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Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill.

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Abstract

Understanding conceptual relationships is an important aspect of learning arithmetic. Most studies of arithmetic, however, do not distinguish between children's understanding of a concept, and their ability to identify situations in which it might be relevant. We compared 8- to 9-year-old children's use of a computational shortcut based on the inverse relationship between addition and subtraction, in problems where it was transparently applicable (e.g. $17 + 11 - 11 = \square$) and where it was not (e.g. $15 + 11 - 8 - 3 = \square$). Most children were able to construct inverse transformations and apply the shortcut in at least some situations, although they used the shortcut more for problems where it was transparently applicable. There were individual differences in the relationship between children's understanding of the inverse relationship and computational skill that have implications for theories of mathematical development.

Keywords: arithmetic, conceptual understanding, individual differences, problem solving.

Can children construct inverse relations in arithmetic? Evidence for individual differences in the development of conceptual understanding and computational skill.

Learning arithmetic is challenging. Not only must children learn to add, subtract, multiply and divide: they must also understand how these operations are related. Sophisticated problem solving in mathematics makes use of inferences, analogies and conceptually-based shortcuts to reduce the computational challenges of solving problems. For example, children use inferences and analogies to develop new strategies and approaches to solve proportional relations problems (e.g. Singer-Freeman & Goswami, 2001); children who understand commutativity reduce by half the number of addition or multiplication facts that they must learn (e.g. Baroody, Ginsburg, & Waxman, 1983); children who understand associativity are able to use decomposition strategies such as 5 + 7 = 5 + 5 + 2 = 10 + 2 = 12 (e.g. Canobi, Reeve, & Pattison, 1998); and children who understand the inverse relation between addition and subtraction realize that adding and subtracting the same quantity leaves the initial quantity unchanged and so no computation is necessary (e.g. Bryant, Christie, & Rendu, 1999).

Children's ability to perform computations and to understand these deeper conceptual underpinnings of arithmetic are related in a complex fashion over development. The iterative model of mathematical learning suggests that developments in conceptual and procedural aspects of mathematics are interrelated (e.g. Baroody & Ginsburg, 1986; Carpenter, 1986; Rittle-Johnson & Siegler, 1998; Rittle-Johnson, Siegler, & Alibali, 2001). Improvements in conceptual understanding can lead to advances in procedural skill and vice-versa. Thus, there is a bidirectional, causal relationship between the developments of each type of knowledge (Rittle-Johnson et al., 2001). At any point in time, therefore, children may have partial understanding of particular concepts and knowledge of procedures. Conceptual understanding is both a constraint on the development of procedures and a product of this development (Sophian, 1997).

The aim of mathematical instruction should be to help children to integrate these two types of knowledge. Children should be able to select and perform procedures based on an understanding of when and why these procedures are appropriate. We can distinguish the meaningful use of different strategies (adaptive expertise; cf Baroody & Dowker, 2003; Torbeyns, Verschaffel, & Ghesquière, 2005) from the rote application of procedures (routine expertise). Furthermore, children should understand how arithmetical concepts may be relevant in different situations. With integrated conceptual and procedural knowledge, children would be able to judge when they need to compute and when they can use conceptually-based shortcuts or inferences to solve a problem.

Most investigations of children's conceptual understanding of arithmetic, however, do not distinguish between children's ability to make use of an arithmetical concept from their ability to identify when a concept may be relevant. Problem solving in everyday life nearly always involves identifying whether a concept is relevant. Problem solvers have to impose their solution on the situations confronting them. If the solution depends on an inference, for example, the successful problem-solver must be able to make that inference, but usually he or she has first to recognize that an inference is possible and then to search for or construct the premises that are needed for the inference, in order to make the actual inference. Analogical solutions to problems are much the same: the person solving the problem usually has to understand that an analogy might be useful and to discern the set of relations that could lead to the analogy, as well as to be able to make the analogy itself

The constructive elements of problem-solving play no part in many of the mathematical tasks that psychologists give children in experiments, and which teachers give them in the classroom. When psychologists and teachers give children sums to do, they test the children's ability to make a calculation, but not their ability to organize the information that they have in order to make it amenable to the kind of calculation that they have decided to make. The constructive part of the task has been done for the children already. These tasks have their value, since we often need to know whether or not children can make particular kinds of calculations and whether or not they possess particular kinds of conceptual knowledge. However, we also need to find out how children spot the need for specific types of calculation when this isn't presented to them on a plate, and how they actively reconstruct the information given to them so that it fits the mathematical solution which they wish to apply. Previous research on mathematical development has so far failed to make comparisons between children's ability to perform these two types of tasks. Here we investigate how well children can apply conceptual principles both on tasks where they are transparently applicable and on more complex tasks where children first have to identify that the concept is relevant and then reconstruct the problem in order to make certain conceptually based inferences.

An interesting illustration of why comparisons between transparent and complex tasks are necessary is provided by existing research on children's understanding of inversion. The basis for this research is the quite reasonable assumption that people cannot understand either addition or subtraction effectively unless they also understand the inverse relations between these two operations (e.g. Bryant et al., 1999; Bryant & Nunes, 2002; Piaget, 1952; Piaget & Moreau, 2001). This assumption has led to several studies of performance in tasks in which children were given $a + b - b = \square$ sums (e.g. $27 + 13 - 13 = \square$). These studies established signs of a patchy understanding of inversion among young school children from four years on: some, though not all, of these children were able to solve inverse $a + b - b = \square$ problems more accurately and more rapidly than control problems, such as $a + b - c = \square$, in which an understanding of inversion is no help at all (e.g. Bisanz, LeFevre, & Gilliland, 1989; Bryant et al., 1999; Canobi, 2005; Gilmore, 2006; Gilmore & Bryant, 2006; Klein & Bisanz, 2000; Rasmussen, Ho, & Bisanz, 2003; Robinson, Ninowski, & Gray, 2006; Siegler & Stern, 1998; Stern, 1992).

The consistent evidence for some understanding of inversion in quite young children is important, but it raises new questions. The most important of these concerns the use that children and adults can make of this understanding. In everyday life people have to transform individual problems that face them in some way in order to take advantage of their knowledge of inversion. For example, a problem like $24 + 11 - 7 - 4 = \square$ is not on the face of it an inversion problem but someone who realizes that $-7 - 4 = -11$ can actively transform the problem into the inversion sum $24 + 11 - 11 = \square$. This is a good and simple example of the additional constructive element of problem solving, and it is easy to make the comparison between a problem such as this and an equivalent transparent problem. The difference between children's success with a transparent sum, for example $24 + 11 - 11 = \square$, and a non-transparent one, for example $24 + 11 - 7 - 4 = \square$, would give us a measure of how hard it is for children to impose their knowledge of inversion on the problem by transforming the problem to suit this knowledge.

There may be strong individual differences among school-children in the extent of the extra difficulty that this extra component of tasks impose on them. It has already been demonstrated that there are individual differences among children in the relationship between their understanding of arithmetical concepts and computational skill. While most children's understanding of arithmetical concepts is related to their ability to perform procedures accurately, for other children computational skill may not reflect conceptual understanding of underlying principles (Dowker, 1998; Canobi, 2004). In particular, clear patterns of individual differences have been found in children's performance on transparent inverse and control problems (Gilmore & Bryant, 2006). Three groups of children with different profiles of performance have been identified. One possesses a good understanding of inversion, as measured by their scores in transparent inversion tasks and also good calculation skill, as measured by their performance in control tasks in which they had to calculate the answer. The second group is characterized by poor understanding of inversion and poor calculation skill, and the third group by good understanding of inversion but poor calculation skill. The most interesting of these groups is the third, discrepant, group of children. Given their low calculation scores, they are surprisingly good at solving transparent inversion problems. At present, it is not clear whether this is because these children have better than expected conceptual understanding, or difficulties or delays in acquiring computational skills, and thus interpretation of these groups is problematic. These questions may be answered by examining children's performance on more complex tasks in which they must actively transform problems so that they are amenable to inversion

Gilmore & Bryant (2006) provided the first evidence for the existence of individual differences in understanding of inversion and arithmetic skill, however, alternative interpretations of the different subgroups could not be disentangled. The present paper extends this in several crucial ways. Firstly, it is important to replicate these clusters to demonstrate that they are meaningful and not a result of chance variation in the initial sample. Secondly, the present work distinguishes between alternative possible interpretations of the clusters by examining whether children apply knowledge of inversion in different situations and how this ability differs among children in relation to their computational skill. Finally, the present paper is the first study to examine children's understanding of inversion in situations where the inverse relationship is not transparently presented. Children's understanding of inversion is examined by comparing performance on problems with or without inverse transformations, both in situations where the inverse transformation is transparent and where it must be constructed. This is an important step in our knowledge of children's use of conceptual principles in arithmetic problem-solving.

Method

Participants

Sixty-eight children (33 boys and 35 girls) participated in the study. Their mean age was 8 years 7 months (range 8 years 1 month to 9 years 2 months). The children were in Year 4 classes at two primary schools (48 from School 1 and 20 from School 2). Both schools were in suburban areas and had predominantly white students. The percentage of children who qualified for free school meals was in line with the national average in School 2 and below the national average in School 1. The children were recruited by contacting a wide range of schools in the local area and sending letters to all parents in the relevant year group of schools who agreed to take part. One child declined to attempt a large number of the problems and so his data were discarded. Thus there were complete data from 67 children. All the children spoke English as their first language and none had been identified as having special educational needs.

Design and Materials

Each child completed 24 four-term arithmetic problems ($a + b - c = d$). Half of the questions were transparent inverse problems where $b = c$ (e.g. $15 + 12 - 12 = \square$), and these were matched with a control problem (e.g. $11 + 11 - 7 = \square$). The children also completed 24 five-term arithmetic problems ($a + b - c - d = e$). Half of the questions were complex inverse problems where $b = c + d$ (e.g. $15 + 11 - 8 - 3 = \square$), and these were matched with a control problem (e.g. 13 + 11 – 5 – 4 = □). The matched pairs of inverse and control problems had the same missing number and the range of operands for inverse and control problems was matched across the whole problem set. In control problems, the addend and subtrahend (or sum of subtrahends in 5-term problems) differed by at least 3, to prevent children decomposing the problem and using inversion to solve it (cf. Bryant et al., 1999). The problems were designed to be at the limit of, or just beyond, what could be solved by this age group when using computation.

The composition of the inverse and control problems was varied in two further ways. First, the order of elements in the problem was varied. Order 1 problems had the order used typically in previous studies of inversion (i.e. $a + b - b = a$; $a + b - b_1 - b_2 = a$), and the control problems were matched to this (i.e. $a + b - c = d$; $a + b - c - d = e$). Order 2 problems had the inverse elements at the start of the sum (i.e. $b - b + a = a$; $b - b_1 - b_2 + a = a$), and the control problems were matched to this ($b - c + a = d$; $b - c - d + a = e$). Second, each problem had either the 'a' term or the sum missing (e.g. $a + b - b = \square$ or $\square + b - b = a$). The children completed three examples of each problem type, see Table 1 for examples.

INSERT TABLE 1 ABOUT HERE

The element order and missing number variations were included to test the range of problems on which children could identify and make use of the inverse principle. If children have a thorough understanding of this concept then they should be able to use it in a variety of problem situations, regardless of the surface form of the problem. Previous work has found that children recognize inverse transformations more easily when the inverse elements are at the start of the sum (i.e. order 2) than when the inverse elements are after the initial term (i.e. order 1) and more easily when the sum is missing than a term on the left-hand-side of the sum (Gilmore, 2006). Including these variations in the present study allows us to test whether the same effects are found for complex as well as transparent inverse problems. Finally, including a range of inverse problems prevents children from developing a simple superficial strategy such as responding with the initial number.

Procedure

The participants were tested individually in two 20-minute sessions. In each session the children were given equal numbers of four- and five-term problems, inverse and control problems, problems with each element order and problems with the ‘a’ term or the sum missing. The order in which the sessions were completed was counterbalanced across participants. Within each session the trials were presented in a different random order for each participant.

The problems were presented on a HP laptop running SuperLab Pro (v. 2.0, Cedrus Corp). They appeared in the centre of the screen with an empty box in place of the missing number. The task was introduced as a numbers game in which the participants had to work out the missing number. At the beginning of each session there were four familiarization / practice trials which were all control problems. In each trial the problem was presented on the screen and the experimenter read it aloud twice. The children were given positive encouragement without any specific feedback throughout.

Results

The results are presented in three parts. First, children’s accuracy on inverse and control problems is examined for the whole group. Second, a cluster analysis that examines individual differences in conceptual understanding and arithmetical skill is described. Finally, children’s accuracy on inverse and control problems is examined for each subgroup separately.

Whole Group Analysis

The first analysis examined the effect of different problem factors on accuracy for the whole group of children. The main aim of this analysis was to compare performance on matched sets of inverse and control problems. More accurate responses on inverse than control problems would indicate that children have recognized and exploited the inverse transformation in the inverse problems. To do this they must understand the underlying relationship between addition and subtraction. A second aim was to compare performance on four-term (transparent) and five-term (complex) problems. Finally, the effect of different problem factor (element order, missing element) is considered.

Initial analyses revealed that there were no effects of sex, session order or school and so these factors were removed from the subsequent analyses. The significant effects reported below were also significant when analysed by items, indicating that the effects are consistent across the problem set. Children's accuracy (measured as the proportion of correct responses) on different types of problems was compared using a four-way ANOVA with problem length (four-term, five-term), problem type (inverse, control), element order (order 1, order 2) and missing element (a, sum) as repeated-measures factors.

There was a significant difference between performance on inverse and control problems overall (problem type main effect $F(1, 66) = 183.15$, $p < .001$, partial eta squared $\eta_p^2 = 0.74$): children were more accurate on inverse problems (mean = 0.76) than control problems (mean = 0.47). This main effect was qualified by a significant interaction between problem type, problem length and missing element ($F(1, 66) = 17.24$, $p < .001$, $\eta_p^2 = 0.21$; Figure 1). Simple main effects analysis revealed that children were more accurate on inverse than control problems for four-term problems with the 'a' term missing ($F(1, 66) = 116.45$, $p < .001$, $\eta_p^2 = 0.64$), four-term problems with the sum missing ($F(1, 66) = 100.87$, $p < .001$, $\eta_p^2 = 0.60$), and five-term problems with the 'a' term missing ($F(1, 66) = 145.17$, $p < .001$, $\eta_p^2 =$

0.69). However there was no difference in accuracy between inverse and control problems for five-term problems with the sum missing ($F(1, 66) = 4.94$, n.s.). This pattern of results suggests that in general children were aided by the presence of an inverse transformation in the sum. But not, however, for five-term inverse problems with the sum missing (e.g. 15 + 11 – 8 – 3 = ☐ or 9 – 6 – 3 + 15 = ☐).

The difference between four- and five-term problems varied between inverse and control problems (problem length * problem type interaction $F(1,66) = 23.94$, $p < .001$, $\eta_p^2 = 0.27$). In the inverse problems the children were more accurate with four- than with five-term items ($F(1, 66) = 48.82$, $p < .001$, $\eta_p^2 = 0.43$), but there was no equivalent effect of problem length on accuracy with control problems ($F(1, 66) < 1$). Thus, while children found transparent inverse problems easier than those that involve the active construction of the inverse elements, the addition of an extra computation stage had no effect on accuracy for control problems. This may be because the five-term problems had smaller addend and subtrahend terms than the four-term problems and children were as accurate in performing three smaller computations (e.g. 14 + 9 – 4 – 2 =) as two larger computations (e.g. 21 + 9 – 13 =).

Finally, the element order variation had some effect on accuracy. The children were more accurate for problems with the inverse elements before the ‘a’ term (order 2) than for problems with the inverse elements after the ‘a’ term (order 1; $F(1, 66) = 115.99$, $p < .001$, $\eta_p^2 = 0.64$). For inverse problems, it is possible that children are able to identify and exploit inverse transformations more easily when they are at the start of the sum. For control problems, this effect may arise from the order of operators. In order 2 control problems, the subtraction operation(s) were before the addition operation, while in order 1 control problems the addition operation was first. Children find subtraction in general more difficult than addition. Thus, children may be more accurate on problems where they perform the subtraction operation first, rather than performing it second, while they are already holding in memory the result of the preceding addition operation.

The whole group analyses described above demonstrate that mean accuracy was higher for inverse than control problems. These group effects were not driven just by the performance of a small number of participants, the vast majority of children showed an advantage for inverse problems (four-term problems: 59 children (88.1%) scored higher on the inverse problems, 7 children (10.4%) scored the same and 1 child (1.5%) scored higher on the control problems; five-term problems: 58 children (86.6%) scored higher on the inverse problems, 3 children (4.5%) scored the same and 6 children (9.0%) scored higher on the control problems). These observed frequencies are significantly different from those expected by chance (sign test four-term $z = -6.70$, $p < .001$; five-term $z = -6.20$, $p < .001$).

The analyses of accuracy revealed that at least some of the children recognized and exploited the inverse transformations in all types of transparent (four-term) inverse problem. They were reliably more accurate on these problems than those which require calculation. Furthermore, performance on five-term problems indicated that some children were able to take advantage of an inverse transformation even when they had to construct this for themselves. Their ability to do so, however, was restricted to certain types of five-term problems. Moreover, children were more accurate on inverse problems where the inverse transformation is transparent (four-term) than when it has to be constructed (five-term). Put together, these findings suggest that children are able to use their understanding of the relationship between addition and subtraction more flexibly for problems where it is transparently applicable than for problems where they have to reconstruct the problem to make it applicable. It is easier for children to use their understanding of conceptual relationships to solve problems than it is for them to recognize that the conceptual relations are relevant.

Individual Differences Analysis

Children show wide individual differences in their conceptual understanding and computational skill in arithmetic (Canobi, 2004; Dowker, 1998). These differences may in turn lead to further differences in children's ability to identify when their conceptual understanding is relevant for problem-solving. One way to examine individual differences is to look for subgroups of participants who may be behaving more similarly. This can be achieved by performing cluster analysis. Cluster analysis has been previously used to reveal differences in the relationship between children's conceptual understanding and their procedural skill in arithmetic (e.g. Canobi, 2004; Gilmore & Bryant, 2006).

To investigate individual differences, the children's accuracy scores on the four-term inverse and control problems were entered into a cluster analysis. These scores reflect the children's ability to use inversion on transparent tasks and their general calculation skill. A hierarchical cluster analysis was performed using Ward's method. The two-cluster solution accounted for 50.4% of the variance in scores; the three-cluster solution accounted for 68.4% of the variance in scores; and the four-cluster solution accounted for 76.9% of the variance in scores, but this included one very small group. Therefore, the three-cluster solution was interpreted.

The children in Cluster 1 (n = 26) had high scores on both the inverse and control four-term problems (thus this cluster was labelled 'high score'). The children in Cluster 2 (n = 21) had the lowest scores on both the inverse and control four-term problems (thus this cluster was labelled 'low score'). The children in Cluster 3 (n = 20) had high scores on the inverse problems but low scores on the control four-term problems (thus this cluster was labelled 'difference score').

The characteristics of the children in each of these clusters were compared. First, the proportion of girls and boys in each cluster was considered. There were 17 boys and 9 girls in Cluster 1, 8 boys and 13 girls in Cluster 2, and 7 boys and 13 girls in Cluster 3. These proportions did not significantly differ ($\chi^2(2) = 5.33$, n.s.). Second, the age of the children in each cluster was compared. The mean age of children in Cluster 1 was 8 years 7 months, in Cluster 2 was 8 years 6 months, and in Cluster 3 was 8 years 7 months. There was no difference between these ages ($F(2,64) < 1$).

Table 2 gives accuracy on inverse and control problems by children in each cluster. The children in Cluster 1 'high score' and Cluster 3 'difference score' showed similar levels of accuracy as each other for both four- and five-term inverse problems, while the children in Cluster 3 had lower levels of accuracy for both four- and five-term control problems. The children in Cluster 2 'low score' had the lowest levels of accuracy for all types of problems. The similar pattern in accuracy for four-and five-term problems suggests that these were meaningful clusters of children, since the cluster-analysis was performed only on scores for four-term problems. Moreover, the differences between the clusters apply both to situations in which children can simply apply their conceptual understanding and to situations in which they must first recognize that it is relevant.

INSERT TABLE 2 ABOUT HERE

To find out more about the differences between the groups, children's accuracy on different problem types was examined. Accuracy on inverse and control problems was compared separately for each problem length and missing element. Due to potential ceiling and floor effects which may limit the interpretation of F-tests, specific non-parametric analyses were used.

Analysis by Groups

Cluster 1: 'High score'. There were 26 children in this group. Children's accuracy on inverse and control problems was compared using Wilcoxon non-parametric tests. The children in this cluster were significantly more accurate on inverse than control problems for four-term problems with the a-term missing ($z = 3.73$, $p < .001$) and with the sum missing ($z = 4.018$, $p < .001$) and for five-term problems with the a-term missing ($z = 4.341$, $p < .001$). However, they were no more accurate on five-term inverse problems with the sum missing than equivalent controls ($z = 0.582$, n.s.; Figure 3a).

The children in this group were able to identify and take advantage of inverse transformations for all types of transparent inverse problems and solve these without calculation. They were also able to recognize and construct inverse transformations for some types of complex inverse problem. However, they were not able to make use of their understanding of the relationship between addition and subtraction in all situations where it was relevant and used calculation to solve some types of complex inverse problems. While in general they had good conceptual understanding and accurate computational skill, these children were able to use their understanding of inversion more flexibly on transparent tasks than on tasks where they first had to recognize that it was relevant.

Cluster 2: 'Low score'. There were 21 children in this group. The children in this cluster showed the same pattern of results as children in Cluster 1. They were significantly more accurate on inverse than control problems for four-term problems with the a-term missing ($z = 3.457$, $p = .001$), and the sum missing ($z = 3.690$, $p < .001$), and for five-term problems with the a-term missing ($z = 3.485$, $p < .001$). However, there was no difference in accuracy between five-term inverse and control problems with the sum missing ($z = 1.337$, n.s.; Figure 3b).

Despite their low overall levels of accuracy, the children in this group showed evidence of understanding the relationship between addition and subtraction. Indeed the pattern of performance of this group was similar to the children in Cluster 1, although at much lower absolute levels. These children were sometimes able to recognize the inverse transformations present in all types of transparent inverse problems and solve these without calculation, although the low overall level of accuracy suggests that these children were not able to do so as accurately or as consistently as the children in Cluster 1, or only some of them could do so. Notwithstanding the inconsistent performance on transparent inverse problems, these children were able to identify and exploit inverse transformations even in some problems where these were not transparent and had to be constructed. However, they were not able to do this on all relevant complex inverse problems. Overall, these children made more use of their conceptual understanding for problems where it was transparently applicable than where they had to identify that it was relevant and reconstruct the problem before applying it.

Cluster 3: 'Difference score'. There were 20 children in this group. The children in this cluster showed a different pattern of results than the children in Clusters 1 and 2. They were significantly more accurate on all types of inverse problem than equivalent control problems (four-term problems a-term missing $z = 3.976$, $p < .001$; four-term problems sum missing $z = 2.761$, $p < .001$; five-term problems a-term missing $z = 3.946$, $p < .001$; five-term problems sum missing $z = 2.527$, $p = .012$; Figure 3c).

The children in this group made use of their conceptual understanding in the widest range of situations. They could flexibly apply their understanding of the relationship between addition and subtraction to solve inverse problems without calculation both where it was transparently applicable and for problems where they had to first recognize that it was relevant, and then reconstruct the elements in order to apply it. In contrast to this sophisticated conceptual understanding these children showed lower levels of computational skill than the children in Cluster 1. Thus the children in this group had very good understanding of the relationship between addition and subtraction despite difficulty performing these operations accurately.

Discussion

This study has revealed three key aspects of children's conceptual and procedural understanding of arithmetic. First, that children aged 8 to 9 years have, on the whole, sophisticated understanding of the relationship between addition and subtraction and not only can they use this to aid problem-solving in situations where it is transparently applicable, but they can also actively construct inverse transformations in order to apply this understanding. Second, that there are clear individual differences among children in the relationship between their conceptual understanding and computational skill in arithmetic, and these differences may shed light on how conceptual and procedural knowledge are related over development. Finally, that children find it more difficult to recognize situations where conceptual understanding may be relevant to problem-solving than to simply apply conceptually-based inferences. This has implications for both the teaching and assessment of arithmetical skill. Each of these conclusions will be examined in turn.

Previous research has demonstrated that children develop understanding of the inverse relation between addition and subtraction over many years. While some children show nascent understanding of this relationship at age 4 (Klein & Bisanz, 2000; Rasmussen et al., 2003) other children fail to take advantage of inverse transformations in a problem at age 9 – 10 (Gilmore & Bryant, 2006). Furthermore, the format, presentation conditions or characteristics of the problem can substantially affect children's ability to exploit the presence of inverse transformations in a problem (Bryant et al., 1999; Stern, 1992; Gilmore, 2006). This study has demonstrated for the first time that children can do a great deal more than just recognize inverse transformations. They can actively construct an inverse relationship in order to aid problem solving. In many situations, children can use their understanding of inversion in a sophisticated manner and go far beyond simply applying a shortcut in response to a recognized pattern of numbers.

These findings are supported by some previous evidence that children may be able to be active users of inversion in a limited sense. This comes from a study by Bryant et al. (1999) in which children (aged 6- to 8-years-old) were given inversion/decomposition problems. In these problems the inverse relationship was not complete but could be created by using decomposition (i.e. $a + b - (b + 1) = \square$ or $a + b - (b - 1) = \square$ e.g. $12 + 7 - 8 = \square$ or $12 + 7 - 6 = \square$). Thus these problems tested whether children could extend understanding of inversion to problems that are not on the surface inverse problems. The children were more accurate on these inversion/decomposition problems than on standard control problems. The present study advances this finding by demonstrating that children can use their understanding of inversion in an active way on problems that require the inverse elements to be constructed using addition.

The body of literature on children's understanding of arithmetical inversion shows many of the features of conceptual knowledge suggested by Vergnaud (1982, 1990, 1997, 1998) and Baroody (Baroody and Ginsburg, 1986; Baroody and Tiilikainen, 2003). Vergnaud (1982, 1990, 1997, 1998) proposed the theory of conceptual fields of knowledge. He suggested that children acquire different properties of the same concept, or are able to apply a concept in different situations over a long period of time. Some aspects of a concept may be mastered many years before other. Baroody and colleagues (Baroody and Ginsburg, 1986; Baroody and Tiilikainen, 2003) described the development of children's conceptual knowledge as a progression of increasingly abstract schemata. At first children's conceptual knowledge is example-driven and context-based, later it becomes increasingly principle-driven, generalized and abstract. Some aspects of a concept may develop later than others. Both these theories highlight that children will often have partial understanding of concepts and so at any point in time it may be inappropriate to try and judge whether children do or do not ‘have’ a concept. As the present study reveals, children develop the ability to apply the concept of inversion to complex problem solving situations many years after they may initially display understanding of this relationship.

This formalization of conceptual knowledge is also consistent with the iterative model of arithmetical development. According to this model, conceptual and procedural knowledge develop together, with advances in one leading to advances in the other. As a result, children may have partial knowledge and procedures, which at first may not be integrated (Bisanz & LeFevre, 1992; Carpenter, 1986). As this study has demonstrated, however, as well as changes in the relationship between conceptual and procedural knowledge within an individual over time there may also be differences in this relationship among individuals.

Using cluster analysis this study revealed that there were three subgroups of children with different patterns of performance. Most children showed conceptual understanding that was in line with their computational skill. However, a subgroup of children had good understanding of the relationship between addition and subtraction despite difficulty performing these operations. Previous research that found evidence of these subgroups (Gilmore & Bryant, 2006) was unable to determine whether this group of children had good conceptual understanding given their computational skills, or poor computational skills given their conceptual understanding. This study revealed that these children showed more sophisticated and flexible use of the inverse relationship on complex inverse problems than children in the other groups. They were the only group to show evidence of recognizing and exploiting the inverse transformations present in all problem types. This suggests that the discrepant performance of the children in this group is due to advanced conceptual understanding rather than delayed or deficient calculation skills.

An important question to arise from this study concerns the meaning of the three clusters. We need to consider why children fall into these groups. There are two main alternatives. The first possible explanation is that these groups represent progression along a single path of development. Children may initially have low conceptual understanding and low calculation skills. Their conceptual understanding may then develop first. Later there may be an improvement in their computational skills. Thus the children in the difference score group are at a particular point on this developmental trajectory with respect to learning inversion and proficiency with addition and subtraction. If this explanation were correct we would expect to see children moving from low scores on both measures of inversion and computation to having high scores on inversion and low scores on computation and then finally having high scores on both measures. There are two findings from the present study that suggest this may be the wrong interpretation of the groups. First, the children in Cluster 3 'difference score' made more sophisticated use of inversion than the children in Cluster 1 'high score'. Thus, it is unlikely that Cluster 1 is a later stage of development than Cluster 3. One way this could happen, however, is if the children in Cluster 1 have as sophisticated understanding of inversion as the children in Cluster 3 but choose not to use it in all situations, perhaps because of their more advanced computational skill. This does not explain why these children would apply their conceptual understanding in most situations but choose to use computation for certain problems. A second piece of evidence that does not support the interpretation that these groups form a single path of development is that there was no difference in the age of the children in each group. A wider range of ages, however, might be needed to find this difference.

The second possible explanation of the three clusters is that these groups represent individual differences among children and alternative developmental paths. It could be that for most children conceptual understanding and computational skill develop together. These children would move from having low scores on both measures of inversion and computation to having high scores on both measures. At the same time a substantial subgroup of children show a different pattern. Their conceptual understanding is more advanced than their computational skill. In this case, either these children have surprisingly good conceptual understanding or they are failing to learn efficient computation. There is evidence from this study that these children may have better than expected conceptual understanding. These children may follow this alternative developmental path due to either educational experiences, or a particular cognitive profile (e.g. memory capacity, IQ, language skills) or the fit between the two.

Canobi (2005) provides further evidence to suggest that there may be individual differences in the development of understanding of inversion. She demonstrated that some children may learn about inversion through understanding that an addition transformation and a subtraction transformation cancel each other out (i.e. 15 + 8 – 8 = 15) while other children may learn about inversion by recognizing the relationship between a given addition sum and a complementary subtraction sum (i.e. 5 + 3 = 8 implies that 8 – 5 = 3). Thus, there may be different routes to developing inversion understanding.

If the clusters represent stable individual differences this would have implications for theories of mathematical development. At present, developmental theories (e.g. Baroody & Ginsburg, 1986; Fuson, 1992; Rittle-Johnson et al., 2001; Vergnaud, 1997) do not consider individual differences in the way that children develop mathematical expertise. Instead these theories tend to propose just a single developmental path. Theories about mathematical development will need to be expanded to incorporate the possibility of alternative developmental paths.

The final conclusions of the present study concern the distinction between being able to make a conceptually-based inference, and realizing that one is relevant. This study showed that children find it more difficult to use conceptual understanding in problem-solving when they have to identify that it is relevant, and reconstruct the elements of the problem to make it applicable first. Since problem solving in everyday life almost always involves the extra elements of identifying relevant situations and reconstructing elements, to get a true picture of children's conceptual competence, we must do more than just use artificial tasks on which a concept is transparently applicable. Moreover, arithmetical instruction should encourage children to consider different ways in which a problem might be solved, and whether different conceptually-based inferences or analogies might provide shortcuts. As highlighted by Vergnaud (1990) it is important to consider the application of arithmetical concepts in a range of situations. This will help children to develop conceptual understanding that is generalized and abstracted rather than tied to particular situations. The aim of mathematical instruction should be to help children integrate their procedural and conceptual knowledge, so that they approach new problems by considering how they might simplify the problem on the basis of their conceptual understanding, rather than performing procedures in a rote and meaningless fashion.

References

Baroody, A. J., & Dowker, A. (2003). *The development of arithmetic concepts and skills: Constructing adaptive expertise.* Mahwah, NJ: Erlbaum.

Baroody, A. J., & Ginsburg, H. P. (1986). The relationship between initial meaningful and mechanical knowledge of arithmetic. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 75-112). Hillsdale, NJ: Erlbaum.

Baroody, A. J., Ginsburg, H. P., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education, 14*, 156-168.

Baroody, A. J., & Tiilikainen, S. H. (2003). Two perspectives on addition development. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 75-125). Mahwah, NJ: Erlbaum.

Bisanz, J., & LeFevre, J.-A. (1992). Understanding elementary mathematics. In J. I. D. Campbell (Ed.), *The nature and origins of mathematical skills* (pp. 113-136). Amsterdam: Elsevier.

Bisanz, J., LeFevre, J.-A., & Gilliland, S. (1989, April). *Developmental changes in the use of logical principles in mental arithmetic.* Paper presented at the biennial meeting of the Society for Research in Child Development, Kansas City.

Bryant, P., Christie, C., & Rendu, A. (1999). Children's understanding of the relation between addition and subtraction: Inversion, identity and decomposition. *Journal of Experimental Child Psychology, 74*, 194-212.

Bryant, P., & Nunes, T. (2002). Children's understanding of mathematics. In U. Goswami (Ed.), *Blackwell handbook of children's cognitive development* (pp. 412-439). Oxford, UK: Blackwell.

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (1998). The role of conceptual understanding in children's addition problem solving. *Developmental Psychology, 34*, 882-891.

Canobi, K. H. (2004). Individual differences in children's addition and subtraction knowledge. *Cognitive Development, 19*, 81-93.

Canobi, K.H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology*, *92*, 220-246.

Carpenter, T. P. (1986). Conceptual knowledge as a foundation for procedural knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 113-132). Hillsdale, NJ: Erlbaum.

Dowker, A. (1998). Individual differences in normal arithmetical development. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 275-302). Hove, UK: Psychology Press.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 243-275). New York: Macmillan.

Gilmore, C.K. (2006). Investigating children's understanding of inversion using the missing number paradigm. *Cognitive Development, 21,* 301 – 316.

Gilmore, C.K. & Bryant, P. (2006), Individual differences in children's understanding of inversion and arithmetical skill. *British Journal of Educational Psychology,* 76, 309 – 331.

Klein, J. S., & Bisanz, J. (2000). Preschoolers doing arithmetic: The concepts are willing but the working memory is weak. *Canadian Journal of Experimental Psychology, 54*, 105-115.

Piaget, J. (1952). *The child's conception of number*. London: Routledge & Kegan Paul.

Piaget, J., & Moreau, A. (2001). The inversion of arithmetic operations. In J. Piaget (Ed.), *Studies in reflecting abstraction (R.L. Campbell Trans., pp. 69-86)*. Hove, UK: Psychology Press.

Rasmussen, C., Ho, E., & Bisanz, J. (2003). Use of the mathematical principle of inversion in young children. *Journal of Experimental Child Psychology, 85*, 89-102.

Rittle-Johnson, B., & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 75-110). Hove, UK: Psychology Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*, 346-362.

Robinson, K. M., Ninowski, J. E., & Gray, M. L. (2006). Children's understanding of the arithmetic concepts of inversion and associativity. *Journal of Experimental Child Psychology, 94,* 349 – 362.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General, 127*, 377-397.

Singer-Freeman, K.E. & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development*, *16*, 811-829.

Sophian, C. (1997). Beyond competence: The significance of performance for conceptual development. *Cognitive Development, 12*, 281-303.

Stern, E. (1992). Spontaneous use of conceptual mathematical knowledge in elementary school children. *Contemporary Educational Psychology, 17*, 266-277.

Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2005). Simple addition strategies in a first-grade class with multiple strategy instruction. *Cognition and Instruction, 23,* 1-21.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39-59). Hillsdale, NJ: Erlbaum.

Vergnaud, G. (1990). Problem solving and concept-formation in the learning of mathematics. In H. Mandl, E. De Corte, S. N. Bennett & H. F. Friedrich (Eds.), *Learning and instruction: European research in an international context* (Vol. 2.2, pp. 399-413). Oxford, UK: Pergamon Press.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 5-28). Hove, UK: Psychology Press.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior, 17*, 167-181.

Table 1. Example four- and five-term inverse and control problems with each element order and the 'a' term or sum missing.

Problem type			Inverse	Control
4-term	Order 1	a	$\square + 9 - 9 = 26$	$\square + 11 - 6 = 31$
		Sum	$17 + 11 - 11 = \square$	$21 + 9 - 13 = \square$
	Order 2	a	$12 - 12 + \square = 26$	$8 - 4 + \square = 30$
		Sum	$12 - 12 + 17 = \square$	$12 - 7 + 12 = \square$
5-term	Order 1	a	$\square + 12 - 9 - 3 = 18$	$\square + 11 - 5 - 2 = 20$
		Sum	$15 + 11 - 8 - 3 = \square$	$13 + 11 - 5 - 4 = \square$
	Order 2	a	$13 - 8 - 5 + \square = 18$	$11 - 3 - 3 + \square = 23$
		Sum	$9 - 6 - 3 + 15 = \square$	$9 - 5 - 2 + 13 = \square$

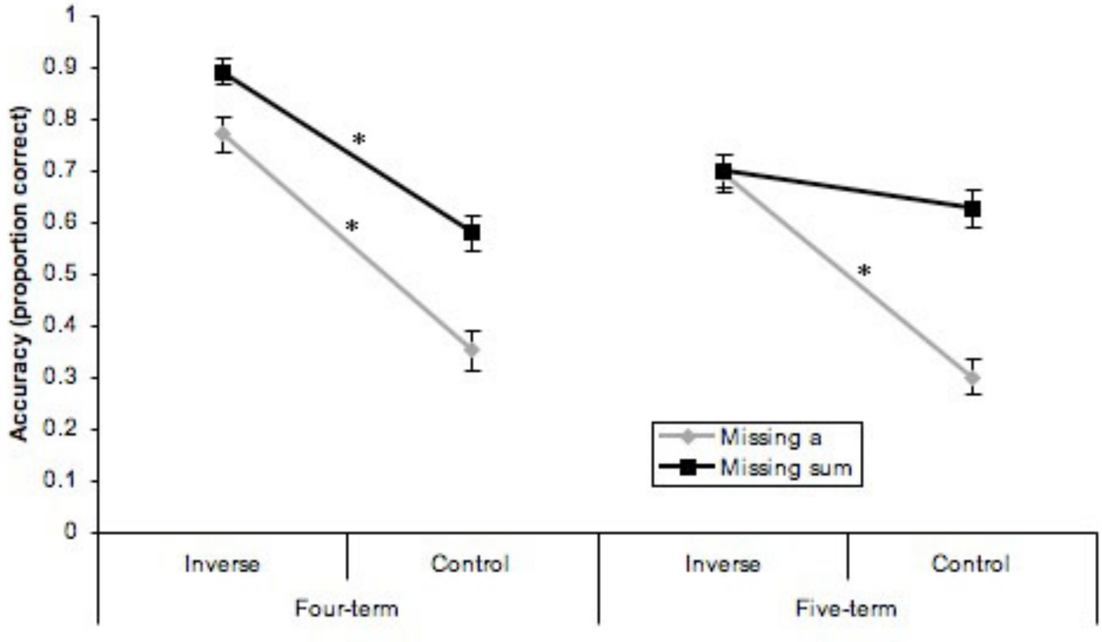
Table 2. Mean accuracy (proportion correct) on four-term and five-term inverse and control problems for each cluster.

Group		Four-term		Five-term	
		Inverse	Control	Inverse	Control
Cluster 1					
	Mean	0.95	0.73	0.83	0.67
	SD	0.07	0.12	0.14	0.14
Cluster 2					
	Mean	0.58	0.21	0.42	0.24
	SD	0.21	0.19	0.24	0.24
Cluster 3					
	Mean	0.94	0.38	0.81	0.43
	SD	0.07	0.11	0.12	0.16

Figure Captions

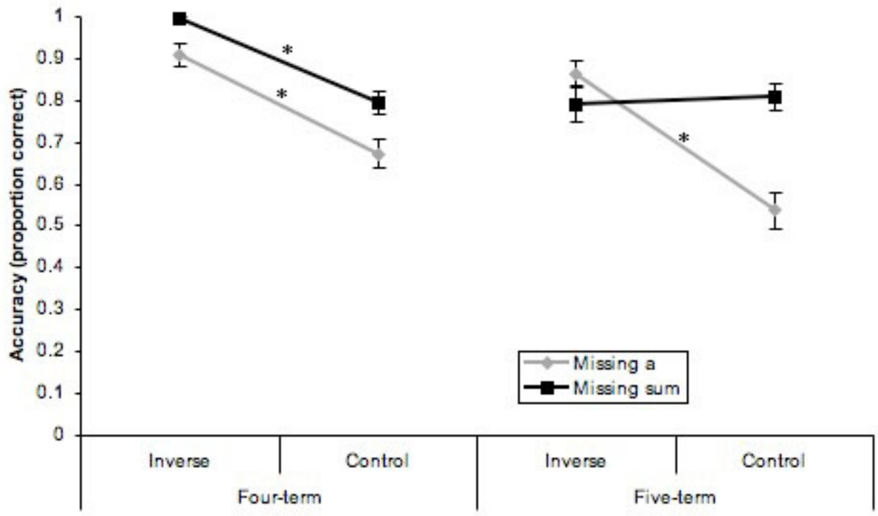
Fig 1: Mean accuracy for different problem types for the whole group (error bars show s.e.m.)

Fig 2: Mean accuracy for different problem types by children in a) Cluster 1 'high score', b) Cluster 2 'low score' and c) Cluster 3 'difference score' (error bars show s.e.m.)

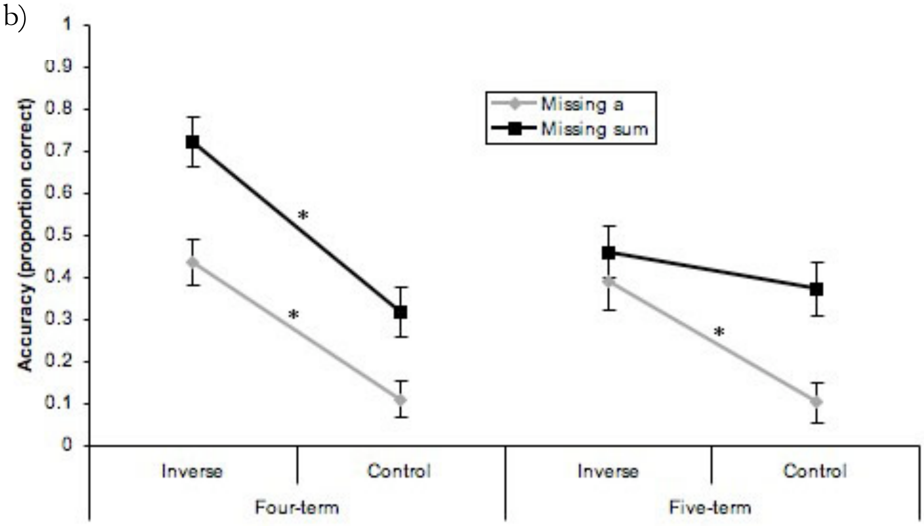


* significant difference between accuracy on inverse and control problems (p < .05)

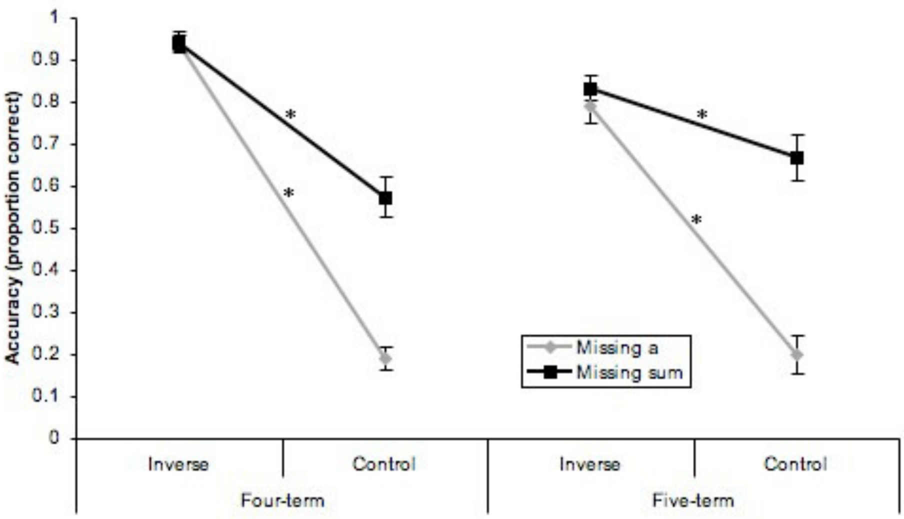
a)



b)



c)



* significant difference between accuracy on inverse and control problems (p < .05)