimates for ρ_∞ and its derivatives in terms of Φ_1, Φ_2 and Ψ (see Theorem 2.11).

(ii) One solves a fixed-point problem of the form

$$\Psi = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, \rho_\infty(\Phi_1, \Phi_2, \Psi))$$

in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$, in which $\mathcal{G}_i : W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{\delta,p}(\Sigma)$ is a Fourier-multiplier operator and $\mathcal{N}_i : U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma) \times W_\varepsilon^{\delta,p}(\Sigma) \times V_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{\delta,p}(\Sigma)$ is a 'local' nonlinear function. This fixed-point problem is solved using the iteration scheme

$$\begin{aligned}\Psi_0 &= \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, 0, \rho_\infty(\Phi_1, \Phi_2, 0)),\\ \Psi_{n+1} &= \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi_n, \rho_\infty(\Phi_1, \Phi_2, \Psi_n)), \qquad n = 1, 2, \ldots,\end{aligned}$$

which converges uniformly in Φ_1, Φ_2 to the unique solution $\Psi_\infty(\Phi_1, \Phi_2)$. There are estimates for Ψ_∞ and its derivatives in terms of Φ_1 and Φ_2 (see Theorem 2.16).

(iii) One solves a fixed-point problem of the form

$$\Phi_2 = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi_\infty(\Phi_1, \Phi_2), \rho_\infty(\Phi_1, \Phi_2, \Psi_\infty(\Phi_1, \Phi_2)))$$

in $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$, in which $\mathcal{G}_i : W_\varepsilon^{\delta,p}(\Sigma) \to W_\varepsilon^{1+\delta,p}(\Sigma)$ is a Fourier-multiplier operator and $\mathcal{N}_i : U_\varepsilon^{\delta,p}(\mathbb{R}^2) \times W_\varepsilon^{1+\delta,p}(\Sigma) \times W_\varepsilon^{\delta,p}(\Sigma) \times V_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{\delta,p}(\Sigma)$ is a 'local' nonlinear function. This fixed-point problem is solved using the iteration scheme

$$\begin{aligned}\Phi_{2,0} &= \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, 0, \Psi_\infty(\Phi_1, 0), \rho_\infty(\Phi_1, 0, \Psi_\infty(\Phi_1, 0)))\\ \Phi_{2,n+1} &= \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_{2,n}, \Psi_\infty(\Phi_1, \Phi_{2,n}), \rho_\infty(\Phi_1, \Phi_{2,n}, \Psi_\infty(\Phi_1, \Phi_{2,n}))),\\ &\qquad\qquad n = 1, 2, \ldots,\end{aligned}$$

which converges uniformly in Φ_1 to the unique solution $\Phi_{2,\infty}(\Phi_1)$. There are estimates for $\Phi_{2,\infty}$ and its derivatives in terms of Φ_1 (see Theorem 2.18).

(iv) A supplementary argument shows that $\Psi_n = \partial_y \Phi_{2,n}$ for each $n \in \mathbb{N}$ and $\Psi_\infty = \partial_y \Phi_{2,\infty}$.

Choose $\tilde{\varepsilon} > 0$. It follows from the uniform convergence described in step (i) that

$$|\rho_\infty(\Phi_1, \Phi_2, \Psi) - \rho_n(\Phi_1, \Phi_2, \Psi)|_{\delta,p,\varepsilon} \leq \tilde{\varepsilon} \tag{106}$$

for all sufficiently large values of n, where ρ_n satisfies the same estimates as ρ_∞. Next consider the fixed-point problem

$$\Psi = \sum_i \mathcal{G}_i \mathcal{N}_i(\Phi_1, \Phi_2, \Psi, \rho_n(\Phi_1, \Phi_2, \Psi))$$

obtained by replacing ρ_∞ with ρ_n in step (ii). Applying the iteration scheme described there to this modified fixed-point problem, we obtain a solution $\tilde{\Psi}_\infty$ which satisfies the same estimates as Ψ_∞, and the argument used above for ρ shows that

$$\|\tilde{\Psi}_\infty(\Phi_1, \Phi_2) - \Psi_n(\Phi_1, \Phi_2)\|_{\delta,p,\varepsilon} \leq \tilde{\varepsilon}$$

for all sufficiently large values of n, where Ψ_n satisfies the same estimates as Ψ_∞. Moreover, we find that

$$\begin{aligned}
&\|\Psi_\infty(\Phi_1,\Phi_2)-\tilde{\Psi}_\infty(\Phi_1,\Phi_2)\|_{\delta,p,\varepsilon}\\
&\quad\leq \sum_i \sup \|\mathcal{G}_i\partial_4\mathcal{N}_i(\Phi_1,\Phi_2,\Psi,\rho)\|_{\delta,p,\varepsilon}\|\rho_\infty(\Phi_1,\Phi_2,\Psi)-\rho_n(\Phi_1,\Phi_2,\Psi)\|_{\delta,p,\varepsilon}\\
&\quad\leq c_\varepsilon\tilde{\varepsilon},
\end{aligned}$$

and it follows from the previous two estimates that

$$\|\Psi_\infty(\Phi_1,\Phi_2)-\Psi_n(\Phi_1,\Phi_2)\|_{\delta,p,\varepsilon}\leq c_\varepsilon\tilde{\varepsilon}. \tag{107}$$

Similarly, examining the fixed-point problem

$$\Phi=\sum_i \mathcal{G}_i\mathcal{N}_i(\Phi_1,\Phi_2,\Psi_n(\Phi_1,\Phi_2),\rho_n(\Phi_1,\Phi_2,\Psi_n(\Phi_1,\Phi_2)))$$

obtained by replacing ρ_∞, Ψ_∞ with ρ_n, Ψ_n in step (iii), we find that

$$\|\Phi_{2,\infty}(\Phi_1)-\Phi_{2,n}(\Phi_1)\|_{1+\delta,p,\varepsilon}\leq c_\varepsilon\tilde{\varepsilon} \tag{108}$$

for sufficiently large values of n, where $\Phi_{2,n}$ satisfies the same estimates as $\Phi_{2,\infty}$; by construction we have that $\Psi_n=\partial_y\Phi_{2,n}$.

Let $K_n(\Phi_1,\Psi_1)$ be the functional obtained by replacing each occurrence of ρ_∞ and $\Phi_{2,\infty}$ in the integrand defining $⟪J_{\mathrm{NL}}'(\Phi_1),\Psi_1⟫$ with respectively ρ_n and $\Phi_{2,n}$. (The $W_\varepsilon^{\delta,p}(\Sigma)$-norm of the integrand defining $K_n(\Phi_1,\Psi_1)$ is finite, and, since Ψ_1 has compact support, the same is true of its $W_\varepsilon^{\delta,p}(B_N(0))$-norm, where N denotes the radius of support of Ψ_1; its integrability follows from the embedding $W_\varepsilon^{\delta,p}(B_N(0))\subset L^1(B_N(0))$.) It follows from (106)–(108) that the difference between the two integrands is bounded in the $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$-norm and hence in the $W_\varepsilon^{\delta,p}(B_N(0))$-norm by $c_\varepsilon\tilde{\varepsilon}$, and using the continuity of the embedding $W_\varepsilon^{\delta,p}(B_N(0))\subset L^1(B_N(0))$, we find that

$$|K_n(\Phi_1,\Psi_1)-⟪J_{\mathrm{NL}}'(\Phi_1),\Psi_1⟫|\leq c_\varepsilon\tilde{\varepsilon}.$$

In order to establish that

$$⟪(J_3+J_4)'(\Phi_{1m}^{(1)}+\Phi_{1m}^{(2)}),\Psi_1⟫-⟪(J_3+J_4)'(\Phi_{1m}^{(1)}),\Psi_1⟫\to 0$$

as $m\to\infty$ it therefore suffices to prove that

$$K_n(\Phi_{1m}^{(1)}+\Phi_{1m}^{(2)},\Psi_1)-K_n(\Phi_{1m}^{(1)},\Psi_1)\to 0$$

as $m\to\infty$.

The integrand defining $K_n(\Phi_{1m}^{(1)}+\Phi_{1m}^{(2)},\Psi_1)-K_n(\Phi_{1m}^{(1)},\Psi_1)$ is a finite sum, each term of which is constructed recursively as follows. A *level 0 formula* has the form

$$\mathcal{G}_i\mathcal{N}_i(\Phi_{1m}^{(1)},\Phi_{1m}^{(2)}),$$

while a *level s formula*, $s=1,2,\ldots$ has the form

$$\mathcal{G}_i\mathcal{N}_i(\Phi_{1m}^{(1)},\Phi_{1m}^{(2)},\text{level } 0 \text{ formulae, level } 1 \text{ formulae, } \ldots, \text{ level } s-1 \text{ formulae});$$

here

$$\mathcal{G}_i : \left\{ \begin{array}{c} W^{\delta,p}_\varepsilon(\mathbb{R}^2) \\ W^{\delta,p}_\varepsilon(\Sigma) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} W^{1+\delta,p}_\varepsilon(\mathbb{R}^2) \\ W^{1+\delta,p}_\varepsilon(\Sigma) \end{array} \right\}$$

is a Fourier-multiplier operator and

$$\mathcal{N}_i : \left\{ \begin{array}{c} W^{1+\delta,p}_\varepsilon(\mathbb{R}^2) \\ W^{1+\delta,p}_\varepsilon(\Sigma) \end{array} \right\} \times \ldots \times \left\{ \begin{array}{c} W^{1+\delta,p}_\varepsilon(\mathbb{R}^2) \\ W^{1+\delta,p}_\varepsilon(\Sigma) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} W^{\delta,p}_\varepsilon(\mathbb{R}^2) \\ W^{\delta,p}_\varepsilon(\Sigma) \end{array} \right\}$$

is a 'local' nonlinear function. Each term in our integrand is the product of Ψ_1 and a level s formula for some $s \geq 0$; the target space of its nonlinearity at level s is $W^{\delta,p}_\varepsilon(\mathbb{R}^2)$, the Fourier-multiplier operator at level s may be replaced by the identity and $\Phi^{(2)}_{1m}$ appears in at least one nonlinearity, that is at least one nonlinearity in the recursion scheme satisfies $\mathcal{N}_i(\cdot, 0, \ldots) = 0$. We now show that each term in our integrand tends to zero in $W^{\delta,p}_\varepsilon(\mathbb{R}^2)$ as $m \rightarrow \infty$ for sufficiently large values of p; by replacing $W^{\delta,p}_\varepsilon(\mathbb{R}^2)$ by $W^{\delta,p}_\varepsilon(B_N(0))$ (see above) and using the continuity of the embedding $W^{\delta,p}_\varepsilon(B_N(0)) \subset L^1(B_N(0))$, one concludes that $K_n(\Phi^{(1)}_{1m} + \Phi^{(2)}_{1m}, \Psi_1) - K_n(\Phi^{(1)}_{1m}, \Psi_1) \rightarrow 0$ as $m \rightarrow \infty$.

Consider the expression

$$\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \text{level } 0 \text{ formulae, level } 1 \text{ formulae, } \ldots, \text{ level } s-1 \text{ formulae}). \tag{109}$$

Suppose first that $\Phi^{(2)}_{1m}$ appears in the nonlinearity at level s, which therefore satisfies

$$\mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots) = \chi_m \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots),$$

so that

$$\Psi_1 \mathcal{G}_s \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots) = \Psi_1 \mathcal{G}^{N,m}_s \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots).$$

It follows that

$$\begin{aligned} \|\Psi_1 \mathcal{G}_s \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots)\|_{\delta,p,\varepsilon} &\leq \|\Psi_1 \mathcal{G}^{N,m}_s \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots)\|_{1+\delta,p,\varepsilon} \\ &\leq c^{N,m}_\varepsilon \|\mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots)\|_{\delta,p,\varepsilon} \\ &\leq c^{N,m}_\varepsilon \end{aligned}$$

for sufficiently large values of p, in which Lemma 3.11 and the fact that all arguments of $\mathcal{N}_s$ are bounded in $W^{1+\delta,p}_\varepsilon(\mathbb{R}^2)$ or $W^{1+\delta,p}_\varepsilon(\Sigma)$ have been used. (Recall that the symbol $c^{N,m}_\varepsilon$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \rightarrow \infty$.) The same result clearly holds when $\mathcal{G}_s$ is replaced by the identity since $\Psi_1 \mathcal{N}_s(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots)$ is identically zero for sufficiently large values of m.

Next suppose that $\Phi^{(2)}_{1m}$ appears in a nonlinearity at level $s-1$, so that (109) takes the form

$$\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi^{(1)}_{1m}, \mathcal{G}_{s-1} \mathcal{N}_{s-1}(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots), \ldots),$$

where $\mathcal{N}_{s-1}(\cdot, 0, \ldots) = 0$. The above expression is clearly equal to

$$\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi^{(1)}_{1m}, (1 - \chi_{N_1}) \mathcal{G}_{s-1} \mathcal{N}_{s-1}(\Phi^{(1)}_{1m}, \Phi^{(2)}_{1m}, \ldots), \ldots)$$

$$+\,\Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \tilde{\mathcal{N}}_s(\Phi_{1m}^{(1)}, \chi_{N_1}\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots),$$
$$(1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots), \tag{110}$$

in which

$$\begin{aligned}
&\tilde{\mathcal{N}}_s(\Phi_{1m}^{(1)}, \chi_{N_1}\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), (1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots) \\
&\quad = \mathcal{N}_s(\Phi_{1m}^{(1)}, \mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots) \\
&\qquad - \mathcal{N}_s(\Phi_{1m}^{(1)}, (1-\chi_{N_1})\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots), \ldots)
\end{aligned}$$

and N_1 is any positive number greater than N; note that $\tilde{\mathcal{N}}_s(\cdot, 0, \ldots) = 0$. The previous argument shows that

$$\|\chi_{N_1}\mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N_1,m}$$

for sufficiently large values of p, and by continuity the second term in (110) tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$ for each fixed value of N_1. The previous argument also shows that

$$\left\| \Psi_1 \left\{ \begin{array}{c} \mathcal{G}_s \\ I \end{array} \right\} \mathcal{N}_s(\Phi_{1m}^{(1)}, (1-\chi_{N_1})u_m, \ldots) \right\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N,N_1}$$

uniformly in m for any bounded sequence $\{u_m\}$ in $W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$ or $W_\varepsilon^{1+\delta,p}(\Sigma)$, and in particular for $u_m = \mathcal{G}_{s-1}\mathcal{N}_{s-1}(\Phi_{1m}^{(1)}, \Phi_{1m}^{(2)}, \ldots)$. Taking the limit $m \to \infty$ followed by the limit $N_1 \to \infty$ in (110), one concludes that this expression tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$.

An appearance of $\Phi_{1m}^{(2)}$ in a level $s-2$ nonlinearity is similarly handled using two new 'cut-off' functions χ_{N_1}, χ_{N_2} with $N_2 > N_1 > N$, and proceeding recursively in this fashion we find that each term in the integrand defining $K_n(\Phi_{1m}^{(1)} + \Phi_{1m}^{(2)}, \Psi_1) - K_n(\Phi_{1m}^{(1)}, \Psi_1)$ tends to zero in $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ as $m \to \infty$ for sufficiently large values of p. □

4 Fourier-multiplier operators

It remains to establish the results stated in Sections 2.2–2.4 and Section 3.2 concerning Fourier-multiplier operators, namely their mapping properties (in particular the estimates on their norms given in Lemmata 2.10, 2.15 and 2.20) and the convergence properties given in Lemma 3.11.

4.1 Basic tools

Here we present our basic tools for studying Fourier-multiplier operators in L^p-based spaces, $p \neq 2$, beginning with well-known results known as 'Marcinkiewicz's theorem' (Lemma 4.1) and 'Mikhlin's theorem' (Lemma 4.2); detailed proofs are given by Stein [34, Chapter IV].

Lemma 4.1 *Consider the operator T defined by*

$$(Tu)(x,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-x_1, z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose that the kernel K satisfies $|\hat{K}| \leq c\varepsilon^{\alpha}$ and

$$\begin{aligned}
\sup_{j\in\mathbb{Z}} \int_{I_j} |\partial_\mu \hat{K}|\,\mathrm{d}\mu &\leq c\varepsilon^{\alpha}, \\
\sup_{j\in\mathbb{Z}} \int_{I_j} |\partial_k \hat{K}|\,\mathrm{d}k &\leq c\varepsilon^{\alpha}, \\
\sup_{j_1,j_2\in\mathbb{Z}} \int_{I_{j_1}} \int_{I_{j_2}} |\partial_\mu \partial_k \hat{K}|\,\mathrm{d}\mu\,\mathrm{d}k &\leq c\varepsilon^{\alpha},
\end{aligned}$$

where I_j is the dyadic interval $(2^j, 2^{j+1})$ or $(-2^{j+1}, -2^j)$. The operator T maps $L^p(\mathbb{R}^2)$ continuously into itself and

$$\|Tu\|_p \leq c\varepsilon^{\alpha}\|u\|_p.$$

Lemma 4.2 *Consider the operator T defined by*

$$(Tu)(x,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-x_1, z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose that the kernel K satisfies

$$\begin{aligned}
|\hat{K}| &\leq c\varepsilon^{\alpha}, \\
|\partial_\mu \hat{K}| + |\partial_k \hat{K}| &\leq \frac{c\varepsilon^{\alpha}}{(\mu^2+k^2)^{1/2}}, \\
|\partial_\mu^2 \hat{K}| + |\partial_\mu \partial_k \hat{K}| + |\partial_k^2 \hat{K}| &\leq \frac{c\varepsilon^{\alpha}}{\mu^2+k^2}
\end{aligned}$$

for each $(\mu,k) \neq (0,0)$. The operator T maps $L^p(\mathbb{R}^2)$ continuously into itself and

$$\|Tu\|_p \leq c\varepsilon^{\alpha}\|u\|_p.$$

The next result is a scaled version of Lemma 4.2 which is useful in dealing with scaled function spaces such as $W_\varepsilon^{\delta,p}(\mathbb{R}^2)$.

Lemma 4.3 *Consider the operator T defined by*

$$(Tu)(x,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-x_1, z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose that the kernel K satisfies

$$\begin{aligned}
|\hat{K}| &\leq c\varepsilon^{\alpha}, \\
|\partial_\mu \hat{K}| + \varepsilon^{-\frac{1}{2}}|\partial_k \hat{K}| &\leq \frac{c\varepsilon^{\alpha}}{(\mu^2+\varepsilon k^2)^{1/2}}, \\
|\partial_\mu^2 \hat{K}| + \varepsilon^{-\frac{1}{2}}|\partial_\mu \partial_k \hat{K}| + \varepsilon^{-1}|\partial_k^2 \hat{K}| &\leq \frac{c\varepsilon^{\alpha}}{\mu^2+\varepsilon k^2}
\end{aligned}$$

for each $(\mu,k) \neq (0,0)$. The operator T maps $L^p(\mathbb{R}^2)$ continuously into itself and

$$\|Tu\|_p \leq c\varepsilon^{\alpha}\|u\|_p.$$

We now turn to Fourier-multiplier operators in $L^p(\Sigma)$-based function spaces. Our first result in this direction is obtained by a straightforward application of Hölder's inequality.

Theorem 4.4 *Consider the operator T defined by*

$$(Tu)(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 K(x-x_1, z-z_1; y, \xi) u(x_1, \xi, z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose the kernel $K(x,z;y,\xi)$ satisfies the hypotheses of Lemma 4.1, Lemma 4.2 or Lemma 4.3 uniformly for $y, \xi \in [0,1]$. The operator T maps $L^p(\Sigma)$ continuously into itself and

$$\|Tu\|_p \leq c\varepsilon^\alpha \|u\|_p.$$

A natural tactic in dealing with more general Fourier-multiplier operators on $L^p(\Sigma)$ is to consider them as operators on $L^p(\mathbb{R}^2, L^p(0,1) \to L^p(0,1))$. Unfortunately the multiplier theorems of Marcinkiewicz and Mikhlin do not generalise to this operator-valued setting in a straightforward manner. An operator-valued generalisation of a theorem by Stein [34, p. 29], in which the hypotheses upon derivatives of $\hat{K}$ are replaced by hypotheses upon the derivatives of K itself, is however available (see the discussion in §II5.1 of this reference); the following result is a scaled version of the appropriate theorem.

Theorem 4.5 *Consider the operator T defined by*

$$(Tu)(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 K(x-x_1, z-z_1; y, \xi) u(x_1, \xi, z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1.$$

Suppose the kernel $K(x,z;y,\xi)$ satisfies

$$\left\| \int_0^1 \hat{K} w\,\mathrm{d}\xi \right\|_{L^p(0,1)} \leq c\varepsilon^\alpha \tag{111}$$

and

$$\left\| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{1/2}\partial_z \end{array} \right\} \int_0^1 K w\,\mathrm{d}\xi \right\|_{L^p(0,1)} \leq \frac{c\varepsilon^{\alpha-1/2}}{(x^2+\varepsilon^{-1}z^2)^{3/2}}, \qquad (x,z) \neq (0,0) \tag{112}$$

for each $w \in L^p(0,1)$. The operator T maps $L^p(\Sigma)$ continuously into itself and

$$\|Tu\|_p \leq c\varepsilon^\alpha \|u\|_p.$$

4.2 *Mapping properties*

The next step is to use the results stated above to analyse the mapping properties of the operators $\mathcal{G}_1, \ldots, \mathcal{G}_{16}$ defined in Lemmata 2.10, 2.15 and 2.20. Our first result is the proof of Lemma 2.10(i); parts (ii) and (iii) are proved in a similar fashion.

Lemma 4.6 *Choose $\delta \in [0,1]$ and $p \in (1,\infty)$. For each $u \in W_\varepsilon^{\delta,p}(\mathbb{R}^2)$ the function*

$$\mathcal{G}_1(u) = \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]$$

belongs to $V_\varepsilon^{\delta,p}(\mathbb{R}^2)$ and satisfies the estimate

$$|\mathcal{G}_1(u)|_{\delta,p,\varepsilon} \leq c\|u\|_{\delta,p,\varepsilon}.$$

Proof. Observe that

$$\begin{aligned}
\left|\frac{1}{1+\varepsilon+\beta q^2}\right| &\leq c, \\
\varepsilon^{-\frac{1}{2}}q\left|\frac{\partial}{\partial\mu}\left(\frac{1}{1+\varepsilon+\beta q^2}\right)\right| &= \left|\frac{-2\beta\varepsilon^{1/2}q\mu}{(1+\varepsilon+\beta q^2)^2}\right| \leq c, \\
\varepsilon^{-1}q\left|\frac{\partial}{\partial k}\left(\frac{1}{1+\varepsilon+\beta q^2}\right)\right| &= \left|\frac{-2\beta\varepsilon qk}{(1+\varepsilon+\beta q^2)^2}\right| \leq c, \\
\varepsilon^{-1}q^2\left|\frac{\partial}{\partial\mu^2}\left(\frac{1}{1+\varepsilon+\beta q^2}\right)\right| &= \left|\frac{-2\beta q^2}{(1+\varepsilon+\beta q^2)^2} + \frac{8\beta^2\varepsilon q^2\mu^2}{(1+\varepsilon+\beta q^2)^3}\right| \leq c, \\
\varepsilon^{-2}q^2\left|\frac{\partial}{\partial k^2}\left(\frac{1}{1+\varepsilon+\beta q^2}\right)\right| &= \left|\frac{-2\beta q^2}{(1+\varepsilon+\beta q^2)^2} + \frac{8\beta^2\varepsilon^2 q^2 k^2}{(1+\varepsilon+\beta q^2)^3}\right| \leq c, \\
\varepsilon^{-\frac{3}{2}}q^2\left|\frac{\partial}{\partial\mu\partial k}\left(\frac{1}{1+\varepsilon+\beta q^2}\right)\right| &= \left|\frac{8\beta^2\varepsilon^{\frac{3}{2}}q^2\mu k}{(1+\varepsilon+\beta q^2)^3}\right| \leq c.
\end{aligned}$$

Lemma 4.3 therefore implies that

$$\left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p \leq c\|u\|_p,$$

and repeating this argument with the multiplier $\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}}(1+\varepsilon+\beta q^2)^{-1}$ we find that

$$\left\|\mathcal{F}^{-1}\left[\frac{\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p \leq c\|u\|_p.$$

It follows that

$$\begin{aligned}
|\mathcal{G}_1(u)|_{\delta,p,\varepsilon} &= \left\|\mathcal{F}^{-1}\left[\frac{1+\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}+\frac{\delta}{2}}}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p \\
&\leq \left\|\mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[u]\right]\right\|_p + \left\|\mathcal{F}^{-1}\left[\frac{\varepsilon^{\frac{1}{2}}(\mu^2+\varepsilon k^2)^{\frac{1}{2}}}{1+\varepsilon+\beta q^2}(\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}[u]\right]\right\|_p \\
&\leq c(\|u\|_p + \|\mathcal{F}^{-1}[(\mu^2+\varepsilon k^2)^{\frac{\delta}{2}}\mathcal{F}[u]]\|_p) \\
&\leq c\|u\|_{\delta,p,\varepsilon}.
\end{aligned}$$

□

The next result gives the proof of Lemma 2.20(i); part (ii) is proved in a similar fashion. Parts (iii)-(iv) are deduced from parts (i) and (ii) together with Lemma 2.10. Observe that

$$\|\partial_x \mathcal{G}_{14}(u)\|_{2,p,\varepsilon} = \|\mathcal{G}_1(\partial_x \mathcal{G}_{12}(u))\|_{2,p,\varepsilon} \leq \|\mathcal{G}_1\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\|\mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \leq c\|u\|_p,$$

$$\varepsilon^{\frac{1}{2}}\|\partial_z \mathcal{G}_{14}(u)\|_{2,p,\varepsilon} = \|\mathcal{G}_1(\varepsilon^{\frac{1}{2}}\partial_z \mathcal{G}_{12}(u))\|_{2,p,\varepsilon} \leq \|\mathcal{G}_1\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})}\,\varepsilon^{\frac{1}{2}}\|\mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \leq c\|u\|_p,$$

where the final inequalities in each line follow from Lemma 2.10(i) and Lemma 2.20(i); the estimates for $\mathcal{G}_{15}$ and $\mathcal{G}_{16}$ are obtained by the same method.

Lemma 4.7 *Choose* $p \in (1,\infty)$*. For each* $u \in L^p(\mathbb{R}^2)$ *the function*

$$\mathcal{G}_{12}(u) = \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right]$$

belongs to $U^{2,p}_\varepsilon(\mathbb{R}^2)$ *and satisfies the estimate*

$$\|\mathcal{G}_{12}(u)\|_{U^{2,p}_\varepsilon} \leq c\|u\|_p.$$

Proof. Using the calculations

$$\begin{aligned}
\frac{\partial}{\partial\mu}\left(\frac{1}{Q}\right) &= -\frac{1}{Q^2}\left(2\mu + 4(\beta - \tfrac{1}{3})\varepsilon^{-2}q^3\partial_\mu q + 6c_0\varepsilon^{-2}q^5\partial_\mu q\right),\\
\frac{\partial}{\partial k}\left(\frac{1}{Q}\right) &= -\frac{1}{Q^2}\left(2(1+\varepsilon)k + 4(\beta - \tfrac{1}{3})\varepsilon^{-2}q^3\partial_k q + 6c_0\varepsilon^{-2}q^5\partial_k q\right)
\end{aligned}$$

and the estimates

$$|\partial_\mu q| = \left|\frac{\varepsilon\mu}{q}\right| \leq c\varepsilon^{\frac{1}{2}}, \qquad |\partial_k q| = \left|\frac{\varepsilon^2 k}{q}\right| \leq c\varepsilon,$$

we find that

$$\left|\frac{\partial}{\partial\mu}\left(\frac{1}{Q}\right)\right| \leq \frac{c}{Q^2}(|\mu| + \varepsilon^{-\frac{3}{2}}q^3 + \varepsilon^{-\frac{3}{2}}q^5), \qquad \left|\frac{\partial}{\partial k}\left(\frac{1}{Q}\right)\right| \leq \frac{c}{Q^2}(|k| + \varepsilon^{-1}q^3 + \varepsilon^{-1}q^5).$$

It follows that

$$\begin{aligned}
\left|\frac{\mu^2}{Q}\right| &\leq c,\\
(\mu^2+k^2)^{\frac{1}{2}}\left|\frac{\partial}{\partial\mu}\left(\frac{\mu^2}{Q}\right)\right| &\leq c\left(\frac{|\mu|}{Q}(\mu^2+k^2)^{\frac{1}{2}} + \frac{\mu^2}{Q^2}(\mu^2+k^2)^{\frac{1}{2}}(|\mu| + \varepsilon^{-\frac{3}{2}}q^3 + \varepsilon^{-\frac{3}{2}}q^5)\right) \leq c,\\
(\mu^2+k^2)^{\frac{1}{2}}\left|\frac{\partial}{\partial k}\left(\frac{\mu^2}{Q}\right)\right| &\leq \frac{c\mu^2}{Q^2}(\mu^2+k^2)^{\frac{1}{2}}(|k| + \varepsilon^{-1}q^3 + \varepsilon^{-1}q^5) \leq c,
\end{aligned}$$

and similar calculations show that

$$\left|\partial_\mu^2\left(\frac{\mu^2}{Q}\right)\right| \leq c, \quad \left|\partial_k^2\left(\frac{\mu^2}{Q}\right)\right| \leq c, \quad \left|\partial_\mu\partial_k\left(\frac{\mu^2}{Q}\right)\right| \leq c.$$

Lemma 4.2 therefore asserts that

$$\left\|\partial_x \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right]\right\|_p = \left\|\mathcal{F}^{-1}\left[\frac{-\mu^2}{Q}\mathcal{F}[u]\right]\right\|_p \le c\|u\|_p,$$

and repeating this argument with the multiplier $(\mu^2+\varepsilon k^2)\mu^2/Q$, one finds that

$$\left\|\mathcal{F}^{-1}\left[(\mu^2+\varepsilon k^2)\mathcal{F}\left[\partial_x \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{Q}\mathcal{F}[u]\right]\right]\right]\right\|_p = \left\|\mathcal{F}^{-1}\left[\frac{-\mu^2(\mu^2+\varepsilon k^2)}{Q}\mathcal{F}[u]\right]\right\|_p \le c\|u\|_p.$$

The previous two inequalities imply that

$$\|\partial_x \mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \le c\|u\|_p,$$

and a similar argument yields the complementary estimate

$$\varepsilon^{\frac{1}{2}}\|\partial_z \mathcal{G}_{12}(u)\|_{2,p,\varepsilon} \le c\|u\|_p. \qquad \square$$

It is instructive to compare the proofs of Lemmata 4.6 and 4.7. The former uses the scaled version of Mikhlin's theorem (Lemma 4.3), while the latter relies upon the standard version (Lemma 4.2). In general, the scaled version of Mikhlin's theorem is appropriate for multipliers which depend upon μ and k only through the combination q and for multipliers whose support is bounded away from the origin (e.g. see Lemmata 4.13 and 4.14 below); in all other circumstances one requires the standard version of Mikhlin's theorem.

We now turn to the more involved analysis necessary for Lemma 2.15. The first step in the proof of parts (i) and (ii) of this lemma is to establish the basic estimate that for each $u \in L^p(\Sigma)$ the function

$$\mathcal{G}(u) = \mathcal{F}^{-1}\left[\int_0^1 G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right]$$

belongs to $W^{2,p}_\varepsilon(\Sigma)$ and satisfies

$$\|\mathcal{G}(u)\|_{2,p,\varepsilon} \le c\varepsilon\|u\|_p;$$

to this end we show that

$$\|\mathcal{G}(u)\|_p \le c\varepsilon\|u\|_p, \qquad \|\tilde{\mathcal{G}}(u)\|_p \le c\varepsilon\|u\|_p, \tag{113}$$

where

$$\tilde{\mathcal{G}}(u) = \mathcal{F}^{-1}\left[\int_0^1 (\mu^2+\varepsilon k^2) G_1 \mathcal{F}[u]\,\mathrm{d}\xi\right],$$

and

$$\|\partial_y^2 \mathcal{G}(u)\|_p \le c\varepsilon\|u\|_p. \tag{114}$$

The expansion

$$q^2 - (1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2 = -\varepsilon^2\mu^2 - (1+\varepsilon)\varepsilon^2 k^2 - \left(\beta - \frac{\alpha}{3}\right)q^4 - c_0 q^6 + \mathcal{O}(q^8)$$

as $q \to 0$ implies the existence of a constant q_0 such that

$$|q^2 - (1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2| \geq c\varepsilon^2 Q \tag{115}$$

whenever $q \leq q_0$. Let $\chi \in C_0^\infty([0,\infty),\mathbb{R})$ be a smooth 'cut-off' function with the properties that

$$\begin{array}{ll}\chi(q) = 1, & q \leq q_0/2,\\ \chi(q) = 0, & q \geq q_0\end{array}$$

and consider the decompositions $\mathcal{G} = \mathcal{G}_\mathrm{a} + \mathcal{G}_\mathrm{b}$, $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_\mathrm{a} + \tilde{\mathcal{G}}_\mathrm{b}$, where

$$\mathcal{G}_\mathrm{a}(u) = \mathcal{F}^{-1}\left[\int_0^1 \chi(q)G_1\mathcal{F}[u]\,\mathrm{d}\xi\right], \qquad \mathcal{G}_\mathrm{b}(u) = \mathcal{F}^{-1}\left[\int_0^1 (1-\chi(q))G_1\mathcal{F}[u]\,\mathrm{d}\xi\right]$$

and $\tilde{\mathcal{G}}_\mathrm{a}$, $\tilde{\mathcal{G}}_\mathrm{b}$ are defined in the same way. We establish (113) by proving that it holds for $\mathcal{G}_\mathrm{a}$, $\tilde{\mathcal{G}}_\mathrm{a}$ and $\mathcal{G}_\mathrm{b}$, $\tilde{\mathcal{G}}_\mathrm{b}$ separately and use an auxiliary argument to deduce (114). The first step in this programme is accomplished by Lemmata 4.8 and 4.9 below, which present the required estimates for $\mathcal{G}_\mathrm{a}$ and $\tilde{\mathcal{G}}_\mathrm{a}$.

Lemma 4.8 *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\mathcal{G}_\mathrm{a}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\mathcal{G}_\mathrm{a}(u)\|_p \leq c\varepsilon\|u\|_p.$$

Proof. We show that the hypotheses of Lemma 4.2, namely

$$\begin{aligned}|\chi G_1| &\leq c\varepsilon,\\ (\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}(\chi G_1)\right| &\leq c\varepsilon,\\ (\mu^2+k^2)\left|\left\{\begin{array}{c}\partial_\mu^2\\ \partial_k^2\\ \partial_\mu\partial_k\end{array}\right\}(\chi G_1)\right| &\leq c\varepsilon,\end{aligned}$$

are satisfied uniformly for $y,\xi \in [0,1]$, so that the result follows by an application of Theorem 4.4.

Let us write

$$\frac{G_1}{\varepsilon^2} = \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3, \tag{116}$$

where

$$\begin{aligned}\tilde{G}_1 &= \frac{(1+\varepsilon)\tilde{G}}{q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2k^2},\\ \tilde{G}_2 &= \frac{1+\varepsilon}{q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2k^2}\\ &\qquad + \frac{1+\varepsilon}{\varepsilon^2\mu^2+(1+\varepsilon)\varepsilon^2k^2+(\beta-\frac{1}{3}(1+\varepsilon))q^4+c_0q^6},\\ \tilde{G}_3 &= \frac{1+\varepsilon}{\varepsilon^2\mu^2+(1+\varepsilon)\varepsilon^2k^2+(\beta-\frac{1}{3})q^4+c_0q^6}\\ &\qquad - \frac{1+\varepsilon}{\varepsilon^2\mu^2+(1+\varepsilon)\varepsilon^2k^2+(\beta-\frac{1}{3}(1+\varepsilon))q^4+c_0q^6}\end{aligned}$$

and

$$\tilde{G}=\begin{cases}\dfrac{\cosh qy}{\cosh q}\left(\dfrac{\beta q^2}{1+\varepsilon}\cosh q(1-\xi)+\dfrac{\varepsilon\mu^2}{(1+\varepsilon)q}\sinh q(\xi-1)\right)+\dfrac{\cosh qy}{\cosh q}\cosh q(1-\xi)-1, \\ \hfill 0<y<\xi<1, \\ \dfrac{\cosh q\xi}{\cosh q}\left(\dfrac{\beta q^2}{1+\varepsilon}\cosh q(1-y)+\dfrac{\varepsilon\mu^2}{(1+\varepsilon)q}\sinh q(y-1)\right)+\dfrac{\cosh q\xi}{\cosh q}\cosh q(1-y)-1, \\ \hfill 0<\xi<y<1.\end{cases}$$

Using (115) and the fact that $\tilde{G}=\mathcal{O}(q^2)$ as $q\to 0$ uniformly for $y,\xi\in[0,1]$, we find that

$$\begin{aligned}|\tilde{G}_1| &\leq \frac{cq^2}{\varepsilon^2 Q} \leq c\varepsilon^{-2}\frac{q^2}{Q} \leq c\varepsilon^{-1}, \\ |\tilde{G}_2| &\leq \frac{cq^8}{\varepsilon^4 Q^2} \leq c\varepsilon^{-2}\left(\frac{q^2}{Q}\right)\left(\frac{q^6}{\varepsilon^2 Q}\right) \leq c\varepsilon^{-1}, \\ |\tilde{G}_3| &\leq \frac{q^4}{\varepsilon^3 Q^2} \leq c\varepsilon^{-3}\left(\frac{q^2}{Q}\right)^2 \leq c\varepsilon^{-1}\end{aligned}$$

for $q\leq q_0$ and hence that $|\chi G_1| \leq c\varepsilon$.

It follows from the calculations

$$\begin{aligned}&\partial_\mu\left(\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right) \\ &\qquad = -\frac{(2q-(1+\varepsilon+3\beta q^2)\tanh q-(1+\varepsilon+\beta q^2)q\,\mathrm{sech}^2 q)}{(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)^2}\partial_\mu q, \\ &\partial_k\left(\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right) \\ &\qquad = -\frac{(2q-(1+\varepsilon+3\beta q^2)\tanh q-(1+\varepsilon+\beta q^2)q\,\mathrm{sech}^2 q)}{(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)^2}\partial_k q \\ &\qquad\qquad + \frac{2\varepsilon^2 k}{(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)^2},\end{aligned}$$

that

$$\begin{aligned}\left|\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right| &\leq \frac{c}{\varepsilon^2 Q}, \\ \left|\partial_\mu\left(\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right)\right| &\leq \frac{c}{\varepsilon^4Q^2}(\varepsilon q+q^3)|\partial_\mu q|, \qquad (117) \\ \left|\partial_k\left(\frac{1}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\right)\right| &\leq c\left(\frac{1}{\varepsilon^4Q^2}(\varepsilon q+q^3)|\partial_k q|+\frac{\varepsilon^2|k|}{\varepsilon^4Q^2}\right) \qquad (118)\end{aligned}$$

for $q\leq q_0$. Furthermore, one has that

$$\partial_\mu\tilde{G}=\partial_q\tilde{G}\,\partial_\mu q$$

(where $\varepsilon^2\mu^2$ is replaced by $q^2-\varepsilon^2k^2$ in the formula for $\tilde{G}$),

$$\partial_k\tilde{G} = \partial_q\tilde{G}\,\partial_k q$$

and $\partial_q\tilde{G} = \mathcal{O}(q)$ as $q \to 0$ uniformly for $y, \xi \in [0,1]$, so that

$$|\partial_\mu\tilde{G}| \leq c\varepsilon^{\frac{1}{2}}q, \qquad |\partial_k\tilde{G}| \leq c\varepsilon q.$$

Combining the above estimates, we find that

$$\begin{aligned}
(\mu^2+k^2)^{\frac{1}{2}}|\partial_\mu\tilde{G}_1| &\leq c(\mu^2+k^2)^{\frac{1}{2}}\left(\frac{\varepsilon^{\frac{1}{2}}q}{\varepsilon^2 Q}+\frac{\varepsilon^{\frac{3}{2}}q^3}{\varepsilon^4Q^2}+\frac{\varepsilon^{\frac{1}{2}}q^5}{\varepsilon^4Q^2}\right) \leq c\varepsilon^{-1}, \\
(\mu^2+k^2)^{\frac{1}{2}}|\partial_k\tilde{G}_1| &\leq c(\mu^2+k^2)^{\frac{1}{2}}\left(\frac{\varepsilon q}{\varepsilon^2 Q}+\frac{\varepsilon^2q^3}{\varepsilon^4Q^2}+\frac{\varepsilon q^5}{\varepsilon^4Q^2}+\frac{\varepsilon^2q^2|k|}{\varepsilon^4Q^2}\right) \leq c\varepsilon^{-1}
\end{aligned}$$

for $q \leq q_0$.

Observe that

$$|\partial_\mu\chi| = |\chi'(q)\partial_\mu q| \leq c\varepsilon^{\frac{1}{2}}, \qquad |\partial_k\chi| = |\chi'(q)\partial_k q| \leq c\varepsilon,$$

whence

$$\begin{aligned}
(\mu^2+k^2)^{\frac{1}{2}}|\tilde{G}_1\partial_\mu\chi| &\leq (\varepsilon\mu^2+\varepsilon k^2)^{\frac{1}{2}}|\tilde{G}_1| \\
&\leq |\tilde{G}_1|+\varepsilon^{-1}|q^2\tilde{G}_1| \\
&\leq c\left(\varepsilon^{-1}+\frac{\varepsilon^{-1}q^4}{\varepsilon^2Q}\right) \\
&\leq c\varepsilon^{-1},
\end{aligned}$$

$$\begin{aligned}
(\mu^2+k^2)^{\frac{1}{2}}|\tilde{G}_1\partial_k\chi| &\leq (\varepsilon^2\mu^2+\varepsilon^2k^2)^{\frac{1}{2}}|\tilde{G}_1| \\
&\leq |\tilde{G}_1|+|q^2\tilde{G}_1| \\
&\leq c\varepsilon^{-1},
\end{aligned}$$

in which $q \leq q_0$ on the right-hand side, and altogether we have that

$$(\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}(\chi\tilde{G}_1)\right| = (\mu^2+k^2)^{\frac{1}{2}}\left|\chi\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}\tilde{G}_1+\tilde{G}_1\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}\chi\right| \leq c\varepsilon^{-1}.$$

The corresponding estimates for the derivatives of $\tilde{G}_2$ are obtained using the formula

$$\left(\frac{1}{f}-\frac{1}{g}\right)' = \frac{(g-f)'}{f^2}+\frac{g'(f-g)(f+g)}{f^2g^2}$$

with

$$f = q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2, \quad g = -\varepsilon^2\mu^2-(1+\varepsilon)\varepsilon^2k^2-(\beta-\tfrac{1}{3}(1+\varepsilon))q^4-c_0q^6,$$

in which the prime denotes differentiation with respect to μ or k. One finds that

$$
\begin{aligned}
&(\mu^2+k^2)^{\frac{1}{2}}|\partial_\mu \tilde{G}_2| \\
&\quad \le\ c(\mu^2+k^2)^{\frac{1}{2}}\left(\frac{q^7}{\varepsilon^4 Q^2}|\partial_\mu q| + (\varepsilon^2|\mu| + q^3|\partial_\mu q|)\frac{q^8}{\varepsilon^8 Q^4}(\varepsilon^2\mu^2+\varepsilon^2k^2+q^4)\right)\ \le\ c\varepsilon^{-1}, \quad (119)\\
&(\mu^2+k^2)^{\frac{1}{2}}|\partial_k \tilde{G}_2| \\
&\quad \le\ c(\mu^2+k^2)^{\frac{1}{2}}\left(\frac{q^7|\partial_k q|}{\varepsilon^4 Q^2} + (\varepsilon^2|k| + q^3|\partial_k q|)\frac{q^8}{\varepsilon^8 Q^4}(\varepsilon^2\mu^2+\varepsilon^2k^2+q^4)\right)\ \le\ c \quad (120)
\end{aligned}
$$

for $q \le q_0$, in which the inequalities

$$|g-f|\ \le\ cq^8, \quad |\partial_\mu(g-f)|\ \le\ cq^7|\partial_\mu q|, \quad |\partial_k(g-f)|\ \le\ cq^7|\partial_k q|$$

have been used. Since $|\tilde{G}_2| \le c\varepsilon^{-1}$ and

$$|q^2\tilde{G}_2|\ \le\ \frac{cq^{10}}{\varepsilon^4 Q^2}\ \le\ c\left(\frac{q^4}{\varepsilon^2 Q}\right)\left(\frac{q^6}{\varepsilon^2 Q}\right)\ \le\ c$$

for $q \le q_0$ the argument given above shows that

$$(\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}(\chi\tilde{G}_2)\right| \le c\varepsilon^{-1}.$$

Repeating this calculation with

$$f = \varepsilon^2\mu^2+(1+\varepsilon)\varepsilon^2k^2+(\beta-\tfrac{1}{3})q^4+c_0q^6, \quad g = \varepsilon^2\mu^2+(1+\varepsilon)\varepsilon^2k^2+(\beta-\tfrac{1}{3}(1+\varepsilon))q^4+c_0q^6,$$

one obtains the estimate

$$(\mu^2+k^2)^{\frac{1}{2}}\left|\left\{\begin{array}{c}\partial_\mu\\ \partial_k\end{array}\right\}(\chi\tilde{G}_3)\right| \le c\varepsilon^{-1}$$

for $\tilde{G}_3$.

A similar analysis yields

$$(\mu^2+k^2)\left|\left\{\begin{array}{c}\partial_\mu^2\\ \partial_k^2\\ \partial_\mu\partial_k\end{array}\right\}(\chi\tilde{G}_i)\right| \le c\varepsilon^{-1}, \qquad i=1,2,3,$$

and the required estimates for G_1 are obtained from equation (116). □

Lemma 4.9 *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\tilde{\mathcal{G}}_{\rm a}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\tilde{\mathcal{G}}_{\rm a}(u)\|_p \le c\varepsilon\|u\|_p.$$

Proof. Here we use the formula

$$\tilde{\mathcal{G}}_{\mathrm{a}}(u) = \varepsilon^{-1}\mathcal{F}^{-1}\left[\int_0^1 q^2\chi G_1\mathcal{F}[u]\,\mathrm{d}\xi\right]$$

and show that the hypotheses of Lemma 4.1, namely that

$$\begin{aligned} |q^2\chi G_1| &\leq c\varepsilon^2, && (121)\\ \int_{I_j} |\partial_\mu(q^2\chi G_1)|\,\mathrm{d}\mu &\leq c\varepsilon^2, && (122)\\ \int_{I_j} |\partial_k(q^2\chi G_1)|\,\mathrm{d}k &\leq c\varepsilon^2, && (123)\\ \int_{I_{j_1}}\int_{I_{j_2}} |\partial_\mu\partial_k(q^2\chi G_1)|\,\mathrm{d}\mu\,\mathrm{d}k &\leq c\varepsilon^2 && (124) \end{aligned}$$

uniformly over all dyadic intervals I_j, I_{j_1}, I_{j_2}, hold uniformly for $y, \xi \in [0,1]$, so that the result follows by Theorem 4.4. To this end we again use the decomposition (116), and recall the estimates

$$|q^2\tilde{G}_i| \leq c, \qquad i = 1,2,3$$

for $q \leq q_0$ established in the proof of Lemma 4.8, from which (121) is an immediate consequence.

Using the fact that $|\partial_q(q^2\tilde{G})| \leq cq^3$ for $q \leq q_0$ together with estimates (117), (118), we find that

$$\begin{aligned} |\partial_\mu(q^2\tilde{G}_1)| &\leq c\left(\frac{q^3}{\varepsilon^2 Q}|\partial_\mu q| + (\varepsilon q + q^3)\frac{q^4}{\varepsilon^4 Q^2}|\partial_\mu q|\right),\\ &= c\left(\frac{\varepsilon q^2|\mu|}{\varepsilon^2 Q} + \frac{\varepsilon^2 q^4|\mu|}{\varepsilon^4 Q^2} + \frac{\varepsilon q^6|\mu|}{\varepsilon^4 Q^2}\right)\\ &\leq c\left(\frac{\varepsilon^2|\mu|^3}{\varepsilon^2|\mu|^4} + \frac{\varepsilon^4|\mu|^5}{\varepsilon^4|\mu|^6} + \frac{\varepsilon^4|\mu|^7}{\varepsilon^4|\mu|^8}\right)\\ &\leq \frac{c}{|\mu|},\\ |\partial_k(q^2\tilde{G}_1)| &\leq c\left(\frac{q^3}{\varepsilon^2 Q}|\partial_k q| + (\varepsilon q + q^3)\frac{q^4}{\varepsilon^4 Q^2}|\partial_k q| + \frac{\varepsilon^2|k|q^4}{\varepsilon^4 Q^2}\right),\\ &= c\left(\frac{\varepsilon^2 q^2|k|}{\varepsilon^2 Q} + \frac{\varepsilon^3 q^4|k|}{\varepsilon^4 Q^2} + \frac{\varepsilon^2 q^6|k|}{\varepsilon^4 Q^2} + \frac{\varepsilon^2|k|q^4}{\varepsilon^4 Q^2}\right)\\ &\leq c\left(\frac{\varepsilon^4|k|^3}{\varepsilon^2|k|^4} + \frac{\varepsilon^7|k|^5}{\varepsilon^6|k|^6} + \frac{\varepsilon^8|k|^7}{\varepsilon^8|k|^8} + \frac{\varepsilon^6|k|^5}{\varepsilon^6|k|^6}\right)\\ &\leq \frac{c}{|k|} \end{aligned}$$

for $q \leq q_0$ (because $\partial_\mu q = \varepsilon\mu/q$ and $\partial_k q = \varepsilon^2 k/q$). It follows that

$$\int_{2^j}^{2^{j+1}} |\partial_\mu(q^2\chi\tilde{G}_1)|\,\mathrm{d}\mu$$

$$
\begin{aligned}
&\le \int_{2^j}^{2^{j+1}} (|\partial_\mu(q^2\tilde{G}_1)| + \varepsilon^{\frac{1}{2}}|\chi'(q)||q^2\tilde{G}_1|)\,\mathrm{d}\mu \\
&\le \int_{2^j}^{2^{j+1}} \frac{1}{\mu}\,\mathrm{d}\mu + c\varepsilon^{\frac{1}{2}}\int_0^{q_0\varepsilon^{-\frac{1}{2}}}\mathrm{d}\mu \\
&\le \log 2 + c \\
&\le c
\end{aligned}
$$

(because χ is identically zero for $q \ge q_0$, and in particular for $\mu \ge q_0\varepsilon^{-1/2}$) and similarly

$$
\int_{2^j}^{2^{j+1}} |\partial_k(q^2\chi\tilde{G}_1)|\,\mathrm{d}k \le c
$$

for $j \in \mathbb{N}_0$; the same estimates clearly hold for the dyadic intervals $(2^{-j-1}, 2^{-j})$, $j \in \mathbb{N}_0$ and those in the negative half-line.

Using the estimates (119), (120), we similarly find that

$$
\begin{aligned}
|\partial_\mu(q^2\tilde{G}_2)| &\le \left(\frac{q^9}{\varepsilon^4 Q^2}|\partial_\mu q| + (\varepsilon^2|\mu| + q^3|\partial_\mu q|)\frac{q^{10}}{\varepsilon^8 Q^4}(\varepsilon^2\mu^2 + \varepsilon^2 k^2 + q^4)\right) \le \frac{c}{|\mu|}, \\
|\partial_k(q^2\tilde{G}_2)| &\le \left(\frac{q^9}{\varepsilon^4 Q^2}|\partial_k q| + (\varepsilon^2|k| + q^3|\partial_k q|)\frac{q^{10}}{\varepsilon^8 Q^4}(\varepsilon^2\mu^2 + \varepsilon^2 k^2 + q^4)\right) \le \frac{c}{|k|}
\end{aligned}
$$

for $q \le q_0$, whence

$$
\int_{I_j} |\partial_\mu(q^2\chi\tilde{G}_2)|\,\mathrm{d}\mu \le c, \qquad \int_{I_j} |\partial_k(q^2\chi\tilde{G}_2)|\,\mathrm{d}k \le c
$$

for every dyadic interval I_j, and the same method yields the corresponding estimates for $\tilde{G}_3$. An analogous argument shows that

$$
\int_{I_{j_1}}\int_{I_{j_2}} |\partial_\mu\partial_k(q^2\chi\tilde{G}_i)|\,\mathrm{d}\mu\,\mathrm{d}k \le c\varepsilon^2, \qquad i = 1, 2, 3
$$

for every pair (I_{j_1}, I_{j_2}) of dyadic intervals, and the estimates (122)–(124) for G_1 follow from equation (116). $\square$

To obtain the estimates for $\mathcal{G}_\mathrm{b}$ and $\tilde{\mathcal{G}}_\mathrm{b}$ we write

$$
G_1 = \varepsilon^2 G + \frac{1+\varepsilon}{Q} \tag{125}
$$

and introduce the further decompositions $\mathcal{G}_\mathrm{b} = \mathcal{G}_{\mathrm{b},1} + \mathcal{G}_{\mathrm{b},2}$, $\tilde{\mathcal{G}}_\mathrm{b} = \tilde{\mathcal{G}}_{\mathrm{b},1} + \tilde{\mathcal{G}}_{\mathrm{b},2}$, where

$$
\mathcal{G}_{\mathrm{b},1}(u) = \varepsilon^2\mathcal{F}^{-1}\left[\int_0^1 (1-\chi(q))G_1\mathcal{F}[u]\,\mathrm{d}\xi\right], \qquad \mathcal{G}_{\mathrm{b},2}(u) = \mathcal{F}^{-1}\left[\int_0^1 \frac{1+\varepsilon}{Q}(1-\chi(q))\mathcal{F}[u]\,\mathrm{d}\xi\right]
$$

and $\tilde{\mathcal{G}}_{\mathrm{b},1}$, $\tilde{\mathcal{G}}_{\mathrm{b},2}$ are defined in the same way. We establish the required estimates for each of these operators separately, treating $\mathcal{G}_{\mathrm{b},1}$, $\tilde{\mathcal{G}}_{\mathrm{b},1}$ by singular-integral techniques together with Theorem 4.5 and $\mathcal{G}_{\mathrm{b},2}$, $\tilde{\mathcal{G}}_{\mathrm{b},2}$ by the scaled version of Mikhlin's theorem together with Theorem 4.4. In order to apply Theorem 4.5 to $\mathcal{G}_{\mathrm{b},2}$ and $\tilde{\mathcal{G}}_{\mathrm{b},2}$ it is necessary to verify hypothesis (111) on their Fourier transforms and hypothesis (112) on their kernels. The first of these tasks is undertaken in the following proposition.

Proposition 4.10 *The estimates*

$$\|\mathcal{F}[\mathcal{G}_{\mathrm{b},1}](w)\|_{L^p(0,1)} \le c\varepsilon^2\|w\|_{L^p(0,1)}, \qquad \|\mathcal{F}[\tilde{\mathcal{G}}_{\mathrm{b},1}](w)\|_{L^p(0,1)} \le c\varepsilon\|w\|_{L^p(0,1)}$$

hold for each $w \in L^p(0,1)$.

Proof. A straightforward argument using the differential calculus shows that

$$q^2 - (1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2 \le -c_{q^\star}q^3 \tag{126}$$

for $q \ge q_\star$, where $q_\star$ is any positive real number and $c_{q^\star}$ is a positive constant which depends only upon $q_\star$. It follows from (126), the inequality

$$\frac{\cosh qy}{\cosh q}\left\{\begin{array}{c}\cosh q(1-\xi)\\ \sinh q(\xi-1)\end{array}\right\} \le c\mathrm{e}^{-q(\xi-y)}, \qquad y \le \xi$$

and the corresponding inequality for $\xi \le y$ obtained by interchanging y and ξ that

$$|G| \;\le\; \frac{c}{q^3}(1+q^2)\mathrm{e}^{-q|\xi-y|} \;\le\; \frac{c_{q^\star}}{q}\mathrm{e}^{-q|\xi-y|}, \qquad q \ge q_\star.$$

Using this estimate with $q = q_0$, we find that

$$\begin{aligned}
\|\mathcal{F}[\mathcal{G}_{\mathrm{b},1}](w)\|_{L^p(0,1)}^p &= \int_0^1 \left|\int_0^1 \varepsilon^2 G(1-\chi)w\,\mathrm{d}\xi\right|^p \mathrm{d}y\\
&\le c\int_0^1 \left|\int_0^1 \frac{\varepsilon^2}{q}(1-\chi)\mathrm{e}^{-q|y-\xi|}|w|\,\mathrm{d}\xi\right|^p \mathrm{d}y\\
&\le c\left(\frac{\varepsilon^2}{q}\right)^p(1-\chi)^p\int_0^1\left[\int_0^1 \mathrm{e}^{-q|y-\xi|}|w|\,\mathrm{d}\xi\right]^p\mathrm{d}y\\
&\le c\left(\frac{\varepsilon^2}{q}\right)^p(1-\chi)^p\left(\frac{1}{q}\right)^p\int_0^1 |w|^p\,\mathrm{d}\xi\\
&\le c\varepsilon^{2p}\|w\|_p^p,
\end{aligned}$$

and the estimate for $\tilde{\mathcal{G}}_{\mathrm{b},1}$ is obtained by the same method. □

Lemma 4.11 *Choose* $p \in (1,\infty)$. *For each* $u \in L^p(\Sigma)$ *the function* $\mathcal{G}_{\mathrm{b},1}(u)$ *belongs to* $L^p(\Sigma)$ *and satisfies the estimate*

$$\|\mathcal{G}_{\mathrm{b},1}(u)\|_p \le c\varepsilon^2\|u\|_p.$$

Proof. Observe that

$$\begin{aligned}
&\frac{\cosh qy}{\cosh q}\left\{\begin{array}{c}\cosh q(1-\xi)\\ \sinh q(\xi-1)\end{array}\right\}\\
&\quad= \frac{(\mathrm{e}^{qy}+q^{-qy})(\pm e^{q(1-\xi)}+e^{-q(1-\xi)})}{2(\mathrm{e}^{q}+e^{-q})}\\
&\quad= \pm\frac{e^{-q(\xi-y)}}{2(1+\mathrm{e}^{-2q})}+\frac{e^{-q(\xi+y)}}{2(1+\mathrm{e}^{-2q})}\pm\frac{e^{-q(2-\xi-y)}}{2(1+\mathrm{e}^{-2q})}+\frac{e^{-q(1-(\xi-y))}}{2(e^{q}+\mathrm{e}^{-q})},
\end{aligned}$$

and using this formula and the corresponding formula obtained by interchanging y and ξ, one finds that

$$\begin{aligned}G = {} & \frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\\ & + \frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(\xi+y)}\\ & + \frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(2-\xi-y)}\\ & + \frac{1+\varepsilon+\beta q^2-\varepsilon\mu^2/q}{2(\mathrm{e}^{q}+\mathrm{e}^{-q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(1-|\xi-y|)}.\end{aligned}$$

We now consider the first of these terms in detail; the others are handled in an analogous fashion.

Define

$$\begin{aligned}I &= \varepsilon^2\mathcal{F}^{-1}\left[\frac{(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\right]\\ &= \frac{\varepsilon^2}{2\pi}\int_{\mathbb{R}^2}\frac{(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\mathrm{e}^{-\mathrm{i}\mu x}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}\mu\,\mathrm{d}k.\end{aligned}$$

Introducing polar coordinates (q,θ) and (r,ϕ) defined by

$$\varepsilon^{\frac{1}{2}}\mu = q\cos\theta,\ \varepsilon k = q\sin\theta, \qquad x=\varepsilon^{\frac{1}{2}}r\cos\phi,\ z=\varepsilon r\sin\phi, \tag{127}$$

we find that

$$\begin{aligned}I &= \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_0^{2\pi}\int_0^\infty\frac{q(1+\varepsilon+\beta q^2-q\cos^2\theta)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos(\theta-\phi))}\,\mathrm{d}q\,\mathrm{d}\theta\\ &= \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_0^{2\pi}\int_0^\infty\frac{q(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\end{aligned}$$

where $\psi=\theta-\phi$ and $\tilde{Q}=q^2\cos^2(\phi+\psi)-(1+\varepsilon+\beta q^2)q\tanh q$. Our strategy is to show that

$$|\partial_x I|\leq\frac{c}{r^3},\qquad \varepsilon^{\frac{1}{2}}|\partial_z I|\leq\frac{c}{r^3}$$

uniformly over $\{y,\xi\in[0,1]:y\neq\xi\}$; because

$$\partial_x=\varepsilon^{-\frac{1}{2}}\cos\phi\,\partial_r-\frac{\varepsilon^{-\frac{1}{2}}}{r}\sin\phi\,\partial_\phi,\qquad \varepsilon^{\frac{1}{2}}\partial_z=\varepsilon^{-\frac{1}{2}}\sin\phi\,\partial_r-\frac{\varepsilon^{-\frac{1}{2}}}{r}\cos\phi\,\partial_\phi$$

it suffices to show that

$$|\partial_r I|\leq\frac{c\varepsilon^{\frac{1}{2}}}{r^3},\qquad |\partial_\phi I|\leq\frac{c\varepsilon^{\frac{1}{2}}}{r^2}.$$

(Here, and in the remainder of this proof, all estimates hold uniformly over $\{y,\xi\in[0,1]:y\neq\xi\}$.) Let us write $I=I_1+I_2+I_3$, where I_2, I_3 are obtained from I by replacing the range of integration for ψ by respectively $(\pi/2-\hat{\varepsilon},\pi/2+\hat{\varepsilon})$, $(3\pi/2-\hat{\varepsilon},3\pi/2+\hat{\varepsilon})$ and $\hat{\varepsilon}$ is a small positive constant, and consider each integral separately.

Notice that

$$\begin{aligned}\partial_r I_1 &= \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_J\int_0^\infty \frac{\mathrm{i}q^2\cos\psi(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi\\ &= \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_J \frac{-\mathrm{i}\cos\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3}\\ &\qquad\times\int_0^\infty \partial_q^3\left(\frac{q^2(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\right)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\end{aligned}$$

in which $J=[0,2\pi]\setminus([\pi/2-\hat{\varepsilon},\pi/2+\hat{\varepsilon}]\cup[3\pi/2-\hat{\varepsilon},3\pi/2+\hat{\varepsilon}])$ and the second line is obtained by three integrations by parts with respect to q (the requirement that $y\neq\xi$ is used at this step). Because

$$\frac{1}{\tilde{Q}}=\mathcal{O}(q^{-3}),\qquad \partial_q^i\tilde{Q}=\mathcal{O}(q^{3-i}),\quad i=0,1,2,\ldots$$

as $q\to\infty$, the third derivative of the quantity in large parentheses in the above expression is $\mathcal{O}(q^{-2})$ as $q\to\infty$; it also vanishes near $q=0$ and is therefore integrable. It follows from these observations that

$$|\partial_r I_1|\leq\frac{c\varepsilon^{\frac{1}{2}}}{r^3}\int_J\int_0^\infty\left|\partial_q^3\left(\frac{q^2(1+\varepsilon+\beta q^2-q\cos^2(\phi+\psi))(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\right)\right|\mathrm{d}q\,\mathrm{d}\theta\leq\frac{c\varepsilon^{\frac{1}{2}}}{r^3}.$$

The integral I_2 is dealt with using the substitution $\omega=\cos\psi$, so that

$$\begin{aligned}\partial_r I_2 = {}&\frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty\frac{\mathrm{i}q^2\omega(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega\\ &-\frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty\frac{\mathrm{i}q^3\omega(\omega\cos\phi-\sqrt{1-\omega^2}\sin\phi)^2(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega,\quad (128)\end{aligned}$$

where $\tilde{\varepsilon}=\sin\hat{\varepsilon}$ and

$$\tilde{Q}=q^2(\omega\cos\phi-\sqrt{1-\omega^2}\sin\phi)^2-(1+\varepsilon+\beta q^2)q\tanh q.$$

Examining the first integral on the right-hand side of equation (128), note that

$$\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty\frac{q^2\omega(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega=I_2^1+I_2^2,$$

where

$$\begin{aligned}I_2^1 &= \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty\frac{q^2\omega(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega,\\ I_2^2 &= \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\mathcal{O}(\omega^3)\int_0^\infty\frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega\\ &= -\mathrm{i}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\frac{\mathcal{O}(\omega^3)}{(|\xi-y|+\mathrm{i}r\omega)^3}\int_0^\infty\partial_q^3\left(\frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\right)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega\\ &= \mathcal{O}(r^{-3}).\end{aligned}$$

Using the formulae

$$\int \omega \mathrm{e}^{-\mathrm{i}qr\omega}\,\mathrm{d}\omega = -\frac{\omega}{\mathrm{i}qr}\mathrm{e}^{-\mathrm{i}qr\omega} + \frac{1}{q^2r^2}\mathrm{e}^{-\mathrm{i}qr\omega}$$

and

$$\partial_\omega\left(\frac{1}{\tilde{Q}}\right) = \frac{q^2}{\tilde{Q}^2}g(\omega),$$

where

$$g(\omega) = -2(\omega\cos\phi - \sqrt{1-\omega^2}\sin\phi)\left(\cos\phi + \frac{\omega}{\sqrt{1-\omega^2}}\sin\phi\right) = \sin 2\phi + \mathcal{O}(\omega),$$

we can integrate by parts with respect to ω to find that

$$\begin{aligned}
I_2^1 &= \int_0^\infty \left[\left(-\frac{\omega}{\mathrm{i}qr} + \frac{1}{q^2r^2}\right)\frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\right]_{\omega=\tilde{\varepsilon}}^{\omega=\tilde{\varepsilon}}\mathrm{d}q \\
&\quad - \int_0^\infty\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\left(-\frac{\omega}{\mathrm{i}qr} + \frac{1}{q^2r^2}\right)\frac{q^4(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}g(\omega)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}\omega\,\mathrm{d}q \\
&= \int_0^\infty \left[\left(-\frac{\omega}{\mathrm{i}qr} + \frac{1}{q^2r^2}\right)\frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\right]_{\omega=\tilde{\varepsilon}}^{\omega=\tilde{\varepsilon}}\mathrm{d}q \\
&\quad + \int_0^\infty\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\frac{\omega}{\mathrm{i}r}\sin 2\phi\,\frac{q^3(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}\omega\,\mathrm{d}q \\
&\quad - \int_0^\infty\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\frac{1}{r^2}\sin 2\phi\,\frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}\omega\,\mathrm{d}q \\
&\quad + \int_0^\infty\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\frac{\mathcal{O}(\omega^2)}{\mathrm{i}r}\frac{q^3(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}\omega\,\mathrm{d}q \\
&\quad - \int_0^\infty\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\frac{\mathcal{O}(\omega)}{r^2}\frac{q^2(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}\omega\,\mathrm{d}q.
\end{aligned}$$

Integrations by parts with respect to q show that the first, fourth and fifth terms on the right-hand side of this expression are $\mathcal{O}(r^{-3})$; integrating the second and third terms on its right-hand side by parts with respect to ω and repeating the above calculation shows that they are also $\mathcal{O}(r^{-3})$.

Turning to the second integral on the right-hand side of equation (128), note that

$$\begin{aligned}
&\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty \frac{\omega(\omega\cos\phi - \sqrt{1-\omega^2}\sin\phi)^2 q^3(1-\chi(q))}{2\tilde{Q}\sqrt{1-\omega^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\quad = \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty \frac{\omega q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\qquad - 2\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty \omega^2\sin 2\phi\,\frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\qquad + \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}}\int_0^\infty \mathcal{O}(\omega^3)\frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega.
\end{aligned}$$

The methods used above show that the first and third terms on the right-hand side of this expression are $\mathcal{O}(r^{-3})$. To discuss the second term, we use the formula

$$\int \omega^2 \mathrm{e}^{-\mathrm{i}qr\omega}\,\mathrm{d}\omega = -\frac{\omega^2}{\mathrm{i}qr} + \frac{2\omega}{q^2r^2} + \frac{2}{\mathrm{i}q^3r^3}$$

and integrate by parts with respect to ω; the result is

$$\begin{aligned}
&\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \omega^2 \frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\quad = \int_0^\infty \left[\left(-\frac{\omega^2}{\mathrm{i}qr} + \frac{2\omega}{q^2r^2} + \frac{2}{\mathrm{i}q^3r^3}\right)\frac{q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\right]_{\omega=-\tilde{\varepsilon}}^{\omega=\tilde{\varepsilon}} \mathrm{d}q \\
&\qquad - \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \left(-\frac{\omega^2}{\mathrm{i}qr} + \frac{2\omega}{q^2r^2} + \frac{2}{\mathrm{i}q^3r^3}\right)\frac{q^5(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega,
\end{aligned}$$

and integrations by parts with respect to q show that each term on the right-hand side of this expression is $\mathcal{O}(r^{-3})$. Altogether we have that

$$|\partial_r I_2| \leq \frac{c\varepsilon^{\frac{1}{2}}}{r^3},$$

and the inequality

$$|\partial_r I_3| \leq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}$$

is obtained by the same argument.

Direct calculations yield the formulae

$$\begin{aligned}
&\partial_\phi I_1 = \\
&\quad -\frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_J \int_0^\infty \frac{q^3(1+\varepsilon+\beta q^2 - q\cos^2(\phi+\psi))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}\sin 2(\phi+\psi)(1-\chi(q))\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi \\
&\quad + \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_J \int_0^\infty \frac{q(1+\varepsilon+\beta q^2 - 2q\sin 2(\phi+\psi))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}(1-\chi(q))\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,
\end{aligned}$$

$$\begin{aligned}
&\partial_\phi I_2 = \\
&\quad -\frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{q^3(1+\varepsilon+\beta q^2)(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})} g(\omega)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\quad + \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{q^4(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}(\omega\cos\phi - \sqrt{1-\omega^2}\sin\phi)^2 g(\omega)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\quad + \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{q(1+\varepsilon+\beta q^2)(1-\chi(q))}{2\tilde{Q}\sqrt{1-w^2}(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega \\
&\quad + \frac{\varepsilon^{\frac{1}{2}}}{2\pi}\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \int_0^\infty \frac{q^2(1-\chi(q))}{\tilde{Q}(1+\mathrm{e}^{-2q})} g(\omega)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\omega)}\,\mathrm{d}q\,\mathrm{d}\omega
\end{aligned}$$

together with a similar expression for $\partial_\phi I_3$, and the methods described above show that each of these integrals is $\mathcal{O}(r^{-2})$.

We have therefore proved that

$$\left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}}\partial_z \end{array} \right\} \mathcal{F}^{-1}[\varepsilon^2 G(1-\chi)] \right| \le \frac{c}{r^3} = \frac{c\varepsilon^{\frac{3}{2}}}{(x^2+\varepsilon^{-1}z^2)^{\frac{3}{2}}}, \qquad y \neq \xi,$$

from which it follows that

$$\left\| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}}\partial_z \end{array} \right\} \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 G(1-\chi) w \, \mathrm{d}\xi \right] \right\|_{L^p(0,1)} \le \frac{c\varepsilon^{\frac{3}{2}}}{(x^2+\varepsilon^{-1}z^2)^{\frac{3}{2}}} \|w\|_{L^p(0,1)}$$

for each $w \in L^p(0,1)$. This result, together with Proposition 4.10, shows that the hypotheses of Theorem 4.5 are met, and we conclude that

$$\left\| \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 G(1-\chi) u \, \mathrm{d}\xi \right] \right\|_p \le c\varepsilon^2 \|u\|_p. \qquad \Box$$

Lemma 4.12 *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\tilde{\mathcal{G}}_{\mathrm{b},1}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\tilde{\mathcal{G}}_{\mathrm{b},1}(u)\|_p \le c\varepsilon \|u\|_p.$$

Proof. We again use the formula

$$\begin{aligned} q^2 G = {} & \frac{q^2(1+\varepsilon+\beta q^2) - \varepsilon\mu^2 q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi-y|} \\ & + \frac{q^2(1+\varepsilon+\beta q^2) - \varepsilon\mu^2 q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q(\xi+y)} \\ & + \frac{q^2(1+\varepsilon+\beta q^2) - \varepsilon\mu^2 q}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q(2-\xi-y)} \\ & + \frac{q^2(1+\varepsilon+\beta q^2) - \varepsilon\mu^2 q}{2(\mathrm{e}^{q}+\mathrm{e}^{-q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q(1-|\xi-y|)} \end{aligned}$$

and consider the first of these terms in detail; the others are handled in an analogous fashion.

Define

$$\begin{aligned} I_1 &= \varepsilon^2 \mathcal{F}^{-1}\left[\frac{(1+\varepsilon)q^2(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi-y|} \right], \\ I_2 &= \varepsilon^2 \mathcal{F}^{-1}\left[\frac{\beta q^4(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi-y|} \right], \\ I_3 &= \varepsilon^2 \mathcal{F}^{-1}\left[\frac{\varepsilon\mu^2 q(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi-y|} \right]. \end{aligned}$$

The method employed in Lemma 4.12 shows that

$$\left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}}\partial_z \end{array} \right\} I_1 \right| \le \frac{c\varepsilon^{\frac{3}{2}}}{(x^2+\varepsilon^{-1}z^2)^{\frac{3}{2}}}, \qquad y \neq \xi,$$

from which it follows that

$$\sup_{\xi\in[0,1]}\int_0^1\left|\left\{\begin{array}{c}\partial_x\\ \varepsilon^{\frac12}\partial_z\end{array}\right\}I_1\right|\mathrm{d}y\le\frac{c\varepsilon^{\frac32}}{(x^2+\varepsilon^{-1}z^2)^{\frac32}},\tag{129}$$

$$\sup_{y\in[0,1]}\int_0^1\left|\left\{\begin{array}{c}\partial_x\\ \varepsilon^{\frac12}\partial_z\end{array}\right\}I_1\right|\mathrm{d}\xi\le\frac{c\varepsilon^{\frac32}}{(x^2+\varepsilon^{-1}z^2)^{\frac32}},\tag{130}$$

and out strategy is to show that (129), (130) also hold for I_2 and I_3.

Using the polar coordinates (127), observe that

$$\begin{aligned} I_2 &= (\varepsilon\partial_x^2+\varepsilon^2\partial_z^2)\tilde I_2\\ &= \left(\partial_r^2+\frac1r\partial_r+\frac1{r^2}\partial_\phi^2\right)\tilde I_2, \end{aligned}$$

where

$$\begin{aligned} \tilde I_2 &= \varepsilon^2\mathcal{F}^{-1}\left[\frac{\beta q^2(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\right]\\ &= \frac{\varepsilon^{\frac12}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty\frac{q^3(1-\chi(q))}{2\tilde Q(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi \end{aligned}$$

and $\psi=\theta-\phi$, $\tilde Q=q^2\cos^2(\phi+\psi)-(1+\varepsilon+\beta q^2)q\tanh q$. To show that the estimates (129), (130) also hold for I_2 it therefore suffices to examine the quantities

$$\partial_r^3\tilde I_2,\quad \frac1{r^2}\partial_r\tilde I_2,\quad \frac1r\partial_r^2\tilde I_2,\quad \frac1{r^3}\partial_\phi^2\tilde I_2,\quad \frac1{r^2}\partial_r\partial_\phi^2\tilde I_2,\quad \frac1r\partial_r^2\partial_\phi\tilde I_2,\quad \frac1{r^2}\partial_r\partial_\phi\tilde I_2,\quad \frac1{r^3}\partial_\phi^3\tilde I_2.$$

In order to deal with the integral

$$\begin{aligned} \partial_r^3\tilde I_2 &= \frac{\varepsilon^{\frac12}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty\frac{-\mathrm{i}q^6(1-\chi(q))\cos^3\psi}{2\tilde Q(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi\\ &= \frac{\varepsilon^{\frac12}\beta}{2\pi}\int_0^{2\pi}\frac{\mathrm{i}\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3}\int_0^\infty\partial_q^3\left(\frac{q^6(1-\chi(q))\cos^3\psi}{2\tilde Q(1+\mathrm{e}^{-2q})}\right)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi, \end{aligned}$$

where we have supposed that $y\neq\xi$ in the integration by parts, let us write

$$\partial_q^3\left(\frac{q^6(1-\chi(q))}{2\tilde Q(1+\mathrm{e}^{-2q})}\right)=\ell+\mathcal{R}(q),$$

where

$$\ell=\lim_{q\to\infty}\partial_q^3\left(\frac{q^6(1-\chi(q))}{2\tilde Q(1+\mathrm{e}^{-2q})}\right)$$

and $\mathcal{R}(q)=\mathcal{O}(q^{-2})$ as $q\to\infty$, so that

$$\partial_r^3\tilde I_2=\tilde I_2^1+\tilde I_2^2,\qquad y\neq\xi,$$

in which

$$\begin{aligned}\tilde{I}_2^1 &= \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\frac{\mathrm{i}\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3}\int_0^\infty \ell\,\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\\ \tilde{I}_2^2 &= \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\frac{\mathrm{i}\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3}\int_0^\infty \mathcal{R}(q)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi.\end{aligned}$$

It follows from the fact that $\mathcal{R}(q)$ is integrable that $|\tilde{I}_2^2|\leq c\varepsilon^{\frac{1}{2}}/r^3$ and hence that

$$\sup_{\xi\in[0,1]}\int_0^1|\tilde{I}_2^2|\,\mathrm{d}y\leq\frac{c\varepsilon^{\frac{1}{2}}}{r^3}.$$

Furthermore, one has that

$$\begin{aligned}\sup_{\xi\in[0,1]}\int_0^1|\tilde{I}_2^1|\,\mathrm{d}y &= \varepsilon^{\frac{1}{2}}\beta\sup_{\xi\in[0,1]}\int_0^1\left|\int_0^{2\pi}\frac{\cos^3\psi}{(|\xi-y|+\mathrm{i}r\cos\psi)^3}\int_0^\infty\ell\,\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi\right|\mathrm{d}y\\ &= \varepsilon^{\frac{1}{2}}\beta\sup_{\xi\in[0,1]}\int_0^1\left|\int_0^{2\pi}\frac{-\ell\cos^3\psi}{(|\xi-y|-\mathrm{i}r\cos\psi)^4}\,\mathrm{d}\psi\right|\mathrm{d}y\\ &= \varepsilon^{\frac{1}{2}}\beta\sup_{\xi\in[0,1]}\int_0^1\left|\int_0^{2\pi}-\ell\cos^3\psi\frac{(|\xi-y|-\mathrm{i}r\cos\psi)^4}{(|\xi-y|^2+r^2\cos^2\psi)^4}\,\mathrm{d}\psi\right|\mathrm{d}y\\ &\leq c\varepsilon^{\frac{1}{2}}\sup_{\xi\in[0,1]}\int_0^1\int_0^{2\pi}|\cos\psi|^3\frac{|\xi-y|^4+r^4\cos^4\psi}{(|\xi-y|^2+r^2\cos^2\psi)^4}\,\mathrm{d}\psi\,\mathrm{d}y\\ &\leq c\varepsilon^{\frac{1}{2}}\int_0^1\int_0^{2\pi}|\cos\psi|^3\frac{|w|^4+r^4\cos^4\psi}{(|w|^2+r^2\cos^2\psi)^4}\,\mathrm{d}\psi\,\mathrm{d}w\\ &= c\varepsilon^{\frac{1}{2}}\int_0^{2\pi}\int_0^{\frac{1}{r|\cos\psi|}}\frac{t^4+1}{r^3(t^2+1)^4}\,\mathrm{d}t\,\mathrm{d}\psi\\ &\leq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}\int_0^\infty\frac{t^4+1}{(t^2+1)^4}\,\mathrm{d}t\\ &\leq \frac{c\varepsilon^{\frac{1}{2}}}{r^3}.\end{aligned}$$

A similar argument shows that

$$\sup_{\xi\in[0,1]}\int_0^1|\partial_r\tilde{I}_2|\,\mathrm{d}y\leq\frac{c\varepsilon^{\frac{1}{2}}}{r},\qquad \sup_{\xi\in[0,1]}\int_0^1|\partial_r^2\tilde{I}_2|\,\mathrm{d}y\leq\frac{c\varepsilon^{\frac{1}{2}}}{r^2}.$$

Direct calculations yield the formulae

$$\begin{aligned}\partial_\phi^2\tilde{I}_2 &= \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty\left(\frac{2q^7h^2}{\tilde{Q}^3}-\frac{q^5h_\phi}{\tilde{Q}^2}\right)\frac{1-\phi(q)}{2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\\ \partial_\phi^3\tilde{I}_2 &= \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty\left(\frac{-6q^9h^3}{\tilde{Q}^4}+\frac{6q^7hh_\phi}{\tilde{Q}^3}-\frac{q^5h_{\phi\phi}}{\tilde{Q}^2}\right)\frac{(1-\chi(q))}{2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,\end{aligned}$$

where $h(\phi,\psi) = \sin 2(\phi+\psi)$. Observe that

$$\begin{aligned}
\sup_{\xi\in[0,1]} \int_0^1 \left| \int_0^{2\pi}\int_0^\infty \frac{q^5(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})} \left\{ \begin{array}{c} h_\phi \\ h_{\phi\phi} \end{array} \right\} \mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi \right| \mathrm{d}y \\
\leq\ & c\int_0^1\int_0^\infty \left| \frac{q^5(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right| \mathrm{e}^{-qw}\,\mathrm{d}q\,\mathrm{d}w \\
\leq\ & c\int_0^\infty \frac{1}{q}\left| \frac{q^5(1-\chi(q))}{\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right| \mathrm{d}q, \\
\leq\ & c,
\end{aligned}$$

and since

$$\frac{q^7(1-\chi(q))}{2\tilde{Q}^3(1+\mathrm{e}^{-2q})}, \qquad \frac{q^9(1-\chi(q))}{2\tilde{Q}^4(1+\mathrm{e}^{-2q})}$$

are integrable this result also holds for the remaining terms in the formulae for $\partial_\phi^2\tilde{I}_2$, $\partial_\phi^3\tilde{I}_3$, which therefore satisfy

$$\sup_{\xi\in[0,1]}\int_0^1 |\partial_\phi^2\tilde{I}_2|\,\mathrm{d}y \leq c\varepsilon^{\frac{1}{2}}, \qquad \sup_{\xi\in[0,1]}\int_0^1 |\partial_\phi^3\tilde{I}_2|\,\mathrm{d}y \leq c\varepsilon^{\frac{1}{2}}.$$

We find from the calculation

$$\begin{aligned}
\partial_r\partial_\phi\tilde{I}_2 &= \frac{\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{-\mathrm{i}q^6\cos\psi\, h(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi \\
&= -\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}\beta}{2\pi}\int_0^{2\pi} \frac{\cos\psi\, h}{|\xi-y|+\mathrm{i}r\cos\psi}\int_0^\infty \partial_q\left(\frac{q^6(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right)\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi,
\end{aligned}$$

which is valid for $y\neq\xi$, that

$$\begin{aligned}
\sup_{\xi\in[0,1]}\int_0^1 |\partial_r\partial_\phi\tilde{I}_2|\,\mathrm{d}y &\leq \frac{c\varepsilon^{\frac{1}{2}}}{r}\int_0^1\int_0^\infty \left|\partial_q\left(\frac{q^6(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right)\right|\mathrm{e}^{-qw}\,\mathrm{d}q\,\mathrm{d}w \\
&\leq \frac{c\varepsilon^{\frac{1}{2}}}{r}\int_0^\infty \frac{1}{q}\left|\partial_q\left(\frac{q^6(1-\chi(q))}{2\tilde{Q}^2(1+\mathrm{e}^{-2q})}\right)\right|\mathrm{e}^{-qw}\,\mathrm{d}q \\
&\leq \frac{c\varepsilon^{\frac{1}{2}}}{r},
\end{aligned}$$

and similar arguments show that

$$\sup_{\xi\in[0,1]}\int_0^1 |\partial_r\partial_\phi^2\tilde{I}_2|\,\mathrm{d}y \leq \frac{c\varepsilon^{\frac{1}{2}}}{r}, \qquad \sup_{\xi\in[0,1]}\int_0^1 |\partial_r^2\partial_\phi\tilde{I}_2|\,\mathrm{d}y \leq \frac{c\varepsilon^{\frac{1}{2}}}{r^2}.$$

We have therefore demonstrated that I_2 satisfies (129), and a similar technique shows that the same is true of I_3. Furthermore, we may clearly interchange the roles of y and ξ in the above arguments and hence conclude that I_2 and I_3 also satisfy (130). Altogether we have that

$$\sup_{\xi\in[0,1]}\int_0^1 \left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}}\partial_z \end{array} \right\} \mathcal{F}^{-1}[\varepsilon^2 q^2 G(1-\chi)] \right| \mathrm{d}y \leq \frac{c}{r^3},$$

$$\sup_{y\in[0,1]} \int_0^1 \left| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}}\partial_z \end{array} \right\} \mathcal{F}^{-1}[\varepsilon^2 q^2 G(1-\chi)] \right| \mathrm{d}\xi \leq \frac{c}{r^3},$$

from which it follows that

$$\left\| \left\{ \begin{array}{c} \partial_x \\ \varepsilon^{\frac{1}{2}}\partial_z \end{array} \right\} \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 q^2 G(1-\chi) w \, \mathrm{d}\xi \right] \right\|_{L^p(0,1)} \leq \frac{c\varepsilon^{\frac{3}{2}}}{(x^2+\varepsilon^{-1}z^2)^{\frac{3}{2}}} \|w\|_{L^p(0,1)}$$

for each $w \in L^p(0,1)$ (e.g. see Hutson & Pym [20, Corollary 2.5.4]); using this result and Proposition 4.10, we find from Theorem 4.5 that

$$\left\| \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 q^2 G(1-\chi) u \, \mathrm{d}\xi \right] \right\|_p \leq c\varepsilon^2 \|u\|_p. \qquad \square$$

The estimates for $\mathcal{G}_{\mathrm{b},2}$ and $\tilde{\mathcal{G}}_{\mathrm{b},2}$ are presented in the next two lemmata, the second of which is proved in the same way as the first.

Lemma 4.13 *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\mathcal{G}_{\mathrm{b},2}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\mathcal{G}_{\mathrm{b},2}(u)\|_p \leq c\varepsilon^2 \|u\|_p.$$

Proof. Observe that

$$\begin{aligned}
\left|\frac{1}{Q}\right| &\leq \frac{c}{\varepsilon^{-2}q^6} \leq c\varepsilon^2, \\
\varepsilon^{-\frac{1}{2}}\left|\partial_\mu\left(\frac{1}{Q}\right)\right| &\leq \frac{c\varepsilon^{-\frac{1}{2}}q}{Q^2}(|\mu| + \varepsilon^{-\frac{3}{2}}q^3 + \varepsilon^{-\frac{3}{2}}q^5) \\
&\leq \frac{c}{Q^2}(\mu^2 + \varepsilon^{-1}q^2 + \varepsilon^{-2}q^3 + \varepsilon^{-2}q^5) \\
&\leq c\left(\frac{1}{Q}\frac{\mu^2}{Q} + \frac{\varepsilon^{-2}q^{12}}{Q^2}\right) \\
&\leq c\left(\varepsilon^2\frac{\mu^2}{\mu^2} + \frac{\varepsilon^{-2}q^{12}}{\varepsilon^{-4}q^{12}}\right) \\
&\leq c\varepsilon^2,
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{-1}\left|\partial_k\left(\frac{1}{Q}\right)\right| &\leq \frac{c\varepsilon^{-1}q}{Q^2}(|k| + \varepsilon^{-1}q^3 + \varepsilon^{-1}q^5) \\
&\leq \frac{c}{Q^2}(k^2 + \varepsilon^{-2}q^2 + \varepsilon^{-2}q^3 + \varepsilon^{-2}q^5) \\
&\leq c\left(\frac{1}{Q}\frac{k^2}{Q} + \frac{\varepsilon^{-2}q^{12}}{Q^2}\right) \\
&\leq c\left(\varepsilon^2\frac{k^2}{k^2} + \frac{\varepsilon^{-2}q^{12}}{\varepsilon^{-4}q^{12}}\right) \\
&\leq c\varepsilon^2
\end{aligned}$$

for $q \geq q_0$, and similar calculations show that

$$\varepsilon^{-1}q^2\left|\partial_\mu^2\left(\frac{1}{Q}\right)\right| \leq c\varepsilon^2, \quad \varepsilon^{-2}q^2\left|\partial_k^2\left(\frac{1}{Q}\right)\right| \leq c\varepsilon^2, \quad \varepsilon^{-\frac{3}{2}}q^2\left|\partial_\mu\partial_k\left(\frac{1}{Q}\right)\right| \leq c\varepsilon^2$$

for $q \geq q_0$. Turning to the 'cut-off' function χ, note that

$$\begin{aligned} |\chi| &\leq c, \\ \varepsilon^{-\frac{1}{2}}q|\partial_\mu\chi| &= \varepsilon^{-\frac{1}{2}}q|\chi'(q)\partial_\mu q| \leq q|\chi'(q)| \leq c, \\ \varepsilon^{-1}q|\partial_k\chi| &= \varepsilon^{-1}q|\chi'(q)\partial_k uq| \leq q|\chi'(q)| \leq c, \end{aligned}$$

and similar calculations show that

$$\varepsilon^{-1}q^2|\partial_\mu^2\chi| \leq c, \quad \varepsilon^{-2}q^2|\partial_k^2\chi| \leq c, \quad \varepsilon^{-\frac{3}{2}}q^2|\partial_\mu\partial_k\chi| \leq c;$$

these estimates clearly also hold for $(1-\chi)$. The multiplier $(1-\chi)/Q$ therefore satisfies the hypotheses of Lemma 4.2 uniformly for $y, \xi \in [0,1]$, and it follows from Theorem 4.4 that

$$\left\|\mathcal{F}^{-1}\left[\int_0^1 \frac{1-\varepsilon}{Q}(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right]\right\|_p \leq c\varepsilon^2\|u\|_p. \qquad \square$$

Lemma 4.14 *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\tilde{\mathcal{G}}_{\mathrm{b},2}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\tilde{\mathcal{G}}_{\mathrm{b},2}(u)\|_p \leq c\varepsilon\|u\|_p.$$

Lemmata 4.8, 4.9 and 4.11–4.14 show that

$$\|\mathcal{G}(u)\|_p \leq c\varepsilon\|u\|_p, \qquad \|\tilde{\mathcal{G}}(u)\|_p \leq c\varepsilon\|u\|_p,$$

and we can deduce the remaining estimate for $\partial_y^2\mathcal{G}$ from them.

Corollary 4.15 *Choose $p \in (1,\infty)$. For each $u \in L^p(\Sigma)$ the function $\partial_y^2\mathcal{G}(u)$ belongs to $L^p(\Sigma)$ and satisfies the estimate*

$$\|\partial_y^2\mathcal{G}(u)\|_p \leq c\varepsilon^2\|u\|_p.$$

Proof. Observe that

$$\begin{aligned} \partial_y^2\mathcal{G}_{\mathrm{a}}(u) &= \mathcal{F}^{-1}\left[\int_0^1 \partial_y^2 G_1\chi\mathcal{F}[u]\,\mathrm{d}\xi\right] + \mathcal{F}^{-1}\left[\int_0^1 \partial_y^2 G_1(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right] \\ &= \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2\partial_y^2\tilde{G}_1\chi\mathcal{F}[u]\,\mathrm{d}\xi\right] + \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2\partial_y^2 G(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right] \\ &= \mathcal{F}^{-1}\left[\int_0^1 \frac{\varepsilon^2(1+\varepsilon)\chi q^2(\tilde{G}+1)}{q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2}\mathcal{F}[u]\,\mathrm{d}\xi\right] \\ &\qquad + \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2q^2G(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi\right], \end{aligned}$$

where we have used (116), (125) and the facts that

$$\partial_y^2 \tilde{G} = q^2(\tilde{G}+1), \qquad \partial_y^2 G = q^2 G.$$

The assertion therefore follows from the estimate

$$\left\| \mathcal{F}^{-1}\left[\int_0^1 \frac{\varepsilon^2(1+\varepsilon)\chi q^2(\tilde{G}+1)}{q^2-(1+\varepsilon+\beta q^2)q\tanh q - \varepsilon^2 k^2} \mathcal{F}[u]\,\mathrm{d}\xi \right] \right\|_p \leq c\varepsilon^2 \|u\|_p,$$

which is obtained by noting that $\partial_q^i(q^2(\tilde{G}+1)) = \mathcal{O}(q^{4-i}), i = 0,1,2$ as $q \to 0$ uniformly for $y, \xi \in [0,1]$ and repeating the first part of the proof of Lemma 4.9, and the estimate

$$\left\| \mathcal{F}^{-1}\left[\int_0^1 \varepsilon^2 q^2 G(1-\chi)\mathcal{F}[u]\,\mathrm{d}\xi \right] \right\|_p \leq c\varepsilon^2 \|u\|_p,$$

which is obtained in the proof of Lemma 4.12. □

The above theory establishes the basic estimate

$$\|\mathcal{G}(u)\|_{2,p,\varepsilon} \leq c\varepsilon \|u\|_p, \tag{131}$$

and we now complete our analysis by showing how Lemma 2.15(i), (ii) follow from this result.

Corollary 4.16 *Choose $\delta \in [0,1]$ and $p \in (1,\infty)$.*

(i) For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the function

$$\mathcal{G}_4(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu G_1 \mathcal{F}[u]\,\mathrm{d}\xi \right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\Sigma)$ and satisfies the estimate

$$\|\mathcal{G}_4(u)\|_{\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon}.$$

(ii) For each $u \in W_\varepsilon^{\delta,p}(\Sigma)$ the function

$$\mathcal{G}_5(u) = \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\varepsilon^{\frac{1}{2}} k G_1 \mathcal{F}[u]\,\mathrm{d}\xi \right]$$

belongs to $W_\varepsilon^{1+\delta,p}(\Sigma)$ and satisfies the estimate

$$\|\mathcal{G}_5(u)\|_{\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon}.$$

Proof. Observe that

$$\|\mathcal{G}_4(u)\|_{1,p,\varepsilon} = \|\partial_x \mathcal{G}(u)\|_{1,p,\varepsilon} \leq \|\mathcal{G}(u)\|_{2,p,\varepsilon} \leq \|u\|_p,$$

and

$$\|\mathcal{G}_4(u)\|_{2,p,\varepsilon} = \|\mathcal{G}(u_x)\|_{2,p,\varepsilon} \leq c\varepsilon \|u_x\|_p \leq c\varepsilon \|u\|_{1,p,\varepsilon}.$$

Interpolating between the previous two inequalities, we find that

$$\|\mathcal{G}_4(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon},$$

and we similarly find that

$$\|\mathcal{G}_5(u)\|_{1+\delta,p,\varepsilon} \leq c\varepsilon \|u\|_{\delta,p,\varepsilon}.$$

□

Parts (iii)-(viii) of Lemma 2.15 are established in an analogous fashion.

4.3 Convergence properties

Our final piece of analysis is the proof of Lemma 3.11, which relates to the operators

$$\mathcal{G}_i^{N,m} = \chi_N \mathcal{G}_i (1-\chi_{R_m}), \qquad i=1,\ldots,6,8,\ldots,11.$$

We begin by examining $\mathcal{G}_1^{N,m}$, $\mathcal{G}_2^{N,m}$, $\mathcal{G}_3^{N,m} : W_\varepsilon^{\delta,p}(\mathbb{R}^2) \to W_\varepsilon^{1+\delta,p}(\mathbb{R}^2)$.

Lemma 4.17 *Choose $N>0$, suppose that $\{R_m\}$ is a sequence of positive, real numbers such that $R_m \to \infty$ as $m\to\infty$ and let $\chi_N : \mathbb{R}^2 \to \mathbb{R}$, $\chi_{R_m} : \mathbb{R}^2 \to \mathbb{R}$ be smooth 'cut-off' functions whose support is contained in respectively $\bar{B}_N(0)$ and $\bar{B}_{R_m}(0)$. The functions*

$$\begin{aligned}
\mathcal{G}_1^{N,m}(u) &= \chi_N \mathcal{F}^{-1}\left[\frac{1}{1+\varepsilon+\beta q^2}\mathcal{F}[(1-\chi_{R_m})u]\right],\\
\mathcal{G}_2^{N,m}(u) &= \chi_N \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2}\mathcal{F}[(1-\chi_{R_m})u]\right],\\
\mathcal{G}_3^{N,m}(u) &= \chi_N \mathcal{F}^{-1}\left[\frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2}\mathcal{F}[(1-\chi_{R_m})u]\right]
\end{aligned}$$

satisfy

$$\|\mathcal{G}_i^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}, \qquad i=1,2,3$$

for each $\delta\in[0,1]$ and each sufficiently large value of p, in which the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m\to\infty$.

Proof. Suppose that $f(\mu,k)$ is one of

$$\frac{1}{1+\varepsilon+\beta q^2},\quad \frac{\mathrm{i}\mu}{1+\varepsilon+\beta q^2},\quad \frac{\mathrm{i}\varepsilon^{\frac{1}{2}}k}{1+\varepsilon+\beta q^2},\quad \frac{-\mu^2}{1+\varepsilon+\beta q^2},\quad \frac{-\varepsilon k^2}{1+\varepsilon+\beta q^2},\quad \frac{-\varepsilon^{\frac{1}{2}}\mu k}{1+\varepsilon+\beta q^2}$$

and define

$$\mathcal{G}^{N,m}(u) = \chi_N\mathcal{F}^{-1}[f(\mu,k)\mathcal{F}[(1-\chi_{R_m})u]],$$

so that

$$\mathcal{G}^{N,m}(u)(x,z) = \chi_N(x,z)\int_{\mathbb{R}^2} K(x-x_1,z-z_1)(1-\chi_{R_m}(x_1,z_1))u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1,$$

where $K(x,z)=\mathcal{F}^{-1}[f(\mu,k)]$. (Note that $f\notin L^1(\mathbb{R}^2)$, so that K is only well-defined as part of the above convolution.) The $L^p(\mathbb{R}^2)$-norm of $\mathcal{G}^{N,m}(u)$ is given by

$$\|\mathcal{G}^{N,m}(u)\|_p = \left(\int_{N_1^{x,z}}\left|\int_{N_2^{x_1,z_1}} K(x-x_1,z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1\right|^p \mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}},$$

where

$$N_1^{x,z} = \{(x,z) : x^2+z^z \leq N\}, \quad N_2^{x_1,z_1} = \{(x,z) : x^2+z^2 \geq R_m\},$$

and using the generalised version of Hölder's inequality (Hardy, Littlewood & Pólya [19, Theorem 188]), one finds that

$$
\begin{aligned}
&\|\mathcal{G}^{N,m}(u)\|_p \\
&= \left(\int_{N_1^{x,z}}\left|\int_{N_2^{x_1,z_1}}\frac{|x-x_1|^2+|z-z_1|^2}{|x-x_1|^2+|z-z_1|^2}K(x-x_1,z-z_1)u(x_1,z_1)\,\mathrm{d}x_1\,\mathrm{d}z_1\right|^p\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\leq \int_{N_1^{x,z}}\left(\int_{N_2^{x_1,z_1}}\left(\frac{1}{|x-x_1|^2+|z-z_1|^2}\right)^{q_1}\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{q_1}} \\
&\qquad\times\left(\int_{N_2^{x_1,z_1}}((|x-x_1|^2+|z-z_1|^2)|K(x-x_1,z-z_1)|)^{q_2}\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{q_2}}\mathrm{d}x\,\mathrm{d}z\Bigg)^{\frac{1}{p}}\|u\|_p,
\end{aligned}
$$

where

$$
\frac{1}{p}+\frac{1}{q_1}+\frac{1}{q_2}=1, \qquad 1<q_1<2, \quad q_2>2
$$

(choices of q_1, q_2 in the indicated ranges are possible for sufficiently large values of p).

A direct calculation shows that $\partial_\mu^2 f$, $\partial_k^2 f$ are bounded as $q\to 0$ and $\mathcal{O}(q^{-2})$ as $q\to\infty$; they therefore belong to $L^s(\mathbb{R}^2)$ for all $s>1$. Using this fact, we find that

$$
\begin{aligned}
&\left(\int_{N_2^{x_1,z_1}}((|x-x_1|^2+|z-z_1|^2)|K(x-x_1,z-z_1)|)^{q_2}\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{1}{q_2}} \\
&\qquad\leq \left(\int_{\mathbb{R}^2}((|x-x_1|^2+|z-z_1|^2)|K(x-x_1,z-z_1)|)^{q_2}\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{1}{q_2}} \\
&\qquad= \left(\int_{\mathbb{R}^2}((|x|^2+|z|^2)|K(x,z)|)^{q_2}\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{q_2}} \\
&\qquad\leq \left(\int_{\mathbb{R}^2}|x^2K(x,z)|^{q_2}\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{q_2}}+\left(\int_{\mathbb{R}^2}|z^2K(x,z)|^{q_2}\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{q_2}} \\
&\qquad\leq \left(\int_{\mathbb{R}^2}|\partial_\mu^2 f(\mu,k)|^{q_2'}\,\mathrm{d}\mu\,\mathrm{d}k\right)^{\frac{1}{q_2'}}+\left(\int_{\mathbb{R}^2}|\partial_k^2 f(\mu,k)|^{q_2'}\,\mathrm{d}\mu\,\mathrm{d}k\right)^{\frac{1}{q_2'}} \\
&\qquad\leq c_\varepsilon,
\end{aligned}
$$

where q_2' is the conjugate index to q_2 and we have used the Hausdorff-Young inequality

$$
\|u\|_{q_2}\leq\|\mathcal{F}[u]\|_{q_2'}, \qquad 1<q_2<2
$$

(e.g. see Hardy, Littlewood & Pólya [19, §§8.5, 8.17]). It follows that

$$
\|\mathcal{G}^{N,m}(u)\|_p\leq c_\varepsilon\int_{N_1^{x,z}}\left(\int_{N_2^{x_1,z_1}}\left(\frac{1}{|x-x_1|^2+|z-z_1|^2}\right)^{q_1}\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{q_1}}\mathrm{d}x\,\mathrm{d}z\Bigg)^{\frac{1}{p}}\|u\|_p,
$$

and this inequality and the calculation

$$
\int_{N_1^{x,z}}\left(\int_{N_2^{x_1,z_1}}\left(\frac{1}{|x-x_1|^2+|z-z_1|^2}\right)^{q_1}\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{q_1}}\mathrm{d}x\,\mathrm{d}z\Bigg)^{\frac{1}{p}}
$$

$$
\begin{aligned}
&\le \int_{N_1^{x,z}} \left(\int_{N_3^{x_1,z_1}} \left(\frac{1}{|x_1|^2+|z_1|^2} \right)^{q_1} \mathrm{d}x_1\,\mathrm{d}z_1 \right)^{\frac{p}{q_1}} \mathrm{d}x\,\mathrm{d}z \right)^{\frac{1}{p}} \\
&= \frac{(\pi N^2)^{\frac{1}{p}}}{(-2+2q_1)^{\frac{1}{q_1}}} (R_m - N)^{-2+\frac{2}{q_1}} \\
&\to 0 \qquad (132)
\end{aligned}
$$

as $m \to \infty$, where $N_3^{x_1,z_1} = \{(x_1, z_1) : x_1^2 + z_1^2 \ge R_m - N\}$, imply that

$$\|\mathcal{G}^{N,m}(u)\|_p \le c_\varepsilon^{N,m} \|u\|_p.$$

Clearly

$$\|\mathcal{G}_i^{N,m}(u)\|_{1,p,\varepsilon} \le \|\mathcal{G}_i^{N,m}(u)\|_p + \|\partial_x \mathcal{G}_i^{N,m}(u)\|_p + \|\partial_z \mathcal{G}_i^{N,m}(u)\|_p, \qquad i = 1, 2, 3,$$

and because $\mathcal{G}_i^{N,m}(u) = \chi_N \mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]$, $i = 1, 2, 3$, where $f_i(\mu, k)$ is the ith choice for $f(\mu, k)$, we have that

$$\|\mathcal{G}_i^{N,m}(u)\|_p \le c_\varepsilon^{N,m} \|u\|_p.$$

Furthermore, the above argument shows that both terms on the right-hand side of the inequalities

$$
\begin{aligned}
&\|\partial_x \mathcal{G}_i^{N,m}(u)\|_p \\
&\qquad \le \|\partial_x \chi_N \mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]\|_p + \|\chi_N \mathcal{F}^{-1}[\mathrm{i}\mu f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]\|_p, \\
&\varepsilon^{\frac{1}{2}} \|\partial_z \mathcal{G}_i^{N,m}(u)\|_p \\
&\qquad \le \varepsilon^{\frac{1}{2}} \|\partial_z \chi_N \mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]\|_p + \|\chi_N \mathcal{F}^{-1}[\mathrm{i}\varepsilon^{\frac{1}{2}} k f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]\|_p
\end{aligned}
$$

are bounded by $c_\varepsilon^{N,m} \|u\|_p$, the first because $\partial_x \chi_N$, $\partial_z \chi_N$ have the same support as χ_N and the second because each of $\mathrm{i}\mu f_i(\mu, k)$ and $\mathrm{i}\varepsilon^{\frac{1}{2}} k f_i(\mu, k)$ is one of the fourth, fifth or sixth choices for $f(\mu, k)$. Altogether we have that

$$\|\mathcal{G}_i^{N,m}(u)\|_{1,p,\varepsilon} \le c_\varepsilon^{N,m} \|u\|_p, \qquad (133)$$

and a similar argument shows that

$$
\begin{aligned}
&\|\partial_{xx} \mathcal{G}_i^{N,m}(u)\|_p \\
&\qquad \le \|\partial_{xx} \chi_N \mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]\|_p + 2\|\partial_x \chi_N \mathcal{F}^{-1}[\mathrm{i}\mu f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]\|_p \\
&\qquad\qquad + \|\chi_N \mathcal{F}^{-1}[\mathrm{i}\mu f_i(\mu, k)\mathcal{F}[\partial_x((1 - \chi_{R_m})u)]]\|_p \\
&\qquad \le c_\varepsilon^{N,m} \|u\|_{1,p,\varepsilon}, \\
&\varepsilon \|\partial_{zz} \mathcal{G}_i^{N,m}(u)\|_p \\
&\qquad \le \varepsilon \|\partial_{zz} \chi_N \mathcal{F}^{-1}[f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]]\|_p + 2\varepsilon^{\frac{1}{2}} \|\partial_z \chi_N \mathcal{F}^{-1}[\mathrm{i}\varepsilon^{\frac{1}{2}} k f_i(\mu, k)\mathcal{F}[(1 - \chi_{R_m})u]\|_p \\
&\qquad\qquad + \|\chi_N \mathcal{F}^{-1}[\mathrm{i}\varepsilon^{\frac{1}{2}} k f_i(\mu, k)\mathcal{F}[\varepsilon^{\frac{1}{2}} \partial_z((1 - \chi_{R_m})u)]]\|_p \\
&\qquad \le c_\varepsilon^{N,m} \|u\|_{1,p,\varepsilon},
\end{aligned}
$$

so that

$$\|\mathcal{G}_i^{N,m}(u)\|_{2,p,\varepsilon} \le c_\varepsilon^{N,m} \|u\|_{1,p,\varepsilon}. \qquad (134)$$

Interpolating between (133) and (134), one finds that

$$\|\mathcal{G}_i^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}. \qquad \square$$

The corresponding results for $\mathcal{G}_4^{N,m}$, $\mathcal{G}_5^{N,m}$, $\mathcal{G}_6^{N,m}$, and $\mathcal{G}_8^{N,m}$, …, $\mathcal{G}_{11}^{N,m}$, are obtained by combining elements of the proof of Lemma 4.17 with the methods used to establish the mapping properties of $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$ and $\mathcal{G}_8$, …, $\mathcal{G}_{11}$ in Section 4.2. We give the details for $\mathcal{G}_4^{N,m}$ and $\mathcal{G}_5^{N,m}$; the remaining operators are treated in an analogous fashion.

Lemma 4.18 *Choose $N > 0$, suppose that $\{R_m\}$ is a sequence of positive, real numbers such that $R_m \to \infty$ as $m \to \infty$ and let $\chi_N : \mathbb{R}^2 \to \mathbb{R}$, $\chi_{R_m} : \mathbb{R}^2 \to \mathbb{R}$ be smooth 'cut-off' functions whose support is contained in respectively $\bar{B}_N(0)$ and $\bar{B}_{R_m}(0)$. The functions*

$$\begin{aligned}
\mathcal{G}_4^{N,m}(u) &= \chi_N\mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu G_1\mathcal{F}[(1-\chi_{R_m})u]\,\mathrm{d}\xi\right],\\
\mathcal{G}_5^{N,m}(u) &= \chi_N\mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\varepsilon^{\frac{1}{2}}kG_1\mathcal{F}[(1-\chi_{R_m})u]\,\mathrm{d}\xi\right]
\end{aligned}$$

satisfy

$$\|\mathcal{G}_4^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}, \qquad \|\mathcal{G}_5^{N,m}(u)\|_{1+\delta,p,\varepsilon} \leq c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}$$

for each $\delta \in [0,1]$ and each sufficiently large value of p, in which the symbol $c_\varepsilon^{N,m}$ denotes a quantity that, for each fixed value of N and ε, tends to zero as $m \to \infty$.

Proof. The first step is to show that

$$\begin{aligned}
\|\mathcal{G}_4^{N,m}(u)\|_p &\leq c_\varepsilon^{N,m}\|u\|_p,\\
\|\bar{\mathcal{G}}_4^{N,m}(u)\|_p &\leq c_\varepsilon^{N,m}\|u\|_p,\\
\|\hat{\mathcal{G}}_4^{N,m}(u)\|_p &\leq c_\varepsilon^{N,m}\|u\|_p,
\end{aligned} \tag{135}$$

where $\bar{\mathcal{G}}_4^{N,m}$ and $\hat{\mathcal{G}}_4^{N,m}$ are the operators obtained by replacing $\mathrm{i}\mu$ with respectively $-\mu^2$ and $-\varepsilon^{\frac{1}{2}}\mu k$ in the definition of $\mathcal{G}_4^{N,m}$; using the argument given at the end of Lemma 4.17 we immediately deduce that

$$\|\partial_x\mathcal{G}_4^{N,m}(u)\|_p \leq c_\varepsilon^{N,m}\|u\|_p, \qquad \varepsilon^{\frac{1}{2}}\|\partial_z\mathcal{G}_4^{N,m}(u)\|_p \leq c_\varepsilon^{N,m}\|u\|_p \tag{136}$$

and

$$\|\partial_{xx}\mathcal{G}_4^{N,m}(u)\|_p \leq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon} \qquad \varepsilon\|\partial_{zz}\mathcal{G}_4^{N,m}(u)\|_p \leq c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon}. \tag{137}$$

To this end we use the decompositions

$$\mathcal{G}_4^{N,m} = \mathcal{G}_{4\mathrm{a}}^{N,m} + \mathcal{G}_{4\mathrm{b}}^{N,m}, \quad \bar{\mathcal{G}}_4^{N,m} = \bar{\mathcal{G}}_{4\mathrm{a}}^{N,m} + \bar{\mathcal{G}}_{4\mathrm{b}}^{N,m}, \quad \hat{\mathcal{G}}_4^{N,m} = \hat{\mathcal{G}}_{4\mathrm{a}}^{N,m} + \hat{\mathcal{G}}_{4\mathrm{b}}^{N,m}$$

and

$$\mathcal{G}_{4\mathrm{b}}^{N,m} = \mathcal{G}_{4\mathrm{b},1}^{N,m} + \mathcal{G}_{4\mathrm{b},2}^{N,m}, \quad \mathcal{G}_{4\mathrm{b}}^{N,m} = \mathcal{G}_{4\mathrm{b},1}^{N,m} + \mathcal{G}_{4\mathrm{b},2}^{N,m}, \quad \mathcal{G}_{4\mathrm{b}}^{N,m} = \mathcal{G}_{4\mathrm{b},1}^{N,m} + \mathcal{G}_{4\mathrm{b},2}^{N,m}$$

which are defined using respectively the 'cut-off' function χ (see the explanation above Lemma 4.8) and the expression (125) (see the explanation above Proposition 4.10).

Let us write

$$G_1 = \varepsilon^2 \tilde{G}_1 + \varepsilon^2 \tilde{G}_2 + \varepsilon^2 \tilde{G}_3$$

(see equation (116)). Calculations similar to those presented in Lemma 4.8 show that

$$\partial_\mu^2(\mathrm{i}\mu\tilde{G}_i), \quad \partial_k^2(\mathrm{i}\mu\tilde{G}_i), \qquad i = 1, 2, 3$$

are bounded at the origin, so that $\partial_\mu^2(\chi\mathrm{i}\mu G_1)$ and $\partial_k^2(\chi\mathrm{i}\mu G_1)$ belong to $L^s(\mathbb{R}^2)$ for each $s > 1$. Noting that all estimates are uniform for $y, \xi \in [0, 1]$, we may apply the method explained in Lemma 4.17 to find that

$$\|\mathcal{G}_{4\mathrm{a}}^{N,m}(u)\|_p \leq c_\varepsilon^{N,m} \|u\|_p,$$

and the same argument shows that

$$\|\bar{\mathcal{G}}_{4\mathrm{a}}^{N,m}(u)\|_p \leq c_\varepsilon^{N,m} \|u\|_p, \qquad \|\hat{\mathcal{G}}_{4\mathrm{a}}^{N,m}(u)\|_p \leq c_\varepsilon^{N,m} \|u\|_p.$$

To obtain the corresponding estimates for $\mathcal{G}_{4\mathrm{b},1}^{N,m}$ we use the expression

$$\begin{aligned} G = {} & \frac{1 + \varepsilon + \beta q^2 - \varepsilon\mu^2/q}{2(1 + \mathrm{e}^{-2q})(q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi - y|} \\ & + \frac{1 + \varepsilon + \beta q^2 - \varepsilon\mu^2/q}{2(1 + \mathrm{e}^{-2q})(q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q(\xi + y)} \\ & + \frac{1 + \varepsilon + \beta q^2 - \varepsilon\mu^2/q}{2(1 + \mathrm{e}^{-2q})(q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q(2 - \xi - y)} \\ & + \frac{1 + \varepsilon + \beta q^2 - \varepsilon\mu^2/q}{2(\mathrm{e}^{q} + \mathrm{e}^{-q})(q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q(1 - |\xi - y|)} \end{aligned}$$

derived in Lemma 4.11. We consider the first of these terms in detail; the others are handled in an analogous fashion. Define

$$I = \varepsilon^2 \mathcal{F}^{-1}\left[\frac{\mathrm{i}\mu(1 + \varepsilon + \beta q^2 - \varepsilon\mu^2/q)(1 - \chi(q))}{2(1 + \mathrm{e}^{-2q})(q^2 - (1 + \varepsilon + \beta q^2) q \tanh q - \varepsilon^2 k^2)} \mathrm{e}^{-q|\xi - y|}\right].$$

In terms of the polar coordinates (127), one has that

$$I = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{(1 + \varepsilon) q^2 (1 - \chi(q))}{2\tilde{Q}(1 + \mathrm{e}^{-2q})} \cos(\phi + \psi) \mathrm{e}^{-q(|\xi - y| + \mathrm{i}r\cos\psi)} \,\mathrm{d}q\,\mathrm{d}\psi, \\ I_2 &= \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{q^3 (1 - \chi(q))}{2\tilde{Q}(1 + \mathrm{e}^{-2q})} \cos^3(\phi + \psi) \mathrm{e}^{-q(|\xi - y| + \mathrm{i}r\cos\psi)} \,\mathrm{d}q\,\mathrm{d}\psi, \\ I_3 &= \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\beta q^4 (1 - \chi(q))}{2\tilde{Q}(1 + \mathrm{e}^{-2q})} \cos(\phi + \psi) \mathrm{e}^{-q(|\xi - y| + \mathrm{i}r\cos\psi)} \,\mathrm{d}q\,\mathrm{d}\psi, \end{aligned}$$

$\psi = \theta - \phi$ and $\tilde{Q} = q^2\cos^2(\phi+\psi) - (1+\varepsilon+\beta q^2)q\tanh q$, and the method used in the proof of Lemma 4.11 shows that

$$|I_1| \leq \frac{c_\varepsilon}{r^2}, \quad |I_2| \leq \frac{c_\varepsilon}{r^2}, \quad |I_3| \leq \frac{c_\varepsilon}{r^3}, \qquad y \neq \xi.$$

The above calculation indicates that

$$\mathcal{F}^{-1}[\mathcal{G}_{4\mathrm{b},1}^{N,m}(\mu,k;y,\xi)] = \sum K_i(x,z;y,\xi),$$

where each summand (of which there are a finite number) satisfies the inequality

$$|K_i(x,y;y,\xi)| \leq \frac{c_\varepsilon}{r^{n_i}}, \qquad n_i \geq 2$$

for $y \neq \xi$. Observe that

$$\begin{aligned}
&\left(\int_{N_1^{x,z}}\int_0^1 \left|\int_{N_2^{x_1,z_1}}\int_0^1 K_i(x-x_1,z-z_1;y,\xi)u(x_1,\xi,z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1\right|^p \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\leq \int_{N_1^{x,z}}\int_0^1 \left(\int_{N_2^{x_1,z_1}}\int_0^1 |K_i(x-x_1,z-z_1;y,\xi)|\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{p'}} \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z \|u\|_p \\
&\leq c_\varepsilon \int_{N_1^{x,z}} \left(\int_{N_2^{x_1,z_1}} \left(\frac{1}{|x-x_1|^2+|z-z_1|^2}\right)^{\frac{p'n_i}{2}} \mathrm{d}x_1\,\mathrm{d}z_1\right)^{\frac{p}{p'}} \|u\|_p \\
&\leq \frac{c_\varepsilon(\pi N^2)^{\frac{1}{p}}(R_m-N)^{-n_i+\frac{2}{p'}}}{p'n_i-2}\|u\|_p \\
&\rightarrow 0
\end{aligned}$$

as $m \rightarrow \infty$, where p' is the conjugate index to p and we have used Hölder's inequality and the calculation (132). It follows that

$$\begin{aligned}
&\|\mathcal{G}^{N,m}(u)\|_p \\
&= \left(\int_{N_1^{x,z}}\int_0^1 \left|\int_{N_2^{x_1,z_1}}\int_0^1 \sum K_i(x-x_1,z-z_1;y,\xi)u(x_1,\xi,z_1)\,\mathrm{d}\xi\,\mathrm{d}x_1\,\mathrm{d}z_1\right|^p \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\leq c_\varepsilon^{N,m}\|u\|_p.
\end{aligned}$$

This technique also yields the estimates for $\bar{\mathcal{G}}_{1\mathrm{b},1}^{N,m}$ and $\hat{\mathcal{G}}_{1\mathrm{b},1}^{N,m}$; here we have to estimate

$$I = \varepsilon^2\mathcal{F}^{-1}\left[\left\{\begin{array}{c} -\mu^2 \\ -\varepsilon^{\frac{1}{2}}\mu k \end{array}\right\} \frac{(1+\varepsilon+\beta q^2-\varepsilon\mu^2/q)(1-\chi(q))}{2(1+\mathrm{e}^{-2q})(q^2-(1+\varepsilon+\beta q^2)q\tanh q-\varepsilon^2k^2)}\mathrm{e}^{-q|\xi-y|}\right]$$

(and three other terms with slightly different exponential factors), and hence

$$\begin{aligned}
I_1 &= \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{(1+\varepsilon)q^3(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\left\{\begin{array}{c}\cos^2(\phi+\psi) \\ \cos(\phi+\psi)\sin(\phi+\psi)\end{array}\right\}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi, \\
I_2 &= \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{q^4(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\left\{\begin{array}{c}\cos^4(\phi+\psi) \\ \cos^3(\phi+\psi)\sin(\phi+\psi)\end{array}\right\}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi, \\
I_3 &= \frac{\varepsilon^{\frac{1}{2}}\mathrm{i}}{2\pi}\int_0^{2\pi}\int_0^\infty \frac{\beta q^5(1-\chi(q))}{2\tilde{Q}(1+\mathrm{e}^{-2q})}\left\{\begin{array}{c}\cos^2(\phi+\psi) \\ \cos(\phi+\psi)\sin(\phi+\psi)\end{array}\right\}\mathrm{e}^{-q(|\xi-y|+\mathrm{i}r\cos\psi)}\,\mathrm{d}q\,\mathrm{d}\psi.
\end{aligned}$$

We find that

$$|I_1| \le \frac{c_\varepsilon}{r^3}, \quad |I_2| \le \frac{c_\varepsilon}{r^3}, \quad |I_3| \le \frac{c_\varepsilon}{r^4}, \qquad y \neq \xi,$$

and the argument given above therefore yields the inequalities

$$\|\bar{\mathcal{G}}_{4\mathrm{b},1}^{N,m}(u)\|_p \le c_\varepsilon^{N,m}\|u\|_p, \qquad \|\hat{\mathcal{G}}_{4\mathrm{b},1}^{N,m}(u)\|_p \le c_\varepsilon^{N,m}\|u\|_p.$$

Calculations similar to those presented in Lemma 4.13 show that $\partial_\mu^2(\mathrm{i}\mu/Q)$, $\partial_k^2(\mathrm{i}\mu/Q)$ are $O(q^{-3})$ as $q \to \infty$, so that $\partial_\mu^2((1-\chi)\mathrm{i}\mu/Q)$, $\partial_k^2((1-\chi)\mathrm{i}\mu/Q)$ belong to $L^s(\mathbb{R}^2)$ for all $s > 1$. Noting that all estimates are uniform for $y, \xi \in [0,1]$, we may apply the method used in Lemma 4.17 to find that

$$\|\mathcal{G}_{4\mathrm{b},2}^{N,m}(u)\|_p \le c_\varepsilon^{N,m}\|u\|_p,$$

and the same method yields the corresponding estimates for $\bar{\mathcal{G}}_{4\mathrm{b},2}^{N,m}$ and $\hat{\mathcal{G}}_{4\mathrm{b},2}^{N,m}$. Finally, we obtain the estimates

$$\begin{aligned}\|\partial_y \mathcal{G}_4^{N,m}(u)\|_p &= \left\| \chi_N \mathcal{F}^{-1}\left[\int_0^1 \mathrm{i}\mu\partial_y G_1 \mathcal{F}[(1-\chi_{R_m})u]\,\mathrm{d}\xi\right]\right\|_p \\ &\le c_\varepsilon^{N,m}\|u\|_p, \end{aligned} \tag{138}$$

$$\begin{aligned}\|\partial_y^2 \mathcal{G}_4^{N,m}(u)\|_p &= \left\| \chi_N \mathcal{F}^{-1}\left[\int_0^1 \partial_y^2 G_1(\mathcal{F}[\partial_x((1-\chi_{R_m})u)])\,\mathrm{d}\xi\right]\right\|_p \\ &\le c_\varepsilon^{N,m}\|\partial_x((1-\chi_{R_m})u)\|_p \\ &\le c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon} \end{aligned} \tag{139}$$

using the method given above for $\bar{\mathcal{G}}_4^{N,m}$, noting that $\partial_\mu^2(\mathrm{i}\mu\partial_y G_1)$, $\partial_k^2(\mathrm{i}\mu\partial_y G_1)$ and $\partial_\mu^2(\partial_y^2 G_1)$, $\partial_k^2(\partial_y^2 G_1)$ are bounded at the origin and the polar-coordinate representation of their kernels differ from those of $\bar{G}_4^{N,m}$ only in the form of the trigonometric factor.

It follows from (135)–(139) that

$$\|\mathcal{G}_4^{N,m}(u)\|_{1,p,\varepsilon} \le c_\varepsilon^{N,m}\|u\|_p, \qquad \|\mathcal{G}_4^{N,m}(u)\|_{2,p,\varepsilon} \le c_\varepsilon^{N,m}\|u\|_{1,p,\varepsilon},$$

and interpolating between these inequalities, we find that

$$\|\mathcal{G}_4^{N,m}(u)\|_{1+\delta,p,\varepsilon} \le c_\varepsilon^{N,m}\|u\|_{\delta,p,\varepsilon}.$$

The same method yields the corresponding estimate for $\mathcal{G}_5^{N,m}$. □

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