Entangling continuous variables with a qubit array

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We show that an array of qubits embedded in a waveguide can emit entangled pairs of microwave photon beams. The quadratures obtained from the homodyne detection of these outputs beams form a pair of correlated continuous variables similar to the Einstein-Podolsky-Rosen experiment. The photon pairs are produced by the decay of plasmonlike collective excitations in the qubit array. The maximum intensity of the resulting beams is bounded by only the number of emitters. We calculate the excitation decay rate both into a continuum of the photon state and into a one-mode cavity. We also determine the frequency of Rabi-like oscillations resulting from a detuning.

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I. INTRODUCTION

The steady improvement of superconducting electronics over the last two decades [1–3] cemented the place of Josephson effect-based devices among the leading platforms for quantum technologies (e.g., quantum computation) [4]. However, the control and observation of essential quantum correlations and entanglement necessary for the operation of these technologies remains a challenging task [5,6]. A convenient test bed for this research is provided by a superconducting qubit array embedded in a coplanar waveguide (a "one-dimensional quantum metamaterial" setup) [1–3,7–10]. Some of these metamaterials are predicted to display interesting nonlinear properties like the two-photon induced transparency [11–13], superradiance [14,15], and lasing [16], but these results have been obtained using approximations short of a full QED treatment.

In this paper we build a consistent theory for a linear array of qubits placed in a waveguide. Specifically, we consider a set of capacitively coupled transmons, but the general results are not going to be sensitive to the particular kind of a qubit. The Josephson junctions are arranged symmetrically in order to ensure a quadratic coupling to the electromagnetic field. The collective excitations are produced by abrupt changes of qubit electric charges, tantamount to a sudden modification of the photon dispersion relation in the waveguide—a quantum analog of the emission of cosmological radiation in a curved space for spontaneous particle pair creations [17,18], also referred to as the dynamical Casimir effect [19]. The symmetry ensures that collective excitations of the array decay into the entangled microwave beams propagating along the waveguide in opposite directions.

Compared to the prior art, the proposed mechanism does not involve the use of an external magnetic field [20] or a pump field within a waveguide [21]. It predicts quantum correlations at a distance and therefore differs from other studies like the two-photon correlations analyzed in [22–24], sub- and superradiance in [25–27], and even phase transition [28].

II. THEORETICAL MODEL

The proposed scheme is presented in Figs. 1 and 2. An array of N transmon qubits is embedded in a ring waveguide at zero temperature. Starting at equilibrium, we adiabatically increase the potential of the qubit island to V and then suddenly drop it to zero. The initial charge on the island is $q_n = C_0 V$, where C_0 is the effective capacitance. Classically, the island charge oscillates at a frequency $\epsilon_0 = \sqrt{8E_J e^2/C_0}/\hbar$, where Josephson energy $E_J \gg e^2/C_0$ for a transmon. In the quantum case these oscillations will decay into the electromagnetic vacuum modes by producing two counterpropagating entangled beams. The subsequent action of a circulator passes these outputs on for a homodyne detection [29], which includes local oscillator mixing and frequency filtering, in order to determine their mutual Einstein-Podolsky-Rosen (EPR)-type quantum correlations [30,31]. The integral of the voltage pulse (flux [32]) and the total induced charge correspond to correlated (anticorrelated) continuous-variable quadratures.

The lumped-element scheme of the device is shown in Fig. 2. The nth transmon's quantum operators are its excess charge $\hat{q}_n = (2e)(\hat{n}_n - N_s)$ measured from the equilibrium value N_s and flux $\hat{\Phi}_n$. Its mutual capacitance with the neutral gate (with gate voltage V) is $C_{n,n}$, and $C_{n,n'}$ is the mutual capacitance between the nth and n'th transmons. The waveguide is described by conjugated operator charges $\Delta\hat{Q}_n$ and fluxes $\hat{\Phi}_n^B$ and is characterized by the mutual capacitance ΔC and inductance $\Delta L/2$ between two adjacent transmons. These operators obey the canonical commutation relations $[\hat{\Phi}_n, \hat{q}_{n'}] = i\hbar\delta_{n,n'}$ and $[\hat{\Phi}_n^B, \Delta\hat{Q}_{n'}] = i\hbar\delta_{n,n'}$. We omit the capacitive couplings to the waveguide as they cancel out for the decay process. The capacitive coupling between the charge $\Delta\hat{Q}_{n'}$ and the qubit charge $\hat{q}_n$ cancels out because of the intrinsic symmetry. Indeed, inserting a capacitance C' parallel to each Josephson junction introduces two opposite terms, $C'\Delta\hat{Q}_n\hat{q}_n$ and $C'(-\Delta\hat{Q}_n)\hat{q}_n$, that do not contribute to the total Hamiltonian. Therefore, these terms do not influence the

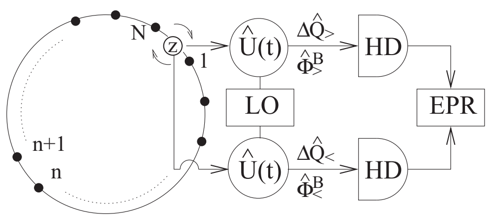

FIG. 1. N qubits are imbedded in a ring waveguide. The emitted signal is collected in a circulator z. One of its quadratures (charge $\Delta\hat{Q}_{\gtrless}$ or flux $\hat{\Phi}^B_{\gtrless}$) or their combination is subject to a homodyne detection (HD) after mixing with a local oscillator (LO) described effectively by a unitary operator $\hat{U}(t)$. The resulting EPR correlation between the output signals is detected.

interaction dynamics between the electromagnetic field and the qubit.

The Hamiltonian of the system is obtained by quantizing the different components' contributions to the total energy [33,34], yielding

$$\hat{H} = \sum_{n=1}^{N} V\hat{q}_n + \sum_{n'=1}^{N} \frac{C^{-1}_{n,n'}}{2}\hat{q}_{n'}\hat{q}_n - E_J^b\left\{\cos\left[\frac{2e}{\hbar}\left(\hat{\Phi}_n - \frac{\hat{\Phi}^B_n}{2}\right)\right] + \cos\left[\frac{2e}{\hbar}\left(\hat{\Phi}_n + \frac{\hat{\Phi}^B_n}{2}\right)\right]\right\} + \frac{\Delta\hat{Q}_n^2}{2\Delta C} + \frac{\left(\hat{\Phi}^B_{n+1} - \hat{\Phi}^B_n\right)^2}{2\Delta L}, \tag{1}$$

where E_J^b is the bare Josephson energy of a junction (see Fig. 2). The effective renormalized Josephson energy $E_J = E_J^b\langle 0|\cos(2e\hat{\Phi}_n/\hbar)\cos(e\hat{\Phi}^B_n/\hbar)|0\rangle$ is defined with respect to the vacuum energy state $|0\rangle$. See Appendix A for more detailed calculations. The essentially nonlinear cosine interaction terms between the transmons and the electromagnetic modes result from the Josephson junctions and have been configured to be an even function of each field amplitude. Therefore, in the symmetric arrangement of the Josephson junctions the first nonvanishing term in the coupling to these modes is quadratic.

The transmons can be close enough to each other to be coupled through the mutual capacitances. Assuming the translational invariance of the ring, $C_{n,n'}$ depends only on the distance $n - n'$ modulo N, and it is convenient to define their Fourier components $C_k = \sum_{n=1}^{N} e^{-i2\pi k(n-n')/N} C_{n,n'}/N$. Here the integer k is defined modulo N, and the capacitance energy can be written as $E_{C,k} = (2e)^2/2C_k$. Under these conditions, we can rewrite the transmon operators in their "wave vector" components as

$$\hat{\Phi}_n = \frac{\hbar}{2e}\sum_{k=1}^{N}\left(\frac{E_{C,k}}{E_J}\right)^{1/4} \frac{e^{i\frac{2\pi kn}{N}}(\hat{b}_k + \hat{b}^\dagger_{-k})}{\sqrt{2N}}, \tag{2}$$

$$\hat{q}_n = (2e)\sum_{k=1}^{N}\left(\frac{E_{C,k}}{E_J}\right)^{-1/4} \frac{e^{i\frac{2\pi kn}{N}}(\hat{b}_k - \hat{b}^\dagger_{-k})}{i\sqrt{2N}}. \tag{3}$$

The creation-annihilation operators $\hat{b}_k^\dagger$ and $\hat{b}_k$ describe plasmonlike collective excitations of charge motion with a wave number given by $K = 2\pi k/N$.

The charge and flux operators can be similarly defined through the electromagnetic field component as $\hat{\Phi}^B_n = \hat{\alpha}_n/\sqrt{\Delta C}$ and $\Delta\hat{Q}_n = \sqrt{\Delta C}\dot{\hat{\alpha}}_n$. Here $\hat{\alpha}_n$ is the vector potential for an ideal waveguide consisting of two parallel infinite planes [11]. It can be expressed through the wave vector components:

$$\hat{\alpha}_n = \sum_{k=1}^{N} e^{i2\pi kn/N}\sqrt{\frac{\hbar}{2\omega_k N}}(\hat{a}_k + \hat{a}^\dagger_{-k}), \tag{4}$$

$$\dot{\hat{\alpha}}_n = \sum_{k=1}^{N} e^{2\pi ikn/N}\sqrt{\frac{\hbar\omega_k}{2N}}(\hat{a}_k - \hat{a}^\dagger_{-k})/i, \tag{5}$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the photon creation-annihilation operators. The vacuum state is, as usual, defined from $\hat{a}_k|0\rangle = 0$ and $\hat{b}_k|0\rangle = 0$. With respect to this vacuum definition and in the weak-coupling approximation, the Hamiltonian (1) is rewritten for $V = 0$ in terms of only the circuit component characteristics, up to fourth order in the field and in a normal-ordered form, as (see Appendix A)

$$:\hat{H}: = \sum_{k=1}^{N} \hbar\epsilon_k\hat{b}_k^\dagger\hat{b}_k + \hbar\omega_k\hat{a}_k^\dagger\hat{a}_k + \frac{E_{\Delta C}}{32N}\sum_{k',l=1}^{N}\sqrt{\frac{\epsilon_{k+l}\epsilon_{k'-l}}{\omega_k\omega_{k'}}} :(\hat{b}^\dagger_{k+l} + \hat{b}_{-k-l}) \times (\hat{b}^\dagger_{k'-l} + \hat{b}_{-k'+l})(\hat{a}_k + \hat{a}^\dagger_{-k})(\hat{a}_{k'} + \hat{a}^\dagger_{-k'}):. \tag{6}$$

The first and second terms correspond, respectively, to the plasmon mode with energy spectrum $\hbar\epsilon_k = \sqrt{4E_J E_{C,k}}$ and the photon mode with $\omega_k = \sqrt{2[1-\cos(K)]/(\Delta C\Delta L) + \omega_0^2}$, where $\hbar\omega_0 = \sqrt{E_J E_{\Delta C}}$ and $E_{\Delta C} = 2e^2/(\hbar^2\Delta C)$. The plasmon spectrum is almost flat. The photon spectrum has a gap resulting from the Josephson energy contribution. Without it, the spectrum would be linear with a light speed $c = D/\sqrt{\Delta C\Delta L}$,

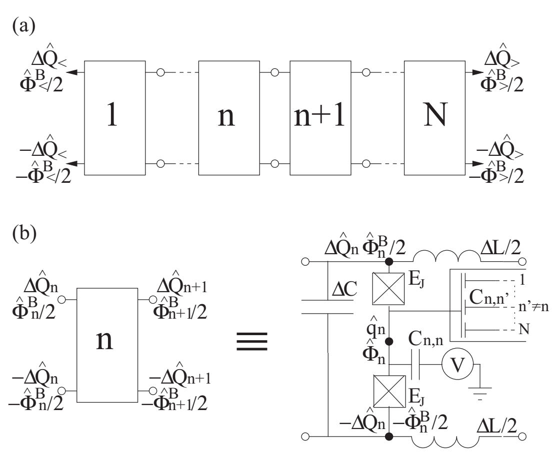

FIG. 2. Quantum circuit corresponding to the Hamiltonian (1). (a) The ring waveguide cut at the circulator is represented by an array of transmons ($n = 1, 2, \ldots, N$) coupled to the microwave radiation mode. The charges $\pm\Delta\hat{Q}_{\gtrless}$ and fluxes $\pm\hat{\Phi}^B_{\gtrless}/2$ are the output components for homodyne detection. (b) Transmon qubit embedded in a waveguide. Each node (solid black dot) is associated with the excess charge $\hat{q}_n$ and flux $\hat{\Phi}_n$ operators describing the transmon and the incremental charge $\pm\Delta\hat{Q}_n$ and flux $\pm\hat{\Phi}^B_n/2$ operators describing the EM field within the waveguide.

where D is the transmon interdistance. The third term is the quartic interaction responsible for the coupling $k + k' \leftrightarrow (k+l) + (k'-l)$ between the radiation and the transmon qubits and is negligible only if $E_{\Delta C} \langle\langle \epsilon_k, \omega_k$. Note that we neglect the quartic self-modulation terms responsible for anharmonicity [2] for both fields since these are even weaker than the coupling. This term would have produced additional (anti)bunching effects in the microwave regime [26,27].

III. THE DISCHARGE EXPERIMENT

A. Mathematical description

We start by adiabatically applying to each qubit the potential V, producing the initial charge with the nonzero expectation $\langle \hat{q}_n \rangle = C_0 V$ interpreted as the displacement of the vacuum state. Then the potential is suddenly dropped to zero. The qubit islands begin discharging, emitting in the process entangled pairs of photons. The corresponding transmon state is a coherent state with $k = 0$. Its amplitude at $t = 0$ is

$$\varphi_0 \equiv \sqrt{N} \langle \hat{b}_0 \rangle = i \sqrt{E_J/\epsilon_0} (eV/E_{C,0}). \tag{7}$$

The qubit regime is recovered in the case of the faint coherent state; that is, the intensity is weak enough to neglect states with a photon number higher than 1 in the coherent superposition. Parametrizing the squeezed radiation mode with the squeezing amplitude $r_k(t)$ and the phase $\theta_k(t)$, the full ansatz for the quantum state of radiation in the waveguide is (see Appendix B)

$$|\Psi(t)\rangle = \prod_{k=1}^{N/2} e^{r_k (e^{-2i\theta_k} \hat{a}_k^\dagger \hat{a}_{-k}^\dagger - e^{2i\theta_k} \hat{a}_k \hat{a}_{-k})} \hat{D}(t)|0\rangle, \tag{8}$$

with the displacement unitary transformation $\hat{D}(t) = \exp(\sqrt{N}\varphi \hat{b}_0^\dagger - \sqrt{N}\varphi^* \hat{b}_0)$.

From the Lagrangian $\mathrm{Re}[\langle \Psi(t)| i\hbar\partial_t - : \hat{H} : |\Psi(t)\rangle]$ we obtain the dynamical equations:

$$i\dot{\varphi} = \epsilon_0 \varphi + \frac{E_{\Delta C}\epsilon_0}{8N}(\varphi + \varphi^*) \times \sum_{k=1}^{N/2} \frac{\cosh(2r_k) - 1 + \cos(2\theta_k)\sinh(2r_k)}{\hbar\omega_k}, \tag{9}$$

$$\dot{\theta}_k = \omega_k + \frac{E_{\Delta C}\epsilon_0}{16}(\varphi + \varphi^*)^2 \frac{1 + \cos(2\theta_k)\coth(2r_k)}{\hbar\omega_k}, \tag{10}$$

$$\dot{r}_k = \frac{E_{\Delta C}\epsilon_0}{16}(\varphi + \varphi^*)^2 \frac{\sin(2\theta_k)}{\hbar\omega_k}. \tag{11}$$

In the short-time limit, assuming that all the capacitances are of the same order of magnitude and taking realistic values for the system parameters ($E_{C,0} \sim E_{\Delta C} \sim 1$ GHz, $E_J \sim 10$ GHz), we can make the following direct estimates. The rate of squeezing is $\dot{r}_k(0) \sim E_J (eV/E_{\Delta C})^2/\hbar \sim 1$ GHz for the initial voltage $V = 1\ \mu$V. The relative charge leakage, $\langle \Delta q_n \rangle / \langle q_n \rangle \sim \sqrt{E_J/E_{C,0}} (eVt/\hbar)^2 \sim 10^{18} t^2$(s), is quadratic in time.

In the long-time limit, the equations are solved in the rotating wave approximation in Appendix B. The decaying of two transmon excitations into two photons satisfies the number and energy conservation $2\epsilon_0 = 2\omega_k$. We consider two distinct cases of a transmon excitation decaying into either a continuum of photon modes or a single mode.

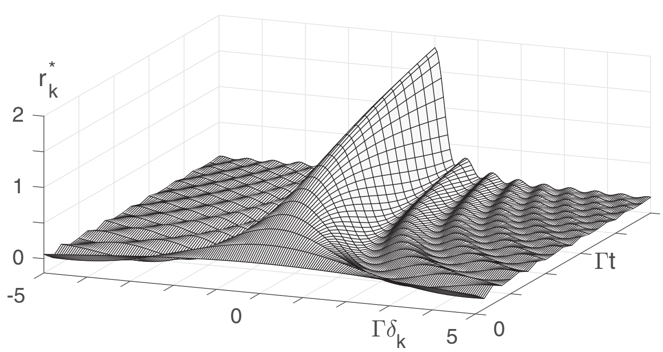

FIG. 3. Dimensionless squeezing parameter $r_k^*(t) = (16\hbar\Gamma/E_{\Delta C})[r_k(t)/|\phi_0|^2]$ as a function of time and detuning frequency for $\epsilon_0 \gg \Gamma$.

B. Decay into a continuum

In the large-N limit and for weak squeezing $r_k(t) \leqslant 1$, we can approximate the phase by $\theta_k(t) = \hbar\omega_k t - \pi/4$. The solution for the transmon field in the continuum limit is then $\varphi(t) = ie^{-i\epsilon_0 t}\varphi_0/\sqrt{1 + \Gamma t}$ with an inverse power decay rate:

$$\Gamma = \left(\frac{E_{\Delta C} eV}{16 E_{C,0}}\right)^2 \frac{E_J/\hbar^3}{\sqrt{\left(\omega_{\frac{N}{2}}^2 - \epsilon_0^2\right)\left(\epsilon_0^2 - \omega_0^2\right)}} \sim \frac{C_0 V^2}{\hbar}. \tag{12}$$

This rate corresponds to the capacitance energy perturbation introduced initially and has to be much less than the plasmon frequency ϵ_0 (typically in the 1–10 GHz range) but much larger than any decoherence rate (in the megahertz range) [1]. For the squeezing parameter, we obtain

$$r_k(t) = \frac{E_{\Delta C}}{16} \int_0^t dt' \frac{|\varphi(t')|^2 \epsilon_0 \cos[2\delta_k t']}{\hbar^2 \omega_k}, \tag{13}$$

where $\delta_k = \omega_k - \epsilon_0$ is the detuning frequency. Figure 3 represents its growth with time and concentration at zero detuning.

The total number of photons in one direction $k \gtrless 0$ can then be estimated as

$$\left\langle \hat{N}_{ph}^{\gtrless} \right\rangle = \sum_{k=1}^{N/2} \sinh^2[r_k(t)] = N \frac{|\varphi_0|^2 - |\varphi(t)|^2}{2}. \tag{14}$$

The photon waves along the two opposite directions have the perfect entanglement correlation $\langle (\delta\hat{N}_{ph}^{>} - \delta\hat{N}_{ph}^{<})^2 \rangle = 0$.

Their quadratures are also correlated and are measured through a homodyne detection after mixing them with a local oscillator of frequency ω that is simply described via the unitary operator $\hat{U}(t) = e^{i(\omega t - \pi/4)\sum_{k=1}^{N} \hat{a}_k^\dagger \hat{a}_k}$. The effective output signals are

$$\hat{\alpha}_n^{\mathrm{out}} = \hat{U}^\dagger(t)\hat{\alpha}_n\hat{U}(t), \quad \dot{\hat{\alpha}}_n^{\mathrm{out}} = \hat{U}^\dagger(t)\dot{\hat{\alpha}}_n\hat{U}(t). \tag{15}$$

Their averages are zero. However, we determine EPR correlations for the Fourier components $\hat{\alpha}_k^{\mathrm{out}}, \dot{\hat{\alpha}}_k^{\mathrm{out}}$ of these continuous variables relative to the shot noise level [30]:

$$\frac{\left\langle \left(\hat{\alpha}_k^{\mathrm{out}} - \hat{\alpha}_{-k}^{\mathrm{out}}\right)^2 \right\rangle}{2\left\langle \hat{\alpha}_k^{\mathrm{out}\,2} \right\rangle|_{r_k=0}} = \frac{\left\langle \left(\dot{\hat{\alpha}}_k^{\mathrm{out}} + \dot{\hat{\alpha}}_{-k}^{\mathrm{out}}\right)^2 \right\rangle}{2\left\langle \dot{\hat{\alpha}}_k^{\mathrm{out}\,2} \right\rangle|_{r_k=0}} \overset{\omega \to \omega_k}{=} e^{-2r_k(t)}. \tag{16}$$

These correlations become important for large squeezing. The corresponding physical quadratures are the charge $\Delta\hat{Q}_{\gtrless} = \sqrt{\Delta C}\dot{\hat{\alpha}}^{\rm out}_{\pm k}$ and the flux $\hat{\Phi}^B_{\gtrless} = \hat{\alpha}^{\rm out}_{\pm k}/\sqrt{\Delta C}$ in the waveguide, which are, respectively, anticorrelated and correlated.

C. Oscillation with two modes

In the case of long wavelengths (gigahertz), the frequency separation between modes in the ring becomes large. We can then select only two entangled modes $\pm k$ only in Eqs. (9), (10), and (11), which interact with a resonant transmon mode. Other photon modes are not perturbed.

Besides the interaction terms for the transition, an additional modulation phase term affects the transition frequency [21]. The maximum squeezing that can be reached is $r_m \overset{N\to\infty}{=} \ln(2N|\varphi_0|^2)/2$, corresponding to the total depletion $|\varphi(t)|^2 = 0$. For simplicity, we shall assume the phase modulation term is constant, which implies the restriction to values $r_k(t) \leqslant r_m - \sqrt{N/2}e^{-r_m}$. We define the dimensionless parameters $\tilde{t} = tE_{\Delta C}\epsilon_0/(32\hbar\omega_k)$ and $\tilde{\delta} = \hbar\delta_k 32\omega_k/E_{\Delta C}\epsilon_0$. Two regimes are considered below.

(1) No phase modulation. For short times, we note that the fastest squeezing rate is achieved if the phase-matching condition $\tilde{\delta} = 4|\varphi_0|^2$ is satisfied. Using this condition, the phase modulation can be neglected, and the squeezing parameter evolves towards r_m. The explicit expression is

$$r_k(t) = \frac{1}{2}\ln\left(\frac{e^{2r_m} + e^{-4\sinh(2r_m)\tilde{t}/N}}{1 + e^{2r_m}e^{-4\sinh(2r_m)\tilde{t}/N}}\right). \tag{17}$$

(2) Weak depletion. When the detuning is not phase matched, the charge leakage from the island can be neglected, i.e., $|\phi(t)| \simeq |\phi_0|$. For a small value of detuning within the interval $-2|\varphi_0|^2 \leqslant \tilde{\delta} \leqslant 6|\varphi_0|^2$, the photon number grows exponentially: $N_{ph}(t) = \sinh^2[r_k(t)] = 8|\varphi_0|^2\sinh^2(\Omega\tilde{t})/\Omega^2$, with the characteristic angular frequency $\Omega = \sqrt{|(\tilde{\delta} - 4|\varphi_0|^2)^2 - 4|\varphi_0|^4|}$. Outside this interval, the solution becomes $N_{ph}(t) = 8|\varphi_0|^2\sin^2(\Omega\tilde{t})/\Omega^2$, which corresponds to a Rabi-like oscillation between the plasmon mode and the photon modes. This Rabi-like superposition of a plasmon state and an EPR photon state illustrates the rich possibilities offered by this qubit line device for quantum design.

IV. CONCLUSIONS

We proposed the superconducting transmon line embedded in a ring waveguide as a generator of entangled beams of microwave radiation. Using the fully quantum description, we can describe the scattering process between photons and the collective transmon excitation. We also showed that high squeezing may be obtained in the long-wavelength regime, allowing for a genuine EPR-like experiment in a microchip device. An interesting extension of this design would be a parametric amplifier analogous to those in the optical range for quantum imaging [21,30].

ACKNOWLEDGMENTS

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APPENDIX A: PHASE AND CHARGE BASIS FORMALISM

We start from the quantum Hamiltonian given by Eq. (1) (see also Fig. 1). The working states of a transmon are not eigenstates of either charge $\hat{q}_n$ or phase $\hat{\phi}_n$. The phase is defined only within the value range $[-\pi, \pi]$, which differs from the range of a continuous quantum variable within the real axis. Therefore, the subsequent developments are valid as long as its uncertainty remains within these bounded values.

A real field like $\hat{\phi}_n$ can be written as a linear combination of creation and annihilation operators $\hat{c}$ and $\hat{c}^\dagger$. Using the identity $\exp(if\hat{c} + if^*\hat{c}^\dagger) = \langle 0|\exp(if\hat{c} + if^*\hat{c}^\dagger)|0\rangle \exp(if^*\hat{c}^\dagger)\exp(if\hat{c})$, with $\langle 0|\exp(if\hat{c} + if^*\hat{c}^\dagger)|0\rangle = \exp(-|f|^2/2)$, we can expand the cosine terms in the Hamiltonian to second order:

$$\cos\hat{\phi}_n = \langle 0|\cos\hat{\phi}_n|0\rangle\left[1 - \frac{:\hat{\phi}_n^2:}{2} + \dots\right], \tag{A1}$$

where we will define the vacuum state $|0\rangle$ later and where we used the normal ordering $:\cdots:$. According to the phase-number uncertainty principle, this procedure is valid as long as the charge number perturbation is large. For the coherent-state ansatz we shall use in the Appendix B, this is the case as long as the transmon capacitance energy $E_{C,k}$ is much smaller than the Josephson energy E_J.

Similarly, for the electromagnetic field,

$$\cos\left[\frac{e}{\hbar}\frac{\hat{\alpha}_n}{\sqrt{\Delta C}}\right] = \langle 0|\cos\left[\frac{e}{\hbar}\frac{\hat{\alpha}_n}{\sqrt{\Delta C}}\right]|0\rangle\left[1 - \frac{e^2 :\hat{\alpha}_n^2:}{2\hbar^2\Delta C} + \dots\right]. \tag{A2}$$

This approximation is valid in the weak-coupling regime $E_{\Delta C}\langle\langle E_J$ and will be justified *a posteriori* at the end of this Appendix.

Using these expansions, the Hamiltonian becomes

$$\hat{H} = \frac{1}{2}\sum_{n,n'=1}^{N} C^{-1}_{n,n'}\hat{q}_n\hat{q}_n - 2E_J\sum_{n=1}^{N}\left(1 - \frac{:\hat{\phi}_n^2:}{2}\right)\left(1 - \frac{e^2:\hat{\alpha}_n^2:}{2\hbar^2\Delta C}\right) + \frac{1}{2}\sum_{n=1}^{N}\left[\dot{\hat{\alpha}}_n^2 + \frac{c^2}{D^2}\hat{\alpha}_n(2\hat{\alpha}_n - \hat{\alpha}_{n-1} - \hat{\alpha}_{n+1})\right] + \mathcal{O}(\hat{\phi}_n, \hat{\alpha}_n)^4, \tag{A3}$$

where we have defined the renormalized Josephson energy $E_J = E_J^b \langle 0| \cos\hat{\phi}_n \cos[\frac{e}{\hbar}\frac{\hat{\alpha}_n}{\sqrt{\Delta C}}]|0\rangle$. Separating the vacuum energy contribution $E_{\rm vac}$ from the normal-ordered part, the Hamiltonian becomes

$$\hat{H} = E_{\rm vac} + E_J \sum_{n=1}^{N} :\hat{\phi}_n^2: + \frac{1}{2}\sum_{n,n'=1}^{N} C_{n,n'}^{-1}(2e)^2 :\hat{q}_n\hat{q}_{n'}: - \frac{E_J E_{\Delta C}}{4\hbar^2}\sum_{n=1}^{N} :\hat{\phi}_n^2\hat{\alpha}_n^2: + \frac{1}{2}\sum_{n=1}^{N} :\left[\hat{\dot{\alpha}}_n^2 + \frac{c^2}{D^2}\hat{\alpha}_n(2\hat{\alpha}_n - \hat{\alpha}_{n-1} - \hat{\alpha}_{n+1}) + \frac{E_J E_{\Delta C}}{\hbar^2}\hat{\alpha}_n^2\right]:, \tag{A4}$$

where $E_{\Delta C} = (2e)^2/(2\Delta C)$ and where

$$E_{\rm vac} = \frac{1}{2}\sum_{n,n'=1}^{N} C_{n,n'}^{-1}\langle 0|\hat{q}_n\hat{q}_{n'}|0\rangle - \sum_{n=1}^{N}\langle 0|2E_J^b \cos\hat{\phi}_n \cos\left[\frac{e}{\hbar}\frac{\hat{\alpha}_n}{\sqrt{\Delta C}}\right] - \frac{1}{2}\left[\hat{\dot{\alpha}}_n^2 + \frac{c^2}{D^2}\hat{\alpha}_n(2\hat{\alpha}_n - \hat{\alpha}_{n-1} - \hat{\alpha}_{n+1})\right]|0\rangle. \tag{A5}$$

Since we are dealing with a ring, we use the periodic boundary condition, so that the capacitance depends on only the relative position $(n - n')$. This amounts to neglecting the edge effect of the devices at the entrance and exit of the waveguides and to concentrating mostly on the bulk effect. We can express the phase and charge operators in terms of the Fourier-transformed creation and annihilation operators. Thus, the field can be rewritten in the form

$$\hat{\phi}_n = \sum_{k=1}^{N}\left(\frac{C_k^{-1}(2e)^2}{2E_J}\right)^{1/4}\frac{e^{i\frac{2\pi kn}{N}}(\hat{b}_k + \hat{b}_{-k}^\dagger)}{\sqrt{2N}}, \quad \hat{q}_n = (2e)\sum_{k=1}^{N}\left(\frac{C_k^{-1}(2e)^2}{2E_J}\right)^{-1/4}\frac{e^{i\frac{2\pi kn}{N}}(\hat{b}_k - \hat{b}_{-k}^\dagger)}{i\sqrt{2N}}, \tag{A6}$$

where k is the wavelength defined modulo N and the capacitance matrix is expressed through its Fourier components:

$$C_{n,n'}^{-1} = \sum_{k=1}^{N}\frac{e^{i\frac{2\pi k(n-n')}{N}}C_k^{-1}}{N}, \quad C_{n,n'} = \sum_{k=1}^{N}\frac{e^{i\frac{2\pi k(n-n')}{N}}C_k}{N}, \quad C_k^{-1} = \sum_{n-n'=1}^{N} e^{-i\frac{2\pi k(n-n')}{N}}C_{n,n'}^{-1} = 1/C_k. \tag{A7}$$

After insertion of these expressions, the Hamiltonian without the electromagnetic field acquires the following form:

$$:\hat{H}: = \sum_{k=1}^{N}\hbar\epsilon_k\hat{b}_k^\dagger\hat{b}_k, \quad \hbar\epsilon_k = \sqrt{4E_J E_{C,k}}, \tag{A8}$$

where we define the intrinsic energy of the capacitance as $E_{C,k} = (2e)^2/(2C_k)$. Similarly, for the electromagnetic field, we find

$$\hat{\alpha}_n = \sum_{k=1}^{N} e^{i2\pi kn/N}\sqrt{\frac{\hbar}{2\omega_k N}}(\hat{a}_k + \hat{a}_{-k}^\dagger), \quad \hat{\dot{\alpha}}_n = \sum_{k=1}^{N} e^{2\pi ikn/N}\sqrt{\frac{\hbar\omega_k}{2N}}(\hat{a}_k - \hat{a}_{-k}^\dagger)/i. \tag{A9}$$

The vacuum (ground) state is defined from the relation $\hat{a}_k|0\rangle = 0$ and $\hat{b}_k|0\rangle = 0$. If we use the dispersion relation $\omega_k = \sqrt{2[1-\cos(2\pi k/N)]/(\Delta C\Delta L) + \omega_0^2}$ and $\omega_0 = \sqrt{E_J E_{\Delta C}}/\hbar$, we can rewrite the total Hamiltonian (A4) for $V = 0$ as Eq. (6). Using a scaling argument, the expansion (A2) is justified by Eq. (A9), provided that the total photon number is of order N and when $E_{\Delta C}\langle\langle\hbar\omega_k$ or taking the lower bound when $E_{\Delta C}\langle\langle\hbar\omega_0$ or, equivalently, when $E_{\Delta C}\langle\langle E_J$. This is the same weak-coupling condition that appears in Eq. (6) if the capacitance C_0 and ΔC have the same order of magnitude.

APPENDIX B: DECAY OF THE PLASMON (COLLECTIVE TRANSMON) EXCITATION

1. The quantum dynamical equations

Let us start with a potential V that has been adiabatically applied. Then the charge perturbation of every island is $N_s = q_n/(2e) = C_0V/(2e) = eV/E_{C,0}$. Without the interaction term in the Hamiltonian (6), the coherent state $\hat{D}(\varphi(t))|0\rangle$ is the solution with initial conditions $\varphi_0 = \varphi(0) = i\sqrt{E_J/\epsilon_0}N_s$, where the displacement operator is

$$\hat{D}(\varphi(t)) = \exp[\sqrt{N}\varphi(t)\hat{b}_0^\dagger - \sqrt{N}\varphi^*(t)\hat{b}_0]. \tag{B1}$$

It changes the operator $\hat{b}_0$ into $\hat{D}^\dagger(\varphi(t))\hat{b}_0\hat{D}(\varphi(t)) = \hat{b}_0 + \varphi(t)$ inside the Hamiltonian. Without this potential, the stationary state in the $\hat{b}_k$ representation is the vacuum state $|0\rangle$. At $t \geqslant 0$, we suddenly drop the potential V to zero. Then the subsequent dynamics is that the initial charge q_n will decrease and emit entangled pairs of photons.

In the new transformed Hamiltonian $\hat{D}^\dagger(\varphi(t))\hat{H}\hat{D}(\varphi(t)) - \hat{D}^\dagger(\varphi(t))\hbar i\partial_t\hat{D}(\varphi(t))$, we can neglect the effect of the operator $\hat{b}_k$ responsible for the fluctuations in comparison to the coherent field $\varphi(t) \gg 1$ in such a way that the resulting new form does not contain any quantum transmon field and is quadratic in only the electromagnetic field operators. Under these conditions, an exact dynamical description of the photon field is achieved using the entangled squeezed state. The full ansatz is

$$|\Psi(t)\rangle = \exp\left[\sum_{k=1}^{N/2} r_k(t)(e^{-2i\theta_k(t)}\hat{a}_k^\dagger\hat{a}_{-k}^\dagger - e^{2i\theta_k(t)}\hat{a}_k\hat{a}_{-k})\right]\hat{D}(\varphi(t))|0\rangle. \tag{B2}$$

The squeezing unitary transformation acts on the operator as $\hat{a}_k \to \cosh[r_k(t)]\hat{a}_k + \sinh[r_k(t)]e^{-2i\theta_k(t)}\hat{a}_{-k}^\dagger$, with which we determine the Lagrangian by calculating the expected value for the Hamiltonian (6) and the time derivative term:

$$\mathrm{Re}[\langle\Psi(t)|i\hbar\partial_t|\Psi(t)\rangle] = (i\hbar N/2)[\varphi^*(t)\partial_t\varphi(t) - \partial_t\varphi^*(t)\varphi(t)] + \hbar\sum_{k=1}^{N/2}\{\cosh[2r_k(t)] - 1\}\partial_t\theta_k(t), \tag{B3}$$

$$\langle\Psi(t)|\hat{H}|\Psi(t)\rangle = E_{\text{vac}} + N\hbar\epsilon_0|\varphi(t)|^2 + 2\sum_{k=1}^{N/2}\hbar\omega_k\frac{\cosh[2r_k(t)] - 1}{2} + \frac{E_{\Delta C}\epsilon_0}{16}\sum_{k=1}^{N/2}[\varphi(t)+\varphi^*(t)]^2\frac{\cosh[2r_k(t)] - 1 + \cos[2\theta_k(t)]\sinh[2r_k(t)]}{\omega_k}. \tag{B4}$$

These lead to the following Lagrange equations [see Eqs. (9), (10), and (11)]:

$$i\partial_t\varphi(t) = \epsilon_0\varphi(t) + \frac{E_{\Delta C}\epsilon_0}{8N}[\varphi(t)+\varphi^*(t)]\sum_{k=1}^{N/2}\frac{\cosh[2r_k(t)] - 1 + \cos[2\theta_k(t)]\sinh[2r_k(t)]}{\hbar\omega_k}, \tag{B5}$$

$$\partial_t\theta_k(t) = \omega_k + \frac{E_{\Delta C}\epsilon_0}{16}[\varphi(t)+\varphi^*(t)]^2\frac{1 + \cos[2\theta_k(t)]\coth[2r_k(t)]}{\hbar\omega_k}, \tag{B6}$$

$$\partial_t r_k(t) = \frac{E_{\Delta C}\epsilon_0}{16}[\varphi(t)+\varphi^*(t)]^2\frac{\sin[2\theta_k(t)]}{\hbar\omega_k}, \tag{B7}$$

or in phase and amplitude:

$$\partial_t|\varphi(t)|^2 = i\frac{E_{\Delta C}\epsilon_0}{8N}[\varphi^2(t) - \varphi^{*2}(t)]\sum_{k=1}^{N/2}\frac{\cosh[2r_k(t)] - 1 + \cos[2\theta_k(t)]\sinh[2r_k(t)]}{\hbar\omega_k}, \tag{B8}$$

$$(\partial_t + \epsilon_0)\arg\varphi(t) = -\frac{E_{\Delta C}\epsilon_0}{16N}\frac{[\varphi(t)+\varphi^*(t)]^2}{|\varphi(t)|^2}\sum_{k=1}^{N/2}\frac{\cosh[2r_k(t)] - 1 + \cos[2\theta_k(t)]\sinh[2r_k(t)]}{\hbar\omega_k}. \tag{B9}$$

For a relative phase much smaller than the phases involved, i.e., $|\theta_k(t) + \arg\varphi(t)|\langle\langle\theta_k(t), -\arg\varphi(t)$, we can use the rotating wave approximation assuming *a posteriori* that terms with a large oscillation frequency do not contribute much. This procedure amounts to eliminating terms on the right-hand sides of Eqs. (9), (10), and (11) that are oscillating much faster than the relative phase, and we obtain

$$i\partial_t\varphi(t) = \epsilon_0\varphi(t) - \sum_{k=1}^{N/2}\frac{E_{\Delta C}\epsilon_0}{16N\hbar\omega_k}\{2\varphi(t)\{\cosh[2r_k(t)] - 1\} + e^{-i2\theta_k(t)}\varphi^*(t)\sinh[2r_k(t)]\}, \tag{B10}$$

$$\partial_t\theta_k(t) = \omega_k - \sum_{k=1}^{N/2}\frac{E_{\Delta C}\epsilon_0}{32\hbar\omega_k}[e^{-i2\theta_k(t)}\varphi^{*2}(t) + e^{i2\theta_k(t)}\varphi^2(t)]\coth[2r_k(t)], \tag{B11}$$

$$\partial_t r_k(t) = -\sum_{k=1}^{N/2}\frac{E_{\Delta C}\epsilon_0}{32i\hbar\omega_k}[e^{-i2\theta_k(t)}\varphi^{*2}(t) - e^{i2\theta_k(t)}\varphi^2(t)]. \tag{B12}$$

2. Decay into a continuum

a. Decay rate

For weak coupling $E_{\Delta C}/16\langle\langle\epsilon_0, \omega_k$, we neglect the last term in Eq. (B11) and solve instead $\partial_t\theta_k(t) = \hbar\omega_k$. We obtain the solution $\theta_k(t) = \omega_k t - \pi/4$. The integration constant has been chosen for consistency if we iterate the approximation in Eq. (B11) by ensuring that $\cos[2\theta(t=0)] = 0$. Then we can deduce that the neglected term in Eq. (B11) does not contribute much at later times. Similarly, the phase can be approximated as $\arg\varphi(t) = \pi/2 - \epsilon_0 t$, and we can write $\varphi(t) = i\exp(-i\epsilon_0 t)|\varphi(t)|$. Defining the detuning $\delta_k = \omega_k - \epsilon_0$, the rotating wave approximation corresponds to the condition $|\delta_k|\langle\langle\epsilon_0, \omega_k$. As a result, we are left with the following equations:

$$\partial_t|\varphi(t)|^2 = \frac{E_{\Delta C}\epsilon_0}{16N}|\varphi(t)|^2\sum_{k=1}^{N/2}\frac{2\cos(2\delta_k t)\sinh[2r_k(t)]}{\hbar\omega_k}, \tag{B13}$$

$$\partial_t r_k(t) = -\frac{E_{\Delta C}\epsilon_0}{16}|\varphi(t)|^2\frac{\cos(2\delta_k t)}{\hbar\omega_k}. \tag{B14}$$

In the absence of coupling, we recover the oscillating solution: $\varphi(t) = i\exp(-i\epsilon_0 t)|\varphi(0)|$. We solve the system of equations in the long-time limit for weak coupling and squeezing $r_k(t) \leqslant 1$, so that $\sinh[2r_k(t)] \simeq 2r_k(t)$. Integrating the second equation, we obtain the solution for $r_k(t)$ in terms of $|\varphi(t)|$:

$$\partial_t|\varphi(t)|^2 = \frac{E_{\Delta C}\epsilon_0}{16N}|\varphi(t)|^2\sum_{k=1}^{N/2}\frac{4\cos(2\delta_k t)r_k(t)}{\hbar\omega_k}, \tag{B15}$$

$$r_k(t) = -\int_0^t dt'\frac{E_{\Delta C}\epsilon_0}{16}|\varphi(t-t')|^2\frac{\cos[2\delta_k(t-t')]}{\hbar\omega_k}. \tag{B16}$$

Substituting $r_k(t)$ by its solution, we obtain a closed linear non-Markovian equation for $|\varphi(t)|^2$:

$$\partial_t|\varphi(t)|^2 = -\left(\frac{E_{\Delta C}}{16}\right)^2\frac{\epsilon_0^2}{N}\sum_{k=1}^{N/2}\int_0^t dt'\frac{2\{\cos(2\delta_k t') + \cos[2\delta_k(2t-t')]\}|\varphi(t-t')|^2|\varphi(t)|^2}{(\hbar\omega_k)^2}. \tag{B17}$$

In the weak-coupling approximation and in the continuum limit $N \to \infty$, the Markovian limit can be taken. In these conditions we can transform the sum into an integral $\frac{1}{N}\sum_{k=1}^{N/2} \to \int_0^\pi dK/(2\pi)$, where $K = 2\pi k/N$ is the wave vector. This integration is made over the detuning frequency with the integral replaced by $\int_0^\pi dK \to \int_{\omega_0-\epsilon_0}^{\omega_{N/2}-\epsilon_0} d\delta_k/(\frac{d\omega_k}{dK})$. Since $E_{\Delta C}\langle\langle\epsilon_0, \omega_k$, we can use a scaling factor λ for the large variables $\epsilon_0 \to \epsilon_0/\lambda$ and $\omega_k \to \omega_k/\lambda$ and take the limit $\lambda \to 0$ in the integral. After all these considerations, Eq. (B17) becomes

$$\partial_t|\varphi(t)|^2 = -\lim_{\lambda\to 0}\left(\frac{E_{\Delta C}\epsilon_0}{16\hbar}\right)^2\int_{\omega_0-\epsilon_0}^{\omega_{N/2}-\epsilon_0}\frac{d\delta_k}{\pi\lambda}g(\delta_k/\lambda)\int_0^t dt'\{\cos(2\delta_k t'/\lambda) + \cos[2\delta_k(2t-t')/\lambda]\}|\varphi(t-t')\varphi(t)|^2, \tag{B18}$$

where $g(\delta_k) = 1/(\omega_k^2\frac{d\omega_k}{dK})$. We proceed to change the integration variable $t' \to \lambda t'$ in order to take the Markovian limit $\varphi(t-\lambda t') \to \varphi(t)$ and to integrate over t':

$$\partial_t|\varphi(t)|^2 = -\lim_{\lambda\to 0}\left(\frac{E_{\Delta C}\epsilon_0}{16\hbar}\right)^2\int_{\omega_0-\epsilon_0}^{\omega_{N/2}-\epsilon_0}\frac{d\delta_k}{\pi}g(\delta_k/\lambda)\left[\frac{2\sin(2\delta_k t/\lambda) - \sin(4\delta_k t/\lambda)}{4\delta_k}\right]|\varphi(t)|^4. \tag{B19}$$

After the variable change $\delta_k = s\lambda/(2t)$, we obtain

$$\begin{aligned}\partial_t|\varphi(t)|^2 &= -\lim_{\lambda\to 0}\left(\frac{E_{\Delta C}\epsilon_0}{16\hbar}\right)^2\int_{(\omega_0-\epsilon_0)t/\lambda}^{(\omega_{N/2}-\epsilon_0)t/\lambda}\frac{ds}{\pi}g(s/2t)\left[\frac{2\sin(s) - \sin(2s)}{4s}\right]|\varphi(t)|^4\\ &= -\left(\frac{E_{\Delta C}\epsilon_0}{16\hbar}\right)^2\int_{-\infty}^{\infty}\frac{ds}{\pi}g(s/2t)\left[\frac{2\sin(s) - \sin(2s)}{4s}\right]|\varphi(t)|^4.\end{aligned} \tag{B20}$$

Now if we assume that $g(\omega)$ is a smooth function over the scale $1/t$, which means in the long-time limit when $t \gg 1/(\omega_{N/2} - \omega_0)$, then we can approximate $g(s/2t) \simeq g(0)$, and after integration over s, we find

$$\partial_t|\varphi(t)|^2 = -\frac{1}{2}\left(\frac{E_{\Delta C}}{16\hbar}\right)^2\left|\frac{d\omega_k}{dK}\right|^{-1}_{\omega_k=\epsilon_0}|\varphi(t)|^4. \tag{B21}$$

Alternatively, the same result could have been obtained more simply but less rigorously from Eq. (B17) after taking the Markovian limit, neglecting the second fast-oscillating cosine, and taking the time integral over the infinite interval,

$$\partial_t|\varphi(t)|^2 = -\left(\frac{E_{\Delta C}}{16}\right)^2\frac{\epsilon_0^2}{N}\sum_{k=1}^{N/2}\int_0^\infty dt'\frac{2\cos[2(\epsilon_0-\omega_k)t']|\varphi(t)|^4}{(\hbar\omega_k)^2} = -\left(\frac{E_{\Delta C}}{16}\right)^2\frac{\epsilon_0^2}{N}\sum_{k=1}^{N/2}\frac{2\pi\delta[2(\epsilon_0-\omega_k)]|\varphi(t)|^4}{(\hbar\omega_k)^2}. \tag{B22}$$

Again, we transform the sum over $k = KN/2\pi$ into an integral over the momentum K to obtain

$$\partial_t|\varphi(t)|^2 = -\frac{|\varphi(t)|^4}{\tau}, \quad \frac{1}{\tau} = \frac{E_{\Delta C}^2}{16^2\hbar^2}\int_0^\pi\frac{dK}{2}\delta(\epsilon_0-\omega_k), \tag{B23}$$

from which we identify a relaxation rate τ. This last equation is identical to (B21) but has the form of Fermi's golden rule: the δ function indicates that the process of transforming two transmon excitations into two photons has to satisfy the energy conservation $2\epsilon_0 = 2\omega_k$. Denoting $\beta = 1/\sqrt{\Delta C \Delta L}$, the integral is calculated easily,

$$\int_0^\pi dK \delta\left(\epsilon_0 - \sqrt{2\beta^2[1-\cos(K)] + \omega_0^2}\right) = \left|\frac{d\omega_k}{dK}\right|^{-1}_{\omega_k=\epsilon_0} = 2\epsilon_0\left[\left(4\beta^2 + \omega_0^2 - \epsilon_0^2\right)\left(\epsilon_0^2 - \omega_0^2\right)\right]^{-1/2}. \tag{B24}$$

Therefore, we find an explicit expression of the relaxation rate in terms of the lower and upper photon energies:

$$\tau = \frac{\hbar^2 16^2}{\epsilon_0 E_{\Delta C}^2}\sqrt{\left(\omega_{\max}^2 - \epsilon_0^2\right)\left(\epsilon_0^2 - \omega_{\min}^2\right)} \sim \hbar\frac{\sqrt{E_J E_{C,0}}}{E_{\Delta C}^2}, \quad \omega_{\min} = \omega_0, \quad \omega_{\max} = \sqrt{\omega_0^2 + 4\beta^2} = \omega_{N/2}. \tag{B25}$$

Note that the condition of timescale separation $\tau\epsilon_0 \gg 1$ amounts to stating that we are in a transmon regime, i.e., $E_{C,0}\langle\langle E_J$ in the case $E_{C,0} \sim E_{\Delta C}$. Solving Eq. (B23), we find the transmon field as $|\varphi(t)|^{-2} = [\varphi(0)]^{-2} + t/\tau$, and using the initial condition field $\varphi_0 = N_s\sqrt{E_J/\epsilon_0}$ and carrier $N_s = eV/E_{C,0}$, we obtain an inverse power law for the decay:

$$|\varphi(t)| = \frac{N_s\sqrt{E_J/\epsilon_0}}{\sqrt{1+t/T}}, \quad T = \frac{\hbar\epsilon_0\tau}{E_J N_s^2} = \frac{\hbar\epsilon_0\tau E_{C,0}^2}{E_J(eV)^2}, \tag{B26}$$

$$\varphi(t) = ie^{-i\epsilon_0 t}\frac{N_s\sqrt{E_J/\epsilon_0}}{\sqrt{1+t/T}}. \tag{B27}$$

More explicitly, we obtain Eq. (12) for the rate $\Gamma = 1/T$.

b. Squeezing parameter

Once we know the explicit form of decay for the transmon field, we can explicitly calculate the squeezing parameter. Integrating Eq. (B16), we find

$$\begin{aligned} r_k(t) &= \frac{E_{\Delta C}}{16}\int_0^t dt' \frac{N_s^2 E_J}{1+t'/T}\frac{\cos[2(\omega_k-\epsilon_0)t']}{\hbar^2\omega_k} = \frac{E_{\Delta C}\epsilon_0\tau}{16}\int_0^t \frac{dt'}{t'+T}\frac{\cos[2(\omega_k-\epsilon_0)t']}{\hbar\omega_k} \\ &= \frac{E_{\Delta C}\epsilon_0\tau}{16\hbar\omega_k}\{\cos[T(\omega_k-\epsilon_0)]\mathrm{Si}[(\omega_k-\epsilon_0)(2t'+T)] + \sin[T(\omega_k-\epsilon_0)]\mathrm{Ci}[2(\omega_k-\epsilon_0)(2t'+T)]\}|_0^t \quad \text{(B28)} \\ &\overset{\omega_k\to\epsilon_0}{=} \frac{E_{\Delta C}\tau}{16\hbar}\ln\left[1+\Gamma t\right] \overset{t\to\infty}{=} +\infty, \quad \text{(B29)} \end{aligned}$$

where we made use of the cosine and sine integral functions. Alternatively, we can use the identity $\lim_{t'\to\infty}\sin(\omega t')/\omega \to \pi\delta(\omega)$ (in the sense of a distribution over ω) in order to obtain limit expressions for long time ($t \gg 1/\epsilon_0$):

$$\begin{aligned} r_k(t) &= \frac{E_{\Delta C}\epsilon_0\tau}{16}\int_0^t \frac{dt'}{t'+T}\frac{d}{dt'}\frac{\sin[2(\omega_k-\epsilon_0)t']}{2\hbar\omega_k(\omega_k-\epsilon_0)} \quad \text{(B30)} \\ &= \frac{E_{\Delta C}}{16}\left(\frac{1}{T} - \frac{1}{t+T}\right)\frac{\tau\pi}{2\hbar}\delta[2(\omega_k-\epsilon_0)] \overset{t\to\infty}{=} \frac{E_{\Delta C}}{16}\frac{N_s^2 E_J\pi\delta(\omega_k-\epsilon_0)}{4\hbar^2\epsilon_0}. \quad \text{(B31)} \end{aligned}$$

The squeezing parameter is related to the average photon production for each mode:

$$\langle\hat{n}_k(t)\rangle = \langle\Psi(t)|\hat{a}_k^\dagger\hat{a}_k|\Psi(t)\rangle = \sinh^2(r_k(t)) \simeq r_k^2(t). \tag{B32}$$

We use this result to estimate the total number of photons in one branch ($k > 0$ or $k < 0$):

$$N_{ph}^>(t) = N_{ph}^<(t) = \sum_{k=1}^{N/2}\langle\hat{n}_k(t)\rangle \simeq N\int_0^\pi \frac{dK}{2\pi}\langle\hat{n}_k(t)\rangle = N\left|\frac{d\omega_k}{dK}\right|^{-1}_{\omega_k=\epsilon_0}\int_{\omega_1}^{\omega_N}\frac{d\omega_k}{2\pi}\langle\hat{n}_k(t)\rangle. \tag{B33}$$

For a smooth function $f(\omega)$, we use the asymptotic equality

$$\begin{aligned} \lim_{t,t'\to\infty}\frac{d}{dt}\frac{d}{dt'}\int_{-\infty}^{\infty}d\omega f(\omega)\frac{\sin(\omega t)\sin(\omega t')}{\omega^2} &= \lim_{t,t'\to\infty}\frac{d}{dt}\frac{d}{dt'}\int_{-\infty}^{\infty}d\omega f(\omega)\frac{\cos[\omega(t-t')] - \cos[\omega(t+t')]}{2\omega^2} \\ &\overset{t,t'\to\infty}{=} \frac{d}{dt}\frac{d}{dt'}\int_{-\infty}^{\infty}d\omega f(\omega)\int_{|t-t'|}^{t+t'}du\frac{\sin(\omega u)}{2\omega} \\ &\overset{t,t'\to\infty}{=} \frac{\pi f(0)}{2}\frac{d}{dt}\frac{d}{dt'}\int_{|t-t'|}^{t+t'}du = \frac{\pi f(0)}{2}\frac{d}{dt}[1 + 1^+(t-t') - 1^+(t'-t)] \\ &\overset{t,t'\to\infty}{=} \pi f(0)\delta(t-t'). \quad \text{(B34)} \end{aligned}$$

Combining this last result with expression (B30), we find for Eq. (B33)

$$N_{ph}^{\gtrless}(t) \overset{t\to\infty}{=} N\frac{2\times 16^2\hbar^2}{\tau E_{\Delta C}^2}\left(\frac{E_{\Delta C}\epsilon_0\tau}{16\hbar\epsilon_0}\right)^2\frac{1}{4}\int_0^t\frac{dt'}{(t'+T)^2} = N\frac{\tau}{2}\left(\frac{1}{T}-\frac{1}{t+T}\right) = N(|\varphi_0|^2-|\varphi(t)|^2)/2. \tag{B35}$$

This relation allows us to identify, in the long-time limit, the conservation law stating that the number of transferred transmons corresponds to the photon number $N|\varphi(t)|^2 + 2N_{ph}(t) = N|\varphi_0|^2$. The photon fluctuations in one mode are determined similarly:

$$\langle\delta^2\hat{n}_k(t)\rangle = \langle\Psi(t)|(\hat{a}_k^\dagger\hat{a}_k)^2|\Psi(t)\rangle - \left\langle n_k^2(t)\right\rangle = \sinh^2[r_k(t)]\{1+\sinh^2[r_k(t)]\} \simeq r_k^2(t). \tag{B36}$$

We note also that photons with opposite momentum have the same correlations as those of equal momentum $\langle\delta^2\hat{n}_k(t)\rangle = \langle\delta\hat{n}_k(t)\delta\hat{n}_{-k}(t)\rangle$, so that we deduce the zero correlation associated with the entanglement $\langle[\delta\hat{n}_k(t)-\delta\hat{n}_{-k}(t)]^2\rangle = 0$. As a result, we find the correlations of the total number of photons on one branch ($k>0$ or $k<0$) for weak squeezing:

$$\langle\delta^2 N_{ph}^{\gtrless}(t)\rangle = \langle\delta N_{ph}^{<}\delta N_{ph}^{>}(t)\rangle = \sum_{k,k'=1}^{N/2}\langle\delta\hat{n}_k(t)\delta\hat{n}_{k'}(t)\rangle \simeq \sum_{k=1}^{N/2} r_k^2(t) \overset{r_k(t)\leqslant 1}{=} N_{ph}^{\gtrless}(t), \tag{B37}$$

which allows us to conclude the perfect entanglement correlation:

$$\langle[\delta N_{ph}^{>}(t) - \delta N_{ph}^{<}(t)]^2\rangle = 0. \tag{B38}$$

The corresponding quadratures are also correlated and are measured through a homodyne detection after mixing them with a local oscillator of frequency ω. The mixing plays the role of a beam splitter and is realized using a coupler between the output signal waveguide and the local oscillator one. It results in two new output signals whose subsequent measurement difference provides an outcome no longer dependent on the fast oscillation in time. Mathematically, using the unitary operator $\hat{U}(t) = e^{i(\omega t-\pi/4)\sum_{k=1}^N \hat{a}_k^\dagger\hat{a}_k}$, this coupler transforms the output quadratures into

$$\hat{\alpha}_n^{\text{out}} = \hat{U}^\dagger(t)\hat{\alpha}_n\hat{U}(t) = \sum_{k=1}^N\frac{e^{i2\pi kn/N}}{\sqrt{N}}\hat{\alpha}_k^{\text{out}} = \sum_{k=1}^N e^{i2\pi kn/N}\sqrt{\frac{\hbar}{2\omega_k N}}(e^{i(\omega t-\pi/4)}\hat{a}_k + e^{-i(\omega t-\pi/4)t}\hat{a}_{-k}^\dagger), \tag{B39}$$

$$\dot{\hat{\alpha}}_n^{\text{out}} = \hat{U}^\dagger(t)\dot{\hat{\alpha}}_n\hat{U}(t) = \sum_{k=1}^N\frac{e^{2\pi ikn/N}}{\sqrt{N}}\dot{\hat{\alpha}}_k^{\text{out}} = \sum_{k=1}^N e^{2\pi ikn/N}\sqrt{\frac{\hbar\omega_k}{2N}}(e^{i(\omega t-\pi/4)}\hat{a}_k - e^{-i(\omega t-\pi/4)}\hat{a}_{-k}^\dagger)/i. \tag{B40}$$

Their averages are all zero, but their fluctuations are present even for an output vacuum field and are referred to as the shot noise level. More precisely, $\langle(\hat{\alpha}_k^{\text{out}}-\hat{\alpha}_{-k}^{\text{out}})^2\rangle|_{r_k=0} = \hbar/2\omega_k$, and $\langle(\dot{\hat{\alpha}}_k^{\text{out}}+\dot{\hat{\alpha}}_{-k}^{\text{out}})^2\rangle|_{r_k=0} = \hbar\omega_k/2$. We determine the EPR entanglement correlations for the Fourier components $\hat{\alpha}_k^{\text{out}}$, $\dot{\hat{\alpha}}_k^{\text{out}}$ of these continuous variables relative to the shot noise level as a reference:

$$\frac{\left\langle\left(\hat{\alpha}_k^{\text{out}}-\hat{\alpha}_{-k}^{\text{out}}\right)^2\right\rangle}{\left\langle\left(\hat{\alpha}_k^{\text{out}}-\hat{\alpha}_{-k}^{\text{out}}\right)^2\right\rangle|_{r_k=0}} = \frac{\left\langle\left(\dot{\hat{\alpha}}_k^{\text{out}}+\dot{\hat{\alpha}}_{-k}^{\text{out}}\right)^2\right\rangle}{\left\langle\left(\dot{\hat{\alpha}}_k^{\text{out}}+\dot{\hat{\alpha}}_{-k}^{\text{out}}\right)^2\right\rangle\Big|_{r_k=0}} = \cosh[2r_k(t)] - \sinh[2r_k(t)]\cos[2(\omega-\omega_k)t] \overset{\omega\to\omega_k}{=} \exp[-2r_k(t)]. \tag{B41}$$

For this frequency ω, we find the corresponding correlated quadratures for the flux average and the total charge propagating to the right ($k>0$) or to the left ($k<0$):

$$\hat{\phi}_{\gtrless}^B = \frac{1}{\sqrt{\Delta C}}\hat{\alpha}_{\pm k}^{\text{out}}\Big|_{\omega=\omega_k}, \quad \Delta\hat{Q}_{\gtrless} = \sqrt{\Delta C}\,\dot{\hat{\alpha}}_{\pm k}^{\text{out}}\Big|_{\omega=\omega_k}. \tag{B42}$$

To assess the entanglement produced between the two output beams, we determine the partial entropy of one of the beams. If B is the Hilbert space associated with the transmon field, we find the reduced matrix density from the partial trace:

$$\hat{\rho}_{k>0} = \text{tr}_{k<0,B}[|\Psi(t)\rangle\langle\Psi(t)|] = \prod_{k=1}^{N/2}\cosh[r_k(t)]\sum_{n_k=0}^{\infty}|n_k\rangle\tanh^{n_k}[r_k(t)]\langle n_k|. \tag{B43}$$

With this expression, we determine the entanglement entropy:

$$\begin{aligned} S(t) &= -\text{tr}(\hat{\rho}_{k>0}\ln\hat{\rho}_{k>0}) = 2\sum_{k>0}\cosh^2[r_k(t)]\ln\{\cosh[r_k(t)]\} - \sinh^2[r_k(t)]\ln\{\sinh[r_k(t)]\} \\ &\overset{r_k(t)\ll 1}{\simeq} -2\sum_{k>0}\ln[r_k(t)]r_k^2(t). \end{aligned} \tag{B44}$$

When $r_k \neq 0$, this entanglement is always positive, and for large r_k, it tends towards infinity.

3. Two electromagnetic entangled mode regimes

a. Two-mode dynamical equations

In a small-size ring, the quantization of the wave number restricts the number of electromagnetic modes with a larger energy level spacing that does not form a continuum anymore. Therefore, we can select two particular entangled modes interacting in resonance with the transmon mode. The frequency of these modes $\omega_k = \epsilon_0 + \delta_k$ has a detuning frequency that is adjusted for the optimal entanglement. By applying the two-mode approximation, the system (B10), (B11), and (B12) simplifies into

$$i\partial_t\varphi(t) = \epsilon_0\varphi(t) - \frac{E_{\Delta C}\epsilon_0}{16N\hbar\omega_k}\{2\varphi(t)\{\cosh[2r_k(t)] - 1\} + e^{-i2\theta_k(t)}\varphi^*(t)\sinh[2r_k(t)]\}, \tag{B45}$$

$$\partial_t\theta_k(t) = \omega_k - \frac{E_{\Delta C}\epsilon_0}{32\hbar\omega_k}[e^{-i2\theta_k(t)}\varphi^{*2}(t) + e^{i2\theta_k(t)}\varphi^2(t)]\coth[2r_k(t)], \tag{B46}$$

$$\partial_t r_k(t) = -\frac{E_{\Delta C}\epsilon_0}{32i\hbar\omega_k}[e^{-i2\theta_k(t)}\varphi^{*2}(t) - e^{i2\theta_k(t)}\varphi^2(t)]. \tag{B47}$$

We define the pair numbers $N_{ph}(t) = \cosh[2r_k(t)] - 1$. This set of equations is solved by using the conservation of the particle numbers and energy:

$$\mathcal{N} = N|\varphi(t)|^2 + \cosh[2r_k(t)] - 1, \tag{B48}$$

$$\begin{aligned}\mathcal{E} &= \hbar\epsilon_0 N|\varphi(t)|^2 + \hbar\omega_k\{\cosh[2r_k(t)] - 1\}\\ &\quad - \frac{E_{\Delta C}\epsilon_0}{32\omega_k}\{4|\varphi(t)|^2\{\cosh[2r_k(t)] - 1\} + [e^{-i2\theta_k(t)}\varphi^{*2}(t) + e^{i2\theta_k(t)}\varphi^2(t)]\sinh[2r_k(t)]\}\end{aligned} \tag{B49}$$

Besides the interaction terms for the transfer between the transmon modes and the mode pair, the energy term also contains an interaction with a modulation phase term that affects the transition frequency [the first term in the last line of Eq. (B49)]. In order to go further, we define the set of dimensionless parameters: $\tilde{t} = tE_{\Delta C}\epsilon_0/(32\hbar\omega_k)$ and $\tilde{\delta} = \hbar\delta_k 32\omega_k/E_{\Delta C}\epsilon_0$. At time $t = 0$, the initial conditions are $\mathcal{E} = \hbar\epsilon_0\mathcal{N} = \hbar\epsilon_0 N|\varphi_0|^2$. By eliminating the angle $2\arg\varphi(t) + 2\theta_k(t)$ between Eq. (B12) and the energy conservation relation (B49), we obtain

$$[\tilde{\delta} - 4|\varphi(t)|^2]\{\cosh[2r_k(t)] - 1\} = \pm\sinh[2r_k(t)]\sqrt{4|\varphi(t)|^4 - [\partial_{\tilde{t}}r_k(t)]^2}. \tag{B50}$$

Rearranging the terms, we obtain the closed set of equations:

$$\begin{aligned}\partial_{\tilde{t}}r_k(t) &= \sqrt{4|\varphi(t)|^4 - [\tilde{\delta} - 4|\varphi(t)|^2]\coth[r_k(t)]},\\ N|\varphi_0|^2 &= N|\varphi(t)|^2 + \cosh[2r_k(t)] - 1.\end{aligned} \tag{B51}$$

In terms of the pair population, these are rewritten as

$$\begin{aligned}\partial_{\tilde{t}}N_{ph}^{1/2}(t) &= \sqrt{4[N_{ph}(t) + 2]|\varphi(t)|^4 - [\tilde{\delta} - 4|\varphi(t)|^2]^2 N_{ph}(t)},\\ N|\varphi_0|^2 &= N|\varphi(t)|^2 + N_{ph}(t).\end{aligned} \tag{B52}$$

b. No phase modulation

For $t = 0$, we note that the fastest squeezing rate is achieved for $\tilde{\delta} = 4|\varphi_0|^2$. In that case, if we neglect the subsequent phase modulation, the squeezing parameter obeys $\partial_t r_k(t) = 2|\varphi(t)|^2$ and therefore starts from zero until reaching a maximum value r_m obtained by imposing the condition $|\varphi(t)|^2 = |\varphi_0|^2 - \{\cosh[2r_k(t)] - 1\}/N = 0$. We find the solution

$$r_m = \tfrac{1}{2}\ln[1 + N|\varphi_0|^2 + \sqrt{(1 + N|\varphi_0|^2)^2 - 1}] \overset{N\to\infty}{=} \tfrac{1}{2}\ln(2N|\varphi_0|^2). \tag{B53}$$

Next, we solve the equation $\partial_{\tilde{t}}r_k(t)/2 = |\varphi_0|^2 - \{\cosh[2r_k(t)] - 1\}/N$ to obtain Eq. (17). For large N and large $|\phi_0|^2$, the presence of phase modulation at later time restricts the validity of Eq. (17) to squeezing values smaller than r_m. Indeed, using the number conservation equation, we find an upper bound for the neglected term on the right-hand side of Eq. (B51). This term is much smaller in comparison to the first term provided that

$$[\tilde{\delta} - 4|\varphi(t)|^2]\coth[r_k(t)] = 4\frac{\cosh[2r_k(t)] - 1}{N}\coth[r_k(t)] \langle\langle 4|\varphi(t)|^4 = 4\left||\varphi_0|^2 - \frac{\cosh[2r_k(t)] - 1}{N}\right|^2. \tag{B54}$$

Using this inequality in the limit for large squeezing, we find $r_k(t) < r_m - \sqrt{N/2}e^{-r_m}$.

c. Weak depletion

We consider the case where the modulation term is present and the charge depletion in the transmon is weak, i.e., $\varphi(t) \simeq \varphi_0$ or, equivalently, $r_k(t)\langle\langle r_m$. Using this simplification, Eq. (B52) becomes easily solvable:

$$\partial_{\tilde{t}} N_{ph}^{1/2}(t) = \sqrt{4[N_{ph}(t)+2]|\varphi_0|^4 - [\tilde{\delta} - 4|\varphi_0|^2]^2 N_{ph}(t)}. \tag{B55}$$

The result is that for values of detuning within the interval $-2|\varphi_0|^2 \leqslant \tilde{\delta} \leqslant 6|\varphi_0|^2$, the photon population grows exponentially:

$$N_{ph}(t) = 8|\varphi_0|^2 \frac{\sinh^2(\Omega\tilde{t})}{\Omega^2}, \quad \Omega = \sqrt{|4|\varphi_0|^4 - [\tilde{\delta} - 4|\varphi_0|^2]^2|}. \tag{B56}$$

Outside this interval, we obtain instead the oscillatory behavior:

$$N_{ph}(t) = 8|\varphi_0|^2 \frac{\sin^2(\Omega\tilde{t})}{\Omega^2}. \tag{B57}$$

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